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2 The a-number of jacobian varieties

This chapter is a revised and slightly expanded version of our article [31][1].

2.1 Introduction

Let X be a smooth complete irreducible curve defined over an algebraically closed field k of characteristic $p > 0$. The purpose of this chapter is to find limitations on the possible a-numbers of jacobian varieties of dimension g, see definition 1.1. We also know that the a-number of the jacobian variety J of a curve X may be defined as the dimension of the kernel of

$$F^* : H^1(\mathcal{O}_J) \to H^1(\mathcal{O}_J),$$

Using the identification $H^1(\mathcal{O}_J) \cong H^1(\mathcal{O}_X)$, via a Abel-Jacobi map, one may also compute $a(J)$ as the dimension $\dim\ker F^* : H^1(\mathcal{O}_X) \to H^1(\mathcal{O}_X)$. This number is easier to compute via the action of the Cartier operator on regular differential forms, defined in [4]. It is a σ^{-1}-linear operator acting on the sheaf Ω^1_X of differential forms on X, with σ denoting the Frobenius automophism of k. It has the following properties:

1. $\mathcal{C}(\omega_1 + \omega_2) = \mathcal{C}(\omega_1) + \mathcal{C}(\omega_2)$
2. $\mathcal{C}(df) = 0$
3. $\mathcal{C}(f^p\omega) = f\mathcal{C}(\omega)$
4. $\mathcal{C}(f^{p-1}df) = df$
5. $\mathcal{C}(df/f) = df/f$

where ω_1, ω_2 (respectively f) are local sections of Ω^1_X (respectively of $\mathcal{O}_X$).
This operator induces a p-linear map $\mathcal{C} : H^0(\Omega^1_X) \to H^0(\Omega^1_X)$ acting on the space of the regular differential forms. For $f \in H^1(\mathcal{O}_X)$, $\omega \in H^0(\Omega^1_X)$ and $\langle \cdot, \cdot \rangle$ the Serre-duality pairing, it holds

$$\langle f, \mathcal{C}\omega \rangle^p = \langle F^* f, \omega \rangle.$$

This relation can be thought as a duality relation between the Cartier operator and the Hasse-Witt operator F^*. This justifies the use of the Cartier operator for calculating the a-number, as stated above.

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We wish also to draw the reader's attention to the fact that, by the properties above, the kernel of the Cartier operator on a curve is the space of *locally exact regular differential forms* on the curve i.e. the regular differential forms ω such that, locally, $\omega = df$. In characteristic p the numbers g and $a(\mathcal{C}) = \dim \ker \mathcal{C}$ need not to be equal, see chapter 1. We refer to [30] for more information on (sheaves of) locally exact differential forms on curves.

The purpose of this paper is showing that the conditions that the Cartier operator $\mathcal{C}$ has low rank, equivalently $a(J)$ *big*, or that $\mathcal{C}$ is nilpotent, imply strong properties on the geometry of the curve X. Indeed we will prove that the curve X cannot have arbitrary genus. If we impose the condition that the rank of the Cartier operator is small when compared to the genus, or if we impose that the Cartier operator is nilpotent, then the genus of the curve X is smaller than a certain bound depending on the rank of the Cartier operator (see theorem 2.1), or its order of nilpotency (see theorem 2.2). Bounds on the rank of the Cartier operator and of its powers, depending on the ramification behaviour of the canonical linear system at a point $x \in X$, were previously given by K.O. Stöhr and P. Viana in the article [35]. We are able to give unconditional bounds by showing that the hypothesis that the Cartier operator has a certain rank $m < g(X)$ implies the existence of certain base point free linear systems on the curve. This is done in Section 2.2. One is able to estimate the dimensions of these linear systems, and draw some bounds on the genus $g(X)$ from this information. This procedure applies for both the main results of this paper, theorems 2.1 and 2.2. The result in theorem 2.2 is sharp as is shown by some classical examples of curves: see example 7. The result in theorem 2.1 is probably not sharp, but it is a first step towards better and possibly sharp bounds.

2.2 Effect on linear systems

We will prove some simple propositions which link degeneration properties of the Cartier operator of a certain curve to the existence of particular linear systems on that curve. These propositions use only the properties of the Cartier operator stated in the introduction and are basic to all the results of this paper.

Notations.

- X a smooth projective curve over an algebraically closed field k with $\mathrm{char}(k) = p > 0$.

- For D any divisor in X, we denote by $H^0(D)$ (resp. $H^1(D)$) the vector space $H^0(X, \mathcal{O}_X(D))$ (resp. $H^1(X, \mathcal{O}_X(D))$).

- K_X a canonical divisor on X.
- Ω the sheaf of differential forms on X.
- $\mathcal{C}$ the Cartier operator $\mathcal{C} : H^0(\Omega) \to H^0(\Omega)$.

Proposition 2.1. *Suppose that there is a point $x \in X$ and there are integers $0 \leq n_1 < n_2 < \ldots < n_m$ such that for every i one has*

$$h^0((n_i + 1)px) = h^0((n_i p + p - 1)x)$$

(i.e. $(n_i + 1)p$ is a gap *at x). Then* $\mathrm{rk}(\mathcal{C}) \geq m$.

Proof. Under the hypotheses above the Riemann-Roch theorem tells us that

$$h^0(K_X - (n_i p + p - 1)x) = 1 + h^0(K_X - (n_i + 1)px)$$

for every $i = 1, \ldots, m$. In other words, there are global differential forms ω_i on X such that ω_i has multiplicity $n_i p + p - 1$ at x for every $i = 1, \ldots, m$. Because of the universality of the construction of the Cartier operator (see [4]), one may compute $\mathcal{C}(\omega_i)$ on a expansion $\omega_i = (t^{n_i p + p - 1} + \text{(higher order terms)})dt$ with respect to a local parameter t at x, and then apply the properties of the Cartier operator stated in the introduction to get $\mathcal{C}(\omega_i) = t^{n_i}(1 + a)dt$, where a belongs to the ideal (t) of the local ring $\mathcal{O}_{X,x}$. Then one finds immediately that $\mathcal{C}(\omega_1), \ldots, \mathcal{C}(\omega_m)$ are linearly independent, which concludes the proof. □

We note that as a consequence of the proposition above, we have:

Corollary 2.1. *If* $\mathrm{rk}(\mathcal{C}) = m$, *then $h^0((m + 1)px) \geq 2$ for every $x \in X$.*

We will need also a strengthened version of the preceding corollary.

Proposition 2.2. *Suppose that* $\mathrm{rk}(\mathcal{C}) = m$. *Then for a general effective divisor D on X with* $\deg D = m + 1$, *which we also write $D = x_1 + \ldots + x_{m+1}$, one has:*

$$h^0(pD) = 1 + h^0(pD - x_{m+1}).$$

Remark. By a *general* divisor of degree n we will always mean that, if we denote by X_n the n-th symmetric product of the curve X, there is an open subset $U \subset X_n$, such that all the effective divisors $D \in U$ have the stated property.

Proof. Let us consider the linear system induced by the image of the Cartier operator $\mathrm{Im}(\mathcal{C}) \subseteq H^0(K_X)$. This is a linear system of dimension $m-1$ (empty if $m=0$), hence m general points $x_1, \ldots, x_m$ will not simultaneously be in the support of any divisor in this linear system. This implies that for *any* $x_{m+1} \in X$ there is no differential form ω which has zeros of order at least p at $x_1, \ldots, x_m$ and a zero of order $p-1$ at x_{m+1}, since otherwise $\mathcal{C}(\omega)$ would be non-zero and it would give a divisor in $\mathbf{P}(\mathrm{Im}(\mathcal{C}))$ which contains $x_1, \ldots, x_m$. The non-existence of such an ω can be rephrased by saying:

$$h^0(\Omega(-px_1 - \ldots - px_m - (p-1)x_{m+1}) = h^0(\Omega(-px_1 - \ldots - px_m - px_{m+1}))$$

which by Riemann-Roch is equivalent to:

$$h^0(px_1 + \ldots + px_{m+1}) = 1 + h^0(px_1 + \ldots + px_m + (p-1)x_{m+1}).$$

□

The next proposition, similar in spirit to the preceding ones, is needed in Section 2.4.

Proposition 2.3. *If the r-th power C^r of the Cartier operator is zero, then $h^0(p^r x) \geq 2$ for every $x \in X$.*

Proof. Suppose that $h^0(p^r x) = 1$. Then, because

$$h^0(K - (p^r - 1)x) = 1 + h^0(K - p^r x)$$

(cf. Prop. 2.2), one can find a global differential form ω which has multiplicity $p^r - 1$ at x. But then one can easily show by induction on $0 \leq i \leq r$ that $C^i(\omega)$ has multiplicity $p^{r-i} - 1$ at x, by using an expansion of ω with respect to a local parameter at x. In particular one gets that the multiplicity of $C^r(\omega)$ at x is zero, which implies that $C^r(\omega) \neq 0$, a contradiction. □

So far we have shown that some degeneration hypotheses on the Cartier operator imply the existence of certain (infinite) families of linear systems on the curve, which are not expected to exist on arbitrary curves. This will be used in the next sections to get bounds on the genus of the curve. To do this we will need the following proposition on the multiplication of sections of line bundles over a curve.

Proposition 2.4. *Let $\mathcal{L}$, $\mathcal{M}$, $\mathcal{N}$ line bundles on X such that:*

1. *$H^0(\mathcal{N})$ induces a base point free linear system on X;*
2. *$\mathcal{L} = \mathcal{O}_C(D)$ and $\mathcal{M} = \mathcal{O}_C(E)$, where D and E are effective divisors with disjoint supports, D non zero.*
3. *there is a point $y \in \mathrm{Supp}(D)$ and a divisor $F \in \mathbf{P}(H^0(\mathcal{N}))$ such that $\mu_y(F) = 1$, and $\mathrm{Supp}(D) \cap \mathrm{Supp}(F) = \{y\}$.*
4. *for this point y one has $h^0(\mathcal{M}(y)) = h^0(\mathcal{M})$;*
5. *y is not a base point for $\mathcal{L} \otimes \mathcal{M} \otimes \mathcal{N}$.*

Then one has:

$$h^0(\mathcal{L} \otimes \mathcal{M} \otimes \mathcal{N}) - h^0(\mathcal{M} \otimes \mathcal{N}) \geq h^0(\mathcal{L} \otimes \mathcal{M}) - h^0(\mathcal{M}) + 1 \tag{3}$$

Proof. The divisors F and D given in the hypotheses are associated to certain global sections of $\mathcal{N}$ and $\mathcal{L}$ respectively, which we will denote again by F and D. These global sections can be used to construct a commutative diagram, with exact rows:

$$\begin{array}{ccccccc} 0 & \rightarrow & H^0(\mathcal{M}) & \xrightarrow{\cdot D} & H^0(\mathcal{L} \otimes \mathcal{M}) & \rightarrow & H^0(\mathcal{O}_D) \\ & & \downarrow F & & \downarrow F & & \downarrow F|_D \\ 0 & \rightarrow & H^0(\mathcal{M} \otimes \mathcal{N}) & \xrightarrow{\cdot D} & H^0(\mathcal{L} \otimes \mathcal{M} \otimes \mathcal{N}) & \rightarrow & H^0(\mathcal{O}_D). \end{array}$$

The map $H^0(\mathcal{O}_D) \rightarrow H^0(\mathcal{O}_D)$ induced by the multiplication by F has cokernel equal to $H^0(\mathcal{O}_y) \cong k(y)$, because F is the equation of the reduced subscheme $\{y\}$ of D by hypothesis. On another hand the composition

$$H^0(\mathcal{L} \otimes \mathcal{M} \otimes \mathcal{N}) \rightarrow H^0(\mathcal{O}_D) \rightarrow k(y)$$

is surjective, by the property (5). The proof will be complete if we show that the map

$$H^0(\mathcal{L} \otimes \mathcal{M})/D \cdot H^0(\mathcal{M}) \xrightarrow{\bar{F}} H^0(\mathcal{L} \otimes \mathcal{M} \otimes \mathcal{N})/D \cdot H^0(\mathcal{M} \otimes \mathcal{N})$$

induced by F is injective, because the image V of this map cannot generate $\mathcal{O}_D$ at the point y, whereas $H^0(\mathcal{L} \otimes \mathcal{M} \otimes \mathcal{N})/D \cdot H^0(\mathcal{M} \otimes \mathcal{N})$ does. The injectivity of that map is proven if one shows that

$$F \cdot H^0(\mathcal{L} \otimes \mathcal{M}) \cap D \cdot H^0(\mathcal{M} \otimes \mathcal{N}) = F \cdot D \cdot H^0(\mathcal{M}).$$

The left hand side of the formula above is easily seen to be equal to $D \cdot (F - y) \cdot H^0(\mathcal{M}(y))$, hence the assertion follows by the property that $H^0(\mathcal{M}(y)) = y \cdot H^0(\mathcal{M})$. $\square$

2.3 Curves with degenerate Cartier operator.

The hyperelliptic case.
We deal the case of hyperelliptic curves separately, because the bound on the genus we will get in this case will actually be stronger than for a general non-hyperelliptic curve.

Proposition 2.5. *Let X be an hyperelliptic curve over an algebraically closed field in characteristic $p > 0$, and suppose that the Cartier operator C has rank m. Then*

$$g(X) < (p+1)/2 + mp$$

Proof. For $g = 1$ the bound clearly holds. Let us assume $g \geq 2$. Let us consider the case $p \geq 3$ first, and let us suppose that $g \geq (p+1)/2 + mp$. Let x be a ramification point of the g^1_2 on X so that $L = \mathcal{O}_X(2x)$ is the line bundle giving the g^1_2. Since $p > 2$ we will have:

$$\Omega_X(-(2kp + (p-1))x) \cong L^{g-1} \otimes L^{-(p-1)/2-kp} = L^n$$

where $n = g - 1 - (p-1)/2 - kp \geq 0$, for any $k \leq m$. Then $\Omega_X(-(2kp + (p-1))x)$ has no base points, which implies that

$$h^0\Omega_X(-(2kp + (p-1))x) > h^0\Omega_X(-(2k+1)px).$$

Hence there is a differential form ω vanishing in x with order $2kp+p-1$, for any $0 \leq k \leq m$. From proposition 2.1, one finds that rk$C \geq m+1$, a contradiction. If $p = 2$ we will consider instead $x_1, \ldots, x_{m+1}$ *general* points of X. Since $\Omega_X = L^{g-1}$ we see that if $g - 1 \geq 2m+1$ then for any $1 \leq i \leq m+1$ there is a section of Ω_X with double zero at $x_1, \ldots, x_{i-1}$ and a simple zero at x_i. But then, as in the proof of proposition 2.2 one sees that rk$C \geq m+1$. Hence $g \leq 2m+1$. □

The non-hyperelliptic case.
We will now assume that X is a non-hyperelliptic curve such that $\mathrm{rk}(C) = m$. Then by proposition 2.2 we know that $h^0(pD_{m+1}) \geq 2$ for a general effective divisor D_{m+1} of degree $m+1$ on X, and moreover the linear system $|pD_{m+1}|$ will not have base points. From this fact we will draw the following bound on the genus of X:

Theorem 2.1. *If X is non-hyperelliptic and rk $C = m < g$, then the following bound on the genus of X holds:*

$$g(X) \leq (m+1)\frac{p(p-1)}{2} + pm.$$

The proof will be split in some lemmas which will enable us to estimate $g = h^0(\Omega_X)$ by estimating differences of the form $h^0(\Omega_X(-pD_i)) - h^0(\Omega_X(-pD_{i+1}))$ for general divisors D_i, D_{i+1}. By Riemann-Roch this is equivalent to estimating the differences of the form: $h^0(pD + px) - h^0(pD)$. We do this in the following lemmas.

Lemma 2.1. *If x and y are general points and D a divisor on C, then*

$$h^0(pD + px + py) - h^0(pD + px) \geq h^0(pD + px) - h^0(pD)$$

Proof. There is an exact sequence:

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(px) \oplus \mathcal{O}_X(py) \to \mathcal{O}_X(px + py) \to 0,$$

where, if we denote by σ (resp. τ) a section of $\mathcal{O}_X(px)$ (resp. $\mathcal{O}_X(py)$) whose associated divisor is px (resp. py), then the first nonzero map is $f \mapsto (f\sigma, f\tau)$ and the second one is the map $(a, b) \mapsto a\tau - b\sigma$. After tensoring by $\mathcal{O}_X(pD)$ and taking the global sections one immediately gets:

$$h^0(pD + px + py) - h^0(pD + px) \geq h^0(pD + py) - h^0(pD).$$

The proof is then complete when one observes that $h^0(pD + py) = h^0(pD + px)$ because of the generality of x and y. $\square$

Remark 3.1. If for a sufficiently general divisor D of degree n the linear system $|pD|$ is base-point-free, then the same holds for $|pD + px|$, where x a general point. This is because the role of x and any of the points in $\mathrm{Supp}(D)$ may be exchanged, by the genericity assumptions. In particular, if $\mathrm{rk}(C) = m$ then $|pD|$ is base-point-free for every generic D such that $\deg(D) \geq m + 1$.

Now we will see that in certain cases the result of lemma 2.1 can be strengthened.

Lemma 2.2. *Let $m = \mathrm{rk}(C) < g$ as above, and let E be a divisor on X. Then either E is non-special and one has:*

$$h^0(E + py) - h^0(E) = p$$

for any $n \geq 0$, or E is special and there exists an integer $1 \leq k \leq m + 1$ such that for general points $y, z_1, \ldots, z_k$ in C, one has:

$$\begin{aligned} &h^0(E + py + pz_1 + \ldots + pz_k) - h^0(E + pz_1 + \ldots + pz_k) \geq \\ &h^0(E + pz_1 + \ldots + pz_k) - h^0(E + pz_1 + \ldots + pz_{k-1}) + 1. \end{aligned}$$

Proof. The assertion in the case when E is non-special is clear by Riemann-Roch. Let us now assume that E is special and

$$h^0(E+py+\sum_{i=1}^{k} pz_i) - h^0(E+\sum_{i=1}^{k} pz_i) = h^0(E+\sum_{i=1}^{k} pz_i) - h^0(E+\sum_{i=1}^{k-1} pz_i)$$

for every $k = 1, \ldots, m+1$. But then

$$h^0(E+py+\sum_{i=1}^{m+1} pz_i) - h^0(E+\sum_{i=1}^{m+1} pz_i) = h^0(E+py) - h^0(E) \qquad (4)$$

by using recursion on k and by the generality of y and the z_i's.

We set $\mathcal{L} = \mathcal{O}_X(py)$, $\mathcal{M} = \mathcal{O}_X(E)$ and $\mathcal{N} = \mathcal{O}_X(\sum_{i=1}^{m+1} pz_i)$ and proceed to verify the hypotheses (1)-(5) of proposition 2.4 in this case. The assumptions (1), (2) are easy consequences of the generality assumption on y, z_i and remark 3.1 above. (3) clearly holds if the map induced by the base-point free line bundle $\mathcal{N}$ is separable, since the statement in (3) is equivalent to saying that this map is smooth at y. If this map were inseparable, then we would find that $\dim |z_1 + \ldots + z_{m+1}| \geq 1$, which is impossible for $g > 0$ and $z_1, \ldots, z_{m+1}$ general points, since a general effective divisor of degree less or equal than g cannot move. (4) follows if y is taken as a non-base-point of $|K_X - E|$, which is certainly possible since this linear system is non-empty. (5) follows because $\mathcal{L} \otimes \mathcal{N}$ is base-point free by remark 3.1 and y may also avoid any base point of $\mathcal{M}$. Now proposition 2.4 gives us the inequality:

$$h^0(E+py+\sum_{i=1}^{m+1} pz_i) - h^0(E+\sum_{i=1}^{m+1} pz_i) = h^0(E+py) - h^0(E) + 1$$

which contradicts (4). □

We state a numerical consequence of lemma 2.2.

Corollary 2.2. *Under the same hypotheses as above, let us denote by D_k a general divisor of degree k. Then for any $n \geq 1$, one has:*

1. $p \geq h^0(pD_{n(m+1)}) - h^0(pD_{n(m+1)-1}) \geq \min(n,p)$
2. $pD_{(p-1)(m+1)+m}$ *is non-special, i.e.* $h^0(K_X - pD_{(p-1)(m+1)+m}) = 0$
3. *For* $1 \leq n \leq p$ *one has :*

$$h^0(pD_{n(m+1)-1}) - h^0(pD_{(n-1)(m+1)}) \geq (n-1)m$$

4. *For* $1 \leq n \leq p$ *one has:*

$$h^0(pD_{n(m+1)}) - h^0(pD_{(n-1)(m+1)}) \geq (n-1)m + n$$

5. $h^0(K_X - pD_{(n-1)(m+1)}) - h^0(K_X - pD_{n(m+1)}) \leq (p-n)(m+1) + m,$ *for any* $1 \leq n \leq p$

6. $h^0(K_X - pD_{(p-1)(m+1)}) \leq m.$

Proof. (1). First note that the inequality

$$p \geq h^0(pD_{n(m+1)}) - h^0(pD_{n(m+1)-1})$$

is obvious. We will prove the other inequality by induction on n. For $n = 1$, from the fact that $|D_{m+1}|$ has no base points, it is clear that $h^0(pD_{m+1}) \geq 1 + h^0(pD_m)$. Suppose the assertion holds for $n-1$ and let us prove it for n. We apply lemma 2.2 with $E = pD_{(n-1)(m+1)-1}$. If this divisor is nonspecial then $h^0(pD_{n(m+1)}) - h^0(pD_{n(m+1)-1}) = p$. If not, then lemma 2.2 implies

$$h^0(pD_{n(m+1)}) - h^0(pD_{n(m+1)-1}) \geq 1 + h^0(pD_{(n-1)(m+1)}) - h^0(pD_{(n-1)(m+1)-1}),$$

but by the inductive hypothesis the right hand side is greater or equal than n, whence the conclusion follows.
(2). By (1) one has

$$h^0(pD_{p(m+1)}) - h^0(pD_{p(m+1)-1}) \geq p,$$

which implies that

$$h^0(K_X - pD_{(p-1)(m+1)+m}) = h^0(K_C - pD_{p(m+1)})$$

by Riemann-Roch. This means that py is in the base locus of
$H^0(K_X - pD_{(p-1)(m+1)+m})$ for any general $y \in X$. This can only happen when
$H^0(K_X - pD_{(p-1)(m+1)+m}) = 0$.
(3) The step $n = 1$ is clear. If we suppose the assertion true for $n-1$, the inductive step follows from the inequalities:

$$\begin{aligned} &h^0(pD_{n(m+1)-1}) - h^0(pD_{(n-1)(m+1)}) \\ &\geq m(h^0(pD_{(n-1)(m+1)}) - h^0(pD_{(n-1)n(m+1)-1})) \end{aligned}$$

due to lemma 2.1, and

$$h^0(pD_{(n-1)(m+1)}) - h^0(pD_{(n-1)(m+1)-1}) \geq \min(n-1, p) = n-1$$

due to (1).
(4) is clear by (1) and (3).
(5) follows from (4) by Riemann-Roch.
(6) One has

$$h^0(pD_{p(m+1)-1}) - h^0(pD_{(p-1)(m+1)}) \geq m(p-1),$$

in view of (2). By Riemann-Roch and the fact that

$$h^0(\Omega_X(-pD_{(p-1)(m+1)+m})) = 0$$

we can calculate:

$$\begin{aligned}
&h^0(K_X - pD_{(p-1)(m+1)}) \\
&= h^0(pD_{(p-1)(m+1)}) - p(p-1)(m+1) + g - 1 \\
&\leq h^0(pD_{p(m+1)-1}) - m(p-1) - p(p-1)(m+1) + g - 1 \\
&= h^0(pD_{(p-1)(m+1)+m}) - p(p-1)(m+1) - mp + g - 1 + m \\
&= h^0(K_X - pD_{(p-1)(m+1)+m}) + m \\
&= m.
\end{aligned}$$

□

Proof of theorem 2.1. By Corollary 2.2 above, properties (5) and (6), one may compute:

$$\begin{aligned}
g &= h^0(K_X) \\
&= h^0(K_X - pD_{(p-1)(m+1)}) + \\
&+ \sum_{n=1}^{p-1}(h^0(K_X - pD_{(n-1)(m+1)}) - h^0(K_X - pD_{n(m+1)})) \\
&\leq m + \sum_{n=1}^{p-1}((p-n)(m+1) + m) \\
&= pm + (m+1)p(p-1)/2.
\end{aligned}$$

□

We wish to remark that the result in theorem 2.1 is certainly sharp only in the case $m = 0$ which is also a particular case of theorem 2.2 in the next section, and for which one has Example 7. In the other cases the result is almost certainly not a sharp one. One may get slight improvements pushing a bit further the techniques used here, but we expect to have in general a bound of the form:

$$g(X) \leq mp + f(p) \qquad \text{where } m = \mathrm{rk}(\mathcal{C}),$$

for which our techniques seem to be insufficient.

2.4 The case of a nilpotent Cartier operator.

In this section we apply the same techniques introduced in the preceding sections to get the following result:

Theorem 2.2. *Let X be a curve defined over an algebraically closed field of characteristic p with Cartier operator C such that:*

$$C^r = 0 \qquad \textit{for some } r \geq 1.$$

Then one has:

$$g(X) \leq q(q-1)/2 \qquad \textit{where } q = p^r.$$

This result is sharp, as shown by Example 2.1 below, and we think it may be of some interest to the arithmetic theory of function fields. We begin by recalling that, by proposition 2.3, we have for a general point $x \in X$:

$$h^0(qx) \geq 2,$$

and, moreover the linear system $|qx|$ is without base points. For a proof of theorem 2.2 it will be sufficient to assume that $q = p^r$ is the minimum power of p such that $\dim |qx| \geq 1$ (and is without base points), for a general $x \in X$. A simple consequence of proposition 2.4 will be the following:

Lemma 2.3. *If D is a generic divisor of degree $n \geq 0$ in X, and if x and y are generic points, then either qD is non-special, or:*

$$h^0(qD + qx + qy) - h^0(qD + qx) \geq h^0(qD + qx) - h^0(qD) + 1$$

Proof. Since $h^0(qD + qx) = h^0(qD + qy)$ because of the generality of x and y, we may rewrite our assertion as:

$$h^0(qD + qx + qy) - h^0(qD + qx) \geq h^0(qD + qy) - h^0(qD) + 1.$$

This certainly holds if qD is non-special, as a consequence of the Riemann-Roch theorem. Let us then assume that qD is special. We set $\mathcal{L} = \mathcal{O}_X(qy)$, $\mathcal{M} = \mathcal{O}_X(qD)$ and $\mathcal{N} = \mathcal{O}_X(qx)$ and verify the hypotheses of proposition 2.4 in this case. The property (1) has already been observed before. The property (2) is clear. To prove (3) we need to know that the linear system $|qx|$ does not give an inseparable map from X to some projective space, since, in this case, we may take y to be a point over which this map is smooth, and find the divisor F such that $\mu_y(F) = 1$ as required in proposition 2.4. If the map associated to $|qx|$ were inseparable, then we would have $\dim |p^{r-1}x| \geq 1$. But this contradicts the minimality of q with respect to the property $\dim |qx| \geq 1$. This proves (3). The property (4) follows from the speciality of qD since it is equivalent to the fact that y is not a base point of $|K_X - qD|$. Finally (5) holds because $|qD + qx + qy|$ is without base points, containing the sum of linear systems without base points. □

Proof of theorem 2.2. By a repeated application of lemma 2.3 we find that:

$$h^0(qx_1 + \ldots + qx_{i+1}) - h^0(qx_1 + \ldots + qx_i) \geq \min(i+1, q) \quad \text{for every } i \geq 0,$$

for generic $x_1, \ldots, x_{i+1}$. Using Riemann-Roch, this is seen to be equivalent to:

$$\begin{aligned} h^0(K_X - qx_1 - \ldots - qx_i) \quad &-h^0(K_X - qx_1 - \ldots - qx_{i+1}) \\ &\leq \max(0, q - i - 1). \end{aligned} \tag{5}$$

In particular it follows that $h^0(K_X - qx_1 - \ldots - qx_{q-1}) = 0$, since otherwise for every generic $x_q \in X$ we would have qx_q in the base locus of $|K_X - qx_1 - \ldots - qx_{q-1}|$, which is impossible. We now estimate the genus $g(X)$ by the flag of vector spaces

$$H^0(K_X) \supset H^0(K_X - qx_1) \supset \ldots \supset H^0(K_X - qx_1 - \ldots - qx_{q-1}) = 0.$$

by (5) we will have:

$$g(X) \leq (q-1) + (q-2) + \ldots + 1 = q(q-1)/2.$$

□

Example 2.1. *The curve X with affine equation:*

$$y^q + y = x^{q+1} \qquad q = p^r$$

has genus $g(X) = q(q-1)/2$ and Cartier operator such that $C^r = 0$.

This class of curves is indeed well known, see for example [32] for a characterization of them by their arithmetical properties. The genus formula follows because they are completed to smooth projective plane curves. On the other hand, one knows that the eigenvalues of the Frobenius acting on the Tate module T_l of the jacobian variety of X acts as the multiplication by $-q$, see [32] p. 186. Then one may apply prop. 1.2 p. 166 in [6] to conclude that the Frobenius F acting on $H^1(J, \mathcal{O}_J) \cong H^1(X, \mathcal{O}_X)$ satisfies

$F^r = 0$. Finally, since the Cartier operator acting on $H^0(K_X)$ and the Frobenius acting on $H^1(X, \mathcal{O}_X)$ are dual to each other by the Serre duality (see the introduction), one gets also $\mathcal{C}^r = 0$.

Remark. In the article [28], or see also [29], a stratification of the moduli space $\mathcal{A}_g \otimes \mathbb{F}_p$ of principally polarized abelian varieties of dim $= g$, called the Ekedahl-Oort stratification was introduced. Let F and V be the Frobenius and Verschiebung operators acting on $H^1_{DR}(A, k)$, see [22], section 5. The strata of the Ekedahl-Oort stratification can then be defined in terms of the mutual positions of $\ker F^i$ and $\ker V^j$, as done in [9]. In particular, since the a-number of an abelian variety A is the dimension of $\ker F \cap \ker V$, from the cited theory it follows that the subset A_k of $\mathcal{A}_g \otimes \mathbb{F}_p$ where $a \geq k$ is a union of strata of the Ekedahl-Oort stratification. Let now $\mathcal{T}_g \otimes \mathbb{F}_p$ be the "open" Torelli locus, i.e. the set of isomorphism classes of the jacobian varieties of smooth curves of genus g in char$= p$. Our result in theorem 2.1 implies that

$$(\mathcal{T}_g \otimes \mathbb{F}_p) \cap A_k = \emptyset$$

if $k \geq m$ and $g > pm + (m+1)p(p-1)/2$.

Similarly, let us denote by N_r the subset of $\mathcal{A}_g \otimes \mathbb{F}_p$ where $\ker F \subseteq \ker V^r$, by noting that $\ker F \subset H^1_{DR}(X, k)$ is equal to $H^0(\Omega^1_X)$, cf [22], section 5. From theorem 2.2 one can see:

$$(\mathcal{T}_g \otimes \mathbb{F}_p) \cap N_r = \emptyset$$

if $g > p^r(p^r - 1)/2$.