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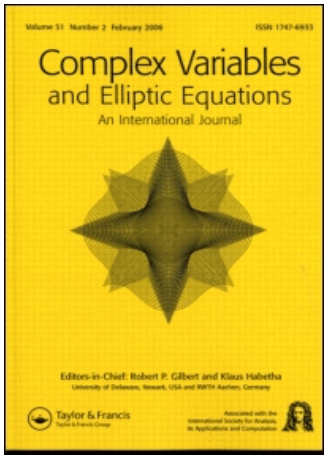

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More approximation on disks

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More approximation on disks

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In this article we study the function algebra generated by z^2 and g^2 on a small closed disk centred at the origin of the complex plane. We prove, using a biholomorphic change of coordinates and already developed techniques in this area, that for a large class of functions g this algebra consists of all continuous functions on the disk.

Keywords: Function algebra; Uniform approximation; Polynomial convexity

2000 Mathematics Subject Classifications: 46J10; 32E20

1. Introduction

Let g be a C^1 function defined in a neighbourhood of the origin in the complex plane, with $g(0) = 0$, $g_z(0) = 0$, $g_{\bar{z}}(0) = 1$ (i.e. g behaves like $\bar{z}$ near 0), and such that z^2 and g^2 separate points near 0. Is it possible to find a small closed disk D about 0 in the complex plane, so that every continuous function on D can be approximated uniformly on D by polynomials in z^2 and g^2? In other words is the function algebra $[z^2, g^2; D]$ on D generated by z^2 and g^2, i.e. the uniform closure in $C(D)$ of the polynomials in z^2 and g^2, equal to $C(D)$? It has been shown that both answers *no* and *yes* are possible, cf. [1,2].

The motivating question for this approximation problem was whether $[z^2, \bar{z}^2 + \bar{z}^3; D]$ equals $C(D)$. The answer has been given recently by O'Farrell and Sanabria-García and is *no*, cf. [3].

The crucial point in showing whether or not the algebra $[z^2, g^2; D]$ coincides with $C(D)$, is to determine whether or not the preimage of $X = (z^2, g^2)(D)$ under the

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map $\Pi(\zeta_1, \zeta_2) = (\zeta_1^2, \zeta_2^2)$ is polynomially convex. Now the set $\Pi^{-1}(X)$ consists of the following four disks:

$$\begin{aligned} D_1 &= \{(z, g(z)) : z \in D\}, \\ D_2 &= \{(-z, -g(z)) : z \in D\} = \{(z, -g(-z)) : z \in D\}, \\ D_3 &= \{(-z, g(z)) : z \in D\}, \\ D_4 &= \{(z, -g(z)) : z \in D\} = \{(-z, -g(-z)) : z \in D\}. \end{aligned}$$

In this situation our problem boils down to the (non-)polynomial convexity of $D_1 \cup D_2$.

An appropriate tool in this context is Kallin's lemma: suppose X_1 and X_2 are polynomially convex subsets of $\mathbb{C}^n$, suppose there is a polynomial p mapping X_1 and X_2 into two polynomially convex subsets Y_1 and Y_2 of the complex plane such that 0 is a boundary point of both Y_1 and Y_2 and with $Y_1 \cap Y_2 = \{0\}$. If $p^{-1}(0) \cap (X_1 \cup X_2)$ is polynomially convex, then $X_1 \cup X_2$ is polynomially convex [4,5].

In [6], Nguyen and the first author obtained a positive answer to our approximation question in a real-analytic situation for a new class of functions g. By using a biholomorphic change of coordinates, it is possible to assume that the first disk is the standard disk $\{(z, \bar{z}) : z \in D\}$ and then apply an approximation result of Nguyen [7]. In the present article the same idea of applying a biholomorphic map near the origin together with already developed techniques in this area is used. We obtain several new results of the form $[z^2, g^2; D] = C(D)$, one of them being a generalisation of the main result of [6], for new and larger classes of functions g (Theorem 2.5).

2. An approximation result

We agree on the following convention: all functions defined in a neighbourhood of the origin are of class C^1, even if we do not mention this explicitly.

Definition 2.1 Let $g(z)$ be an *even* function defined near the origin with $g(z) = o(z)$. Suppose that there exists a polynomial $p(\zeta_1, \zeta_2)$ such that for all functions $R(z)$ with $R(z) = o(g(z))$ both

$$\operatorname{Im} p(z, \bar{z} + g(z) + R(z)) > 0$$

and

$$\operatorname{Im} p(z, \bar{z} - g(z) + R(z)) < 0$$

hold for all $z \neq 0$ sufficiently close to 0. We then say that g satisfies the *polynomial condition* (with respect to p).

We analyse this condition in the next lemma, in particular for the situation where the condition $g(z) = o(z)$ is replaced by the stronger

$$\begin{aligned} &g \text{ is } \textit{homogeneous} \text{ of degree } m > 1, \text{i.e.} \\ &g(tz) = t^m g(z) \quad \text{for } t > 0 \end{aligned}$$

(so in fact g is defined everywhere).

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Example 2.2

- If $m > 1$, then for $g(z) = i|z|^m$ one can take $p(\zeta_1, \zeta_2) = \zeta_1 + \zeta_2$.
- For the function $g(z) = a|z|^2 + b\bar{z}^2$ with $|b| < |a|$ one can take $p(\zeta_1, \zeta_2) = -ia\zeta_1 + i\bar{a}\zeta_2$. From this fact a version of the main result of [2] follows.
- The function $g(z) = |z|^2 + \bar{z}^2$ does not satisfy the polynomial condition because it has non-zero zeroes.
- The function $g(z) = z^3\bar{z}$ satisfies the polynomial condition with respect to $p(\zeta_1, \zeta_2) = -i\zeta_1^3 + i\zeta_2^3$.

LEMMA 2.3

(1) *If g satisfies the polynomial condition with respect to a polynomial p, then g satisfies the polynomial condition with respect to the odd part of the polynomial p.*
(2) *If g is even and of class C^1 near the origin in the complex plane, is homogeneous of order $m > 1$ satisfying*

$$\operatorname{Im} \frac{\partial p}{\partial \zeta_2}(z, \bar{z}) \cdot g(z) > 0 \quad \text{if } |z| = 1,$$

where p is a homogeneous polynomial of odd degree n, which is complex-symmetric i.e. with $p(\zeta_1, \zeta_2) = \sum_{k=0}^{n} a_k \zeta_1^k \zeta_2^{n-k}$ and $a_k = \overline{a_{n-k}}$ for all $k = 0, \ldots, n$, then g satisfies the polynomial condition with respect to p.

Proof We start with the proof of part (1). Fix $R(z) = o(g(z))$ for the moment, then for $z \neq 0$ close to 0, we have:

$$\text{(a)} \quad \operatorname{Im} p(z, \bar{z} + g(z) + R(z)) > 0,$$
$$\text{(b)} \quad \operatorname{Im} p(z, \bar{z} - g(z) - R(-z)) < 0.$$

Replace z by $-z$ in (b) and use the fact that g is even, then also:

$$\text{(c)} \quad \operatorname{Im} p(-z, -\bar{z} - g(z) - R(z)) < 0.$$

Now write p as a sum of homogeneous analytic polynomials, in other words $p = p_s + \cdots + p_n$ where p_j is homogeneous of degree j. Rewrite (c), for small $z \neq 0$, as:

$$\sum_{j=s}^{n} (-1)^{j+1} p_j(z, \bar{z} + g(z) + R(z)) > 0.$$

Combination with (a) shows that all terms with j even in (a) drop out. In a similar way these terms can be removed in the second part of the polynomial condition.

We not only show part (2) of the lemma but also indicate how to arrive at the condition stated in this part. So from (1) we see that if g satisfies the polynomial condition, then there is an odd polynomial p such that

$$\text{(a)} \quad \operatorname{Im} p(z, \bar{z} + g(z) + R(z)) > 0$$

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and

$$\text{(b)} \quad \operatorname{Im} p(z, \bar{z} - g(z) + R(z)) < 0$$

hold for all $z \neq 0$ sufficiently close to 0 if $R(z) = o(g(z))$. Now write p as a sum of homogeneous analytic polynomials, $p = p_{2s-1} + \cdots + p_{2n-1}$ where all p_k are homogeneous of odd degree k.

We assume first that m is not an odd integer. Let $n_0 \leq n$ be maximal such that $2n_0 - 1 < 2s - 2 + m$.

Taking for R the zero function we obtain:

$$\begin{aligned} p(z, \bar{z} + g(z)) = {} & p_{2s-1}(z, \bar{z}) + \cdots + p_{2n_0-1}(z, \bar{z}) \\ & + \frac{\partial p_{2s-1}}{\partial \zeta_2}(z, \bar{z}) \cdot g(z) + O(|z|^{\alpha}), \end{aligned}$$

for some $\alpha > 2s - 2 + m$. Now we restrict z to the unit circle Γ, and obtain for $t > 0$:

$$\begin{aligned} p(tz, t\bar{z} + g(tz)) = {} & t^{2s-1} p_{2s-1}(z, \bar{z}) + \cdots + t^{2n_0-1} p_{2n_0-1}(z, \bar{z}) \\ & + t^{2s-2+m} \frac{\partial p_{2s-1}}{\partial \zeta_2}(z, \bar{z}) \cdot g(z) + O(t^{\alpha}). \end{aligned}$$

Now take the imaginary part, divide by t^{2s-1} and let t tend to 0. We obtain $\operatorname{Im} p_{2s-1}(z, \bar{z}) \geq 0$. Similarly, using the second condition on g, we obtain $\operatorname{Im} p_{2s-1}(z, \bar{z}) \leq 0$, hence $\operatorname{Im} p_{2s-1}(z, \bar{z}) = 0$ for all $z \in \Gamma$ (hence for all $z \in \mathbb{C}$). Writing $p_{2s-1}(\zeta_1, \zeta_2) = \sum_{k=0}^{2s-1} a_k \zeta_1^k \zeta_2^{2s-1-k}$ this means that $a_k = \overline{a_{2s-1-k}}$ for all $k = 0, \ldots, 2s-1$, i.e. the polynomial p_{2s-1} is complex-symmetric.

Repeating this reasoning we successively obtain:

$$\operatorname{Im} p_{2s+1}(z, \bar{z}) = 0, \ldots, \operatorname{Im} p_{2n_0-1}(z, \bar{z}) = 0$$

and

$$\operatorname{Im} \frac{\partial p_{2s-1}}{\partial \zeta_2}(z, \bar{z}) \cdot g(z) \geq 0. \tag{1}$$

Also in the case that m is an odd integer (a) and (b) in a similar way as above lead to (1).

Now suppose that for all $z \in \Gamma$ the inequality (1) is strict then we will show that the polynomial condition is satisfied for g with respect to the polynomial p_{2s-1}. Indeed, if $R(z) = o(g(z))$ it follows for small $z \neq 0$:

$$\begin{aligned} p_{2s-1}(z, \bar{z} + g(z) + R(z)) = {} & p_{2s-1}(z, \bar{z}) \\ & + \frac{\partial p_{2s-1}}{\partial \zeta_2}(z, \bar{z}) \cdot g(z) \cdot \left(1 + \frac{R(z)}{g(z)}\right) + O(|z|^{2s-3+2m}). \end{aligned}$$

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So for $z \in \Gamma$ and small $t > 0$ it follows that:

$$\begin{aligned}&\operatorname{Im} p_{2s-1}(tz, t\bar{z} + g(tz) + R(tz))\\ &\quad = \operatorname{Im} t^{2s-2+m}\left(\frac{\partial p_{2s} - 1}{\partial \zeta_2}(z, \bar{z}) \cdot g(z) \cdot \left(1 + \frac{R(tz)}{g(tz)}\right) + O(t^{m-1})\right).\end{aligned}$$

Since $R(tz)/g(tz)$ is uniformly small on Γ if $t > 0$ is sufficiently small, the above expression is positive on Γ for small $t > 0$. In other words: $\operatorname{Im} p_{2s-1}(z, \bar{z} + g(z) + R(z)) > 0$ if $z \neq 0$ is sufficiently small. Also $\operatorname{Im} p_{2s-1}(z, \bar{z} - g(z) + R(z)) < 0$ for small $z \neq 0$. So g satisfies the polynomial condition with respect to p_{2s-1}.

The result of (2) immediately follows. ■

We need the following lemma which is without doubt well known.

Auxiliary Lemma 2.4 *Let $F(w_1, w_2)$ be holomorphic near the origin, let $l \geq 2$ be an integer and let $F(w_1, w_2) = O(\|(w_1, w_2)\|^l)$.*

Let $A(w_1, w_2)$ be defined near the origin with

$$A(w_1, w_2) = O(\|(w_1, w_2)\|).$$

Then sufficiently close to the origin

$$F(w_1, w_2 + A(w_1, w_2)) = F(w_1, w_2) + A(w_1, w_2)B(w_1, w_2),$$

with $B(w_1, w_2) = O(\|(w_1, w_2)\|^{l-1})$.

Proof As $F(w_1, w_2)$ is holomorphic near the origin,

$$H(w_1, w_2, w_3) = \begin{cases} \dfrac{F(w_1, w_3) - F(w_1, w_2)}{w_3 - w_2}, & \text{if } w_3 \neq w_2, \\ \dfrac{\partial F}{\partial \zeta_2}(w_1, w_2), & \text{if } w_3 = w_2, \end{cases}$$

is holomorphic near the origin, $H(w_1, w_2, w_3) = O(\|(w_1, w_2, w_3)\|^{l-1})$ and

$$F(w_1, w_2 + z) = F(w_1, w_2) + zH(w_1, w_2, w_2 + z).$$

Since $A(w_1, w_2) = O(\|(w_1, w_2)\|)$ it follows that

$$F(w_1, w_2 + A(w_1, w_2)) = F(w_1, w_2) + A(w_1, w_2)B(w_1, w_2),$$

and

$$B(w_1, w_2) = H(w_1, w_2, w_2 + A(w_1, w_2)) = O(\|(w_1, w_2)\|^{l-1}).$$ ■

We now formulate our main result. It consists of a general statement on generators of the algebra of all continuous functions near the origin, and of an application of this result in a special situation.

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THEOREM 2.5

(1) *Consider the functions F, g and h.*

- *Let $F(w_1, w_2)$ be an odd holomorphic function near the origin satisfying $F(w_1, w_2) = O(\|(w_1, w_2)\|^3)$ and let $f(z) = F(z, \bar{z})$.*
- *Suppose that g satisfies the polynomial condition.*
- *Let h be defined near the origin with $h(z) = o(g(z))$.*

Then for all disks D about 0 with sufficiently small radius

$$[z^2,\ (\bar{z} + f(z) + g(z) + h(z))^2 : D] = C(D).$$

(2) *In particular this result can be applied in the following situation: let $g(z) = \sum_{k=-\infty}^{\infty} a_k \bar{z}^k z^{2m-k}$ with m a positive integer. Suppose that $\sum_{k=-\infty}^{\infty} |ka_k| < \infty$ and that one of the following increasingly weaker conditions is met:*

$$\exists\ l \leq m \text{ such that } |a_l| > \sum_{n \neq l} |a_n|,$$

or

$$\exists\ l \leq m \text{ such that } \sum_{n=1}^{\infty} \left| \frac{a_{l+n}}{a_l} + \frac{\bar{a}_{l-n}}{\bar{a}_l} \right| < 1,$$

or

$$\exists\ l \leq m \text{ such that } \operatorname{Re}\left(1 + \sum_{n=1}^{\infty} \left(\frac{a_{l+n}}{a_l} + \frac{\bar{a}_{l-n}}{\bar{a}_l} \right) w^n \right) > 0 \quad \text{on } |w| = 1.$$

Then g is an even homogeneous C^1 function of degree $2m$ that satisfies the polynomial condition.

Proof For part (1), let $X = \{(z^2,\ (\bar{z} + f(z) + g(z) + h(z))^2) : z \in D\}$.

The inverse image of X under the map $\Pi : \mathbb{C}^2 \to \mathbb{C}^2$, defined by $\Pi(\zeta_1, \zeta_2) = (\zeta_1^2, \zeta_2^2)$ consists of

$$\begin{aligned}
D_1 &= \{(z, \bar{z} + f(z) + g(z) + h(z)) : z \in D\}, \\
D_2 &= \{(-z, -(\bar{z} + f(z) + g(z) + h(z))) : z \in D\} \\
&= \{(z, \bar{z} + f(z) - g(z) - h(-z)) : z \in D\}, \\
D_3 &= \{(-z, \bar{z} + f(z) + g(z) + h(z)) : z \in D\}, \\
D_4 &= \{(z, -(\bar{z} + f(z) + g(z) + h(z))) : z \in D\} \\
&= \{(-z, \bar{z} + f(z) - g(z) - h(-z)) : z \in D\}.
\end{aligned}$$

Note that the condition on the existence of the polynomial p implies that g has no non-zero zeroes and that the two functions z^2 and $(\bar{z} + f(z) + g(z) + h(z))^2$ separate the points of D (if D is sufficiently small).

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The techniques developed in the papers [1,2] on approximation on disks give us:

$$\begin{aligned}
&[z^2, (\bar{z} + f(z) + g(z) + h(z))^2 : D] = C(D) \\
\iff &P(X) = C(X) \\
\iff &X \text{ is polynomially convex} \\
\iff &D_1 \cup D_2 \cup D_3 \cup D_4 \text{ is polynomially convex} \\
\iff &D_1 \cup D_2 \text{ is polynomially convex.}
\end{aligned}$$

We comment on these equivalences. The first equivalence is trivial. Since X is totally real except at the origin, the second one follows from a theorem of O'Farrell *et al.* [8]. The next equivalence is a consequence of a theorem of Sibony [9], and the last one is an application of Kallin's lemma using the polynomial $p(\zeta_1, \zeta_2) = \zeta_1 \cdot \zeta_2$.

Later on we will also use the following theorem of Wermer [10]. *If the function F is of class C^1 near the origin in the complex plane, with $F_{\bar{z}}(0) \neq 0$, then $[z, F : D] = C(D)$ if D is a sufficiently small disk around* 0.

For precise statements and use of these theorems [1], in particular the proof of theorem 1.

Now let us show that $D_1 \cup D_2$ is polynomially convex. Consider the map $G(w_1, w_2) = (w_1, w_2 + F(w_1, w_2))$. Since $F(w_1, w_2) = O(\|(w_1, w_2)\|^3)$ it follows that G is biholomorphic near the origin (with inverse called H).

Now $E_1 = H(D_1)$ consists of points of the form $(z, q(z))$ where q is of class C^1 near 0 and $q(0) = 0$. Then there are a and b such that $q(z) = az + b\bar{z} + r(z)$, where $r(z) = o(z)$. Applying G we see

$$(z, q(z) + F(z, q(z))) = (z, \bar{z} + f(z) + g(z) + h(z)). \tag{2}$$

Since $f(z) + g(z) + h(z) = O(z^3) + o(z) + o(z)$ and moreover $F(z, q(z)) = O(z^3)$ we infer that $q(z) = \bar{z} + r(z)$. So (2) translates into

$$(z, \bar{z} + r(z) + F(z, \bar{z} + r(z))) = (z, \bar{z} + f(z) + g(z) + h(z)).$$

Applying the auxiliary lemma to this expression with $w_1 = z, w_2 = \bar{z}$ and $A(w_1, w_2) = r(w_1)$ we obtain:

$$(z, \bar{z} + r(z) + f(z) + r(z)B(z, \bar{z})) = (z, \bar{z} + f(z) + g(z) + h(z)).$$

It follows that

$$r(z) = \frac{g(z) + h(z)}{1 + B(z, \bar{z})} = g(z) + \frac{h(z) - g(z)B(z, \bar{z})}{1 + B(z, \bar{z})}.$$

We conclude that $E_1 = H(D_1)$ consists of points $(z, \bar{z} + g(z) + R_1(z))$ in which $R_1(z) = o(g(z))$ and is of class C^1.

Now E_1 is polynomially convex if D is sufficiently small (Wermer).

Similarly $E_2 = H(D_2)$ consists of points $(z, \bar{z} - g(z) + R_2(z))$ in which $R_2(z) = o(g(z))$ and is of class C^1. Also E_2 is polynomially convex if D is sufficiently small. Since g satisfies the polynomial condition, Kallin's lemma can be applied, showing that

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$E_1 \cup E_2$ is polynomially convex. Applying G it follows that $D_1 \cup D_2$ is polynomially convex for sufficiently small D.

(2) We now show the second part of the theorem using Lemma 2.3.

Let $p(\zeta_1, \zeta_2) = \bar{\alpha}\zeta_1^{2m-2l+1} + \alpha\zeta_2^{2m-2l+1}$ with α to be determined later (and with $l \leq m$). Then for $z \in \Gamma$:

$$\begin{aligned}
&\frac{1}{2m-2l+1}\operatorname{Im}\frac{\partial p}{\partial \zeta_2}(z,\bar{z})\cdot g(z)\\
&\quad = \operatorname{Im}\sum_{k=-\infty}^{\infty}\alpha a_k \bar{z}^{2m-2l+k} z^{2m-k}\\
&\quad = \operatorname{Im}\left\{\sum_{k=-\infty}^{l-1}\alpha a_k \bar{z}^{2m-2l+k} z^{2m-k} + \alpha a_l |z|^{4m-2l} + \sum_{k=l+1}^{\infty}\alpha a_k \bar{z}^{2m-2l+k} z^{2m-k}\right\}\\
&\quad = \operatorname{Im}\left\{\alpha a_l |z|^{4m-2l} + \sum_{n=1}^{\infty}\left(\alpha a_{l+n} - \bar{\alpha}\bar{a}_{l-n}\right)\bar{z}^{2m-l+n} z^{2m-l-n}\right\}\\
&\quad = \operatorname{Im}\left\{i|a_l||z|^{4m-2l}\left(1 + \sum_{n=1}^{\infty}\left(\frac{a_{l+n}}{a_l} + \frac{\bar{a}_{l-n}}{\bar{a}_l}\right)\left(\frac{\bar{z}}{z}\right)^n\right)\right\}.
\end{aligned}$$

In the last equality we chose $\alpha = i(|a_l|/a_l)$. The final expression has positive imaginary part for $|z| = 1$ if the third condition in the statement of the theorem is satisfied. ■

Remark 2.6 This result includes the more restricted case of polynomials $g(z) = \sum_{k=0}^{2m} a_k \bar{z}^k z^{2m-k}$ in z and $\bar{z}$, for which there exists $0 \leq l \leq m$ such that $|a_l| > \sum_{k \neq l} |a_k|$, essentially studied by Nguyen [7], and applied in a real-analytic setting by Nguyen and de Paepe [6]. The condition on the coefficients here is more general. For instance if $m = 1$ the condition is valid if $|(a_2/a_1) + (\bar{a}_0/\bar{a}_1)| < 1$, which is certainly the case for (but is not equivalent to) $|a_1| > |a_0| + |a_2|$.

Example 2.7 Applying both parts of Theorem 2.5 we obtain a result from [11]:

$$[z^2, \bar{z}^2 + z^3; D] = \left[z^2, \left(\bar{z} + \frac{1}{2}\frac{z^3}{\bar{z}} + \text{h.o.t.}\right)^2; D\right] = C(D).$$

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