Downloaded from UvA-DARE, the institutional repository of the University of Amsterdam (UvA)
http://dare.uva.nl/document/61321

File ID	61321
Filename	Chapter 3 A Two-country Monetary-Production Economy

SOURCE (OR PART OF THE FOLLOWING SOURCE):

Type	Dissertation
Title	A term structure model of interest rates and forward premia: an alternative monetary approach
Author	W.H. Daal
Faculty	Faculty of Economics and Business
Year	2002

FULL BIBLIOGRAPHIC DETAILS:
http://dare.uva.nl/record/104767

Copyright

It is not permitted to download or to forward/distribute the text or part of it without the consent of the author(s) and/or copyright holder(s), other then for strictly personal, individual use.

UvA-DARE is a service provided by the library of the University of Amsterdam (http://dare.uva.nl)

Chapter 3

A Two-country Monetary-Production Economy

We present an integrated two-country continuous-time production economy that allows for monetary endogeneity and for the risk averse representative agent of each country to hold money for both transaction purposes and portfolio considerations. Uncertainty in the economy is described by a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where Ω is the sample space, $\mathcal{F}$ is the $\sigma - algebra$ of measurable events, and the stream of information over time is given by the filtration $\mathbb{F}(t) = \{\mathcal{F}(t)\}_{t \geq 0}$. The probability measure $\mathbb{P}$ on $\mathcal{F}$ represents the common probability beliefs held by the representative agents in both countries. There is a five-dimensional Brownian motion $w(t) = \left(w_y(t), w(t)_{x_h}, w(t)_{x_f}, w(t)_{m_h}, w(t)_{m_f}\right)$ on the probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where the subscripts y, x_i, and m_i, for $i = [h, f]$, captures the quantities associated with, respectively, the common, the local real, and the local monetary sources of uncertainty. Henceforth, the subscripts (h) and (f) denote the quantities associated with the domestic and foreign country, respectively. All stochastic processes in the economy are assumed adapted to the filtration $\mathbb{F}$ of the Brownian motion $w(t)$. In this study we assume the existence of three independent state variables, two country-specific and one common state variables. The production process in each country is driven by two independent state variables, namely, a local state variable, $X_i(t)$, and a

common state variable, $Y(t)$. The stochastic processes for the local and common or say international state variables are given by the following stochastic differential equations, respectively:

$$dX_i(t) = \mu_{x_i}(X_i, t)\,dt + S_{x_i}(X_i, t)\,dw_{x_i}(t) \quad i \in \{h, f\} \tag{3.1}$$

and

$$dY(t) = \mu_y(Y, t)\,dt + S_y(Y, t)\,dw_y(t), \tag{3.2}$$

where $\mu_{x_i}(X_i, t) = \kappa_{x_i}(\theta_{x_i} - X_i(t))$ and $\mu_y(Y, t) = \kappa_y(\theta_y - Y(t))$ represents the expected instantaneous change in the local state variable, $X_i(t)$, and the international state variable, $Y(t)$, respectively. The diffusion processes of these state variables are captured by the terms $S_{x_i}(X_i, t) = \sigma_{x_i}\sqrt{X_i(t)}$ and $S_y(Y, t) = \sigma_y\sqrt{Y(t)}$, respectively. These specifications of the state variables imply that they follow mean-reverting square root processes, with θ_{x_i} and θ_y denoting the long term means of the local and common state variables, respectively. The speed of adjustment parameters, κ_{x_i} and κ_y, determine the speed with which these state variables instantaneously revert back to their respective long term means. The parameters θ_{x_i}, κ_{x_i}, σ_{x_i}, θ_y, κ_y, and σ_y are positive constants, such that $2\kappa_{x_i}\theta_{x_i} > \sigma_{x_i}^2$ and $2\kappa_y\theta_y > \sigma_y^2$. As shown by Feller (1951) these conditions imply the strict positiveness of $X_i(t)$ and $Y(t)$, when these variables follow a mean reverting square root process. The diffusion terms of each state variable, which captures the volatility of the process, increases with the square root of the level of the respective state variable and, thus, allowing more variability at higher levels of the respective state variable and vice versa.

We assume that the markets are dynamically complete, such that Pareto efficiency is obtained and that it is possible to use a 'representative agent' for each country [for a proof of Pareto optimality and a representative agent in a complete-markets general equilibrium model see Constantinides (1982), Dumas and Luciano (1990), and particularly Kandori (1988)]. It follows that in this setup of a representative resident for each coun-

try, individuals in each country have homogenous believes and time preferences. There is a single stochastic production technology in each country producing a fully tradeable good, that is partly sold on the local market and partly exported to the foreign market. The representative resident of each country consumes the locally produced good, $c_{ii}(t)$, and the imported good, $c_{ij}(t)$. Each good can be interpreted as a composite good that consists of all the goods produced in that particular country. Following CIR (1985a) and Nielsen and Saà-Raquejo (1993), we assume that the two infinitely divisible physical goods in this economy can be allocated to consumption or investment. Each production technology uses the local homogeneous capital-consumption good as its only input.

In particular, the representative agents of each country can invest in both stochastically constant return to scale production technologies located in these countries. Let η_i be the amount of the i-th good produced (or used in production) by the production technologies, whereby $d\eta_i$ is the aggregate production process of good i, c.q. the transformation of an investment of η_i amount of the i-th good. It is assumed that the transformation of an investment of η_i amount of the good at time t in the production process is governed by the following stochastic differential equation:

$$\frac{d\eta_i(t)}{\eta_i(t)} = \alpha_i(X_i, Y, t)\,dt + S_{\eta_i}(X_i, Y, t)\,dw_{\eta_i}(t) \quad i \in \{h, f\}, \tag{3.3}$$

where

$$\alpha_i(X, Y, t) = \alpha_{x_i}(X_i, t) + \alpha_{y_i}(Y, t) \text{ and}$$
$$S_{\eta_i}(X_i, Y, t)\,dw_{\eta_i}(t) = S_{\eta_{xi}}(X_i, t)\,dw_{x_i}(t) + S_{\eta_{yi}}(Y, t)\,dw_y(t),$$

represent the expected instantaneous real rate of return on the investment in production of good i and the diffusion term of this rate of return, respectively. The instantaneous variance of the real rate of return of the i-th productive investment is given by $S_{\eta_i\eta_i} = S^2_{\eta_{xi}}(X_i, t) + S^2_{\eta_{yi}}(Y, t)$. We allow for the local state variable and the international state variable, $[X_i(t), Y(t)]$, to affect both the mean and variance of the instantaneous

rate of return of the i-th production process. Therefore, the two production processes in these countries are correlated with each other, i.e. $S_{\eta_i \eta_j} \neq 0$, as the common state variable drives both production processes. There are two sources of uncertainties underlying the production process in each country, which are captured by a two-dimensional vector of Wiener processes, $dw_{\eta_i}(t) = [dw_{x_i}(t), dw_y(t)]$. Each transformation process can be regarded as a firm located in each country, each employing a different technology. Equation (3.3) implies that the stochastic investment opportunity sets differ across these two countries when the state of technology results in different rates of return on physical capital in the home country and the foreign country. As both firms are assumed to issue stocks to the amount of the good produced, the domestic and foreign agents can invest in both production technologies. We may consider the two single good production technologies as the optimally diversified portfolio of production technologies. The stochastic differential equation (3.3) represents a system where the output of the production process is reinvested in that same process. The system does not imply anything about which portion of the output is reinvested and which portion is consumed. Neither does it say anything about the allocation of the i-th investment good between the home representative agent and the foreign representative agent, as this is determined endogenously as part of the general equilibrium. The specific functional form of the production process will be provided in the following chapters.

In the present study we allow for endogeneity in the money supply processes, as we assume that the money supply in each country is affected by both the autonomous monetary factors and the productive factors in the economy. Clark, Goodhart, and Huang (1999), based on a standard utility loss function that accommodates both inflation and output target, find that with persistence in inflation, the optimum monetary policy is state contingent and shock dependent, as the monetary authority adjusts the control variable in response to shocks in output. Other authors, such as Christiano and Eichenbaum (1992), also indicate that in models where monetary authorities have greater flexibility to direct cash quickly to the financial sector through open market operations, money

supply reacts to an unanticipated change in the productive sector, such as a technology shock. The money supply process in each country is described by the following general stochastic differential equation:

$$\frac{dM_i^s(t)}{M_i^s(t)} = \mu_{m_i} dt + \sigma_{m_i} dw_{m_i}(t) + \gamma_i \frac{d\eta_i(t)}{\eta_i(t)}, \quad i \in \{h, f\}, \tag{3.4}$$

where μ_{m_i} and σ_{m_i} denotes the autonomous expected rate of growth and the autonomous instantaneous volatility of the money supply process in country i, respectively. The monetary endogeneity, as captured by the inclusion of the production process in the money supply process, ultimately depends on the monetary feedback parameter, γ_i. The feedback parameter γ_i captures the domestic monetary response to shocks in local production. Given that money reacts on all uncertainties underlying the productive activity in the economy, both state variables are allowed to affect the money supply process. Note that monetary disturbances are isolated from both local and international shocks that affects local production, when $\gamma_i = 0$. This modeling approach implies that the money supply processes in both countries, depending on the feedback parameter, are correlated with the production processes in the respective country. Such a modeling technique is also applied by King and Trehan (1984), although in a different framework. In this context it should be noted that the money supply processes across both countries are correlated with each other, $S_{m_i m_j} \neq 0$ for $i \neq j$, as both are driven by the common source of uncertainty in this economy. The general formulation in equation (3.4) allows for the money supply to be stationary around either a deterministic or stochastic time trend as in Bakshi and Chen (1996). Stock and Watson (1989) and Marshall (1992) find evidence that money supply is stationary around a significant time trend. The specific functional form of the money supply process will be provided in the following chapters.

We assume that 'firms' are competitive and act as price takers in the international goods markets. In line with Stulz (1986), Foresi (1990), and Basak and Gallmeyer (1999) we work with the price of money, $\pi_i(t) = 1/P_{c_i}$, i.e. the inverse of the price level. As noted by Foresi (1990), it is more convenient because the price of money reveals the

symmetries between money and other assets, as $d\pi_i/\pi_i$ is the real rate of return on the i-th money. Given the money supply and production processes, we conjecture that the dynamics of the price of the i-th money in this economy, $\pi_i(t)$, is given by the following stochastic differential equation:

$$\frac{d\pi_i(t)}{\pi_i(t)} = \mu_{\pi_i}(X_i, Y, t)\, dt + S_{\pi_i}(X_i, Y, t)\, dw_{\pi_i}(t), \quad i \in \{h, f\}, \tag{3.5}$$

where $\mu_{\pi_i}(X_i, Y, t)$ denotes the expected instantaneous change in the price of the i-th money and $S_{\pi_i}(X_i, Y, t)$ is the diffusion term of $\pi_i(t)$. The last term can be considered as the unanticipated movements in the price of money. Note that the expected rate of change in terms of the price level of the good is equal to minus the expected rate of change of the price of money plus its variance. The price level of money and its dynamics are solved endogenously in equilibrium.

Let $\epsilon_{ij}(t)$ denote the nominal exchange rate, which is defined as the domestic currency price i of one unit of foreign currency j. Given the stochastic processes of production and money supply in this two-country economy, we conjecture that the spot exchange rate process is governed by the following stochastic differential equation :

$$\frac{d\epsilon_{ij}(t)}{\epsilon_{ij}(t)} = \mu_{\epsilon_{ij}}(X, Y, t)\, dt + S_{\epsilon_{ij}}(X, Y, t)\, dw_{\epsilon_{ij}}(t), \tag{3.6}$$

where $X = [X_h, X_f]$, $\mu_{\epsilon_{ij}}(X, Y, t)$ denotes the expected instantaneous rate of depreciation of currency i and $S_{\epsilon_{ij}}(X, Y, t)$ represents the diffusion term of the rate of depreciation of the exchange rate. Both are a function of the state variables in this two-country world economy, when we allow the fundamentals in the economy to be state dependent. The explicit solutions for the exchange rate and its dynamics are determined endogenously as part of the general equilibrium.

Huang and Litzenberger (1993) and Ingersoll (1978) show that Pareto optimal allocation obtains when markets are complete. A dynamically complete securities market structure requires that the set of traded assets permits agents to perfectly hedge against

stochastic changes in the state variables. Since this economy is characterized by five sources of uncertainty, the representative investors must be able to freely trade in at least five securities with no perfectly correlated return uncertainty. Therefore, to obtain dynamically complete securities markets in this economy, we assume that there is a market for riskless lending and borrowing in each country and that the representative agent of each country can frictionlessly trade in the following securities: two real assets, two nominal riskless bonds, and N contingent claims.

We assume that the representative agent of each country only borrows and lends in its own country at the instantaneous *locally* riskless real interest rate, $r_i(t)$. By local instantaneously riskless we mean that the local agent at each moment of time knows with certainty the real interest rate prevailing on the local market. This can be considered as a floating-rate bank account, which is riskless in terms of the locally produced good. The real assets represents the shares of any of the two firms in the economy. We assume that the representative agents of both countries have full access to the equity markets in both countries. As a result the representative investor of each country can trade frictionlessly in both equities, representing the investment in both production technologies. This may be considered as the optimally diversified portfolio of production technologies. Equation (3.3) specifies the instantaneous real rate of return on the investment in the i-th production technology expressed in the i-th currency. The instantaneous real rate of return on foreign real investment j in terms of the local currency i is determined as

$$\frac{d\widetilde{\eta}_j(t)}{\widetilde{\eta}_j(t)} = \widetilde{\alpha}_j(X,Y,t)\,dt + S_{\widetilde{\eta}_j}(X,Y,t)\,dw_{\widetilde{\eta}_j}(t), \qquad \text{for } i \neq j \tag{3.7}$$

where

$$\widetilde{\eta}_j(t) = \epsilon_{ij}(t)\,\eta_j(t)$$
$$\widetilde{\alpha}_j(X,Y,t) = \alpha_j(X_j,Y,t) + \mu_{\epsilon_{ij}}(X,Y,t) + S_{\eta_j\epsilon_{ij}}(X,Y,t),$$
$$S_{\widetilde{\eta}_j}(X,Y,t)\,dw_{\widetilde{\eta}_j}(t) = S_{\eta_j}(X_j,Y,t)\,dw_{\eta_j}(t) + S_{\epsilon_{ij}}(X,Y,t)\,dw_{\epsilon_{ij}}(t),$$

and $S_{\eta_j \epsilon_{ij}}(X,Y,t)$ denotes the covariance between the spot exchange rate depreciation and the real rate of return on the foreign real asset. Throughout this work we follow the notation that the tilde hat denotes foreign quantities expressed in local currency. The drift term, $\mu_{\epsilon_{ij}}(X,Y,t)$, the diffusion term, $S_{\epsilon_{ij}}(X,Y,t)$, and the covariance term, $S_{\eta_j \epsilon_{ij}}(X,Y,t)$, are determined endogenously in equilibrium. The optimal portfolio demand for both equities will also be determined endogenously in equilibrium.

In addition the security markets are characterized by the presence of bond markets, where the representative investors of each country can freely trade in two nominal riskless bonds. These bonds are in zero net supply in equilibrium. The two nominal bonds are locally riskless in each of their respective currencies, with an instantaneous nominal rate of return $R_i(t)dt$. We denote the spot nominal interest rate, that is the instantaneously locally riskless rate at which deposits accumulate interest, by $R_i(t)$. It can be considered as the continuously compounding interest rate. The instantaneous real rate of return of the nominal riskless bond in country i is governed by the following stochastic differential equation (by applying Itô's lemma):

$$\frac{dB_i(t)}{B_i(t)} = \beta_i(X_i,Y,t)dt + S_{\pi_i}(X_i,Y,t)\,dw_{\pi_i}(t) \quad i \in \{h,f\}\,, \tag{3.8}$$

where

$$\beta_i(X_i,Y,t) = R_i(t) + \mu_{\pi_i}(X_i,Y,t).$$

The drift term, $\beta_i(X_i,Y,t)$, i.e. the expected instantaneous real rate of return on the default-free nominal bond, is a stochastic process that is determined endogenously as part of the general equilibrium. The instantaneous real rate of return on the foreign nominal bond expressed in the local currency is given by the following stochastic differential equation:

$$\frac{d\widetilde{B}_j(t)}{\widetilde{B}_j(t)} = \widetilde{\beta}_j(X,Y,t)dt + S_{\widetilde{B}_j}(X,Y,t)\,dw_{\widetilde{B}_j}(t), \tag{3.9}$$

where

$$\widetilde{B}_j(t) = \epsilon_{ij}(t)B_j(t)$$
$$\widetilde{\beta}_j(X,Y,t) = R_j(t) + \mu_{\pi_j}(X_j,Y,t) + \mu_{\epsilon_{ij}}(X,Y,t) + S_{\pi_j\epsilon_{ij}}(X,Y,t)$$
$$S_{\widetilde{B}_j(t)}(X,Y,t)\,dw_{\widetilde{B}_j(t)}(t) = S_{\pi_j}(X_j,Y,t)\,dw_{\pi_j}(t) + S_{\epsilon_{ij}}(X,Y,t)\,dw_{\epsilon_{ij}}(t).$$

Furthermore, we allow for the representative resident of each country to hold real money balances both for portfolio and transaction-related purposes, $m_i^d(t) = \pi_i(t)M_i^d(t)$. The portfolio demand for money arises because of the uncertainty associated with the returns of interest-bearing assets. The rationale for real cash balances as an object of portfolio choice is that it reduces the riskiness of an asset portfolio. For example, holding interest-bearing securities is risky when the holder is uncertain when he wants to undertake future consumption and what the future price of those securities will be at that time. Uncertainty is by itself not enough to explain money holdings, when other riskless interest-bearing assets are available. Here come into play the function of money as a medium of exchange and thereby facilitating transactions in the economy. For example, if the agents were certain about the timing of their future flow of expenditure, they could buy bonds that mature at that particular time. If this was possible they could avoid the investment uncertainty and earn a higher return than would be obtain from holding money. But even if the agents are certain about their future consumption plans, they do not know for certain the price of their bonds at that time.

As pointed out by Basak and Gallmeyer (1999), the nominal rate of return on the bond, $R_i(t)dt$, can be considered as the additional compensation the nominal bond gives over currency i for not providing transaction services to the representative agents in the economy. This can be observed by comparing the real rate of return on money, equation (3.5), with that on the nominal bond, equation (3.8). Thus $R_i(t)$ can be considered as a measure of transaction services provided by currency i. Thus the demand for non-interest-bearing money therefore arises when there exists both uncertainty about the timing of

expenditure and about the rate of return from non-money assets. In this context, as argued by Stulz (1986) and Foresi (1990), it should be noted that money holdings, as an object of portfolio choice, are not distinguishable from nominal bonds, because both nominal assets have the same risk exposure and the same covariance with other assets. In this framework we allow for the representative agent of each country to hold both monies.

Based on the assumption of continuous and frictionless trading opportunities and the dynamics of the existing real and nominal assets, we assume the existence of N freely traded contingent claims, $F^k(t)$. Common types of derivative assets are, among others, put and call options, futures and forward contracts (including forward foreign exchange contracts), and convertible or more exotic types of bonds. The payoffs of these securities, which depend on the payoffs on one or more of the underlying assets in the economy, are determined endogenously as part of the general equilibrium. These securities can be issued and purchased by the representative agent, c.q. firm, in each country. Following CIR (1985a) we assume that the value of these claims, given the dynamic description of the underlying assets, depends in general on all variables necessary to describe the state of the economy. As a result we conjecture the following stochastic differential equation for the k-th contingent claim:

$$\frac{dF^k(t)}{F^k(t)} = \zeta_k(X,Y,t)\,dt + S_{F^k}(X,Y,t)\,dw_F(t) \qquad \text{for } k = 1,\ 2, \cdots, N, \tag{3.10}$$

where $\zeta_k(X,Y,t)\,F^k$ represents the total mean return on the j-th claim, $S_{F^k}(X,Y,t)$ is a N-dimensional vector of diffusion terms, and $X = [X_h, X_f]$. In this specification we allow both parameters to be state dependent, such that they can capture the impact of changes in the state variables in the economy on the expected value and volatility of the claim. The variance-covariance matrix of the real rate of return on claim k is captured by $S_{F^kF^k}$. The values of the parameters, $\zeta_k(X,Y,t)$ and $S_{F^k}(X,Y,t)$, are determined endogenously as part of the general equilibrium.

Money has been incorporated into general equilibrium models (i.e. models based on

optimizing behavior of economic agents) of the term structure of interest rates basically in two ways. Firstly, money has been incorporated by means of imposing transactions costs of money. This results in a demand for money by assuming that asset exchanges or barter trade are costly. Alternatively, the transaction-cost technology has been incorporated by imposing Clower cash-in-advance constraints in term structure models. Secondly, money has been incorporated by providing real money balances as an argument in the agents' utility function [Sidrauski (1967)]. The other ways of incorporating money in general equilibrium models have been used to a lesser extent in term structure models. These alternative ways of incorporating money are the role of money as a sole means of intergenerational transaction (overlapping generations models) and money as an object of portfolio choice for risk averse agents. One of the shortcomings of the overlapping-generations approach and the portfolio approach is that they neglect the transaction function of money.

The cash-in-advance constraint is a formal representation of the transaction demand for money, whereby it is implicitly assumed that money is required for transactions and that nominal consumption in the current period cannot exceed nominal money balances carried over from previous period. In this approach, money is introduced into the optimization problem through the so-called cash-in-advance constraint along with the budget constraint. In equilibrium money would have zero value if the liquidity constraint was not imposed. This approach has largely been used to motivate money holdings in international financial models [for example, Lucas (1982), Svensson (1985), Engel (1992), and Bekaert (1994)] as a way of introducing different moneys into the system. In this study we adopt the second approach, by introducing money into the optimization framework through the so-called money-in-the-utility function approach, whereby real money balances is introduced as an argument in the utility function.

In our model, money is introduced directly in the utility function of the agents in the economy, by assuming that it provides them with liquidity services. Money-in-the-utility accounts for the transaction function of money in a form less extreme than the

Lucas (1982) cash-in-advance constraint, as indicated by Danthine and Donaldson (1986), Kydland (1983), and McCallum (1982). This approach of including cash balances in the representative agents utility function also allows for the precautionary and store-of-value motives for holding cash balances. Feenstra (1986), in a partial equilibrium analysis, shows that money in the utility approach is functionally equivalent to the transaction-cost technology approach. In addition, Poterba and Rotemberg (1987) and Holman (1998) provide empirical evidence in favor of the money-in-the-utility approach.

The representative consumer agent of country i, therefore, maximizes the following von Neumann-Morgenstern utility function:

$$E_t \int_t^{\infty} e^{-\rho s} U\left(c_i(s), m_i^d(s), s\right) ds \quad i \in \{h, f\}, \tag{3.11}$$

where E is the expectation operator, ρ denotes the time preference parameter, $c_i(s)$ is the consumption flow at time s of the i-th representative resident, and $m_i^d(s)$ is its demand for real money balances at time s. We assume that the representative agents in this economy have homogenous time preference parameter. To guarantee that the representative agent's allocation problem possesses a unique solution, the utility function must be twice continuously differentiable, strictly concave, and increasing in both its arguments, i.e. $U_c > 0$, $U_m > 0$, $U_{cc} < 0$, $U_{mm} < 0$, $U_{cm} < 0$, and $U_{cc}U_{mm} - (U_{cm})^2 > 0$, where the subscripts denote the corresponding partial derivatives. Following Stulz (1986), Bakshi and Chen (1996, 1997), and Foresi (1990), the demand for both goods and both monies by the risk averse representative agents in the economy is incorporated in the following separable log utility function, for $i \in \{h, f\}$:

$$U\left(c_{ih}, c_{if}, m_{d_{ih}}, m_{d_{if}}, t\right) = \ln\left[\left[c_{ih}(s)\right]^{\theta_{ih}}\left[c_{if}(s)\right]^{\theta_{if}}\right] + \ln\left[\left[m_{ih}^d(s)\right]^{\delta_{ih}}\left[m_{if}^d(s)\right]^{\delta_{if}}\right], \tag{3.12}$$

where θ_{ij} and δ_{ij}, for $j \in [h, f]$, represents the expenditure share allocated respectively to consumption of the j-th good and to the j-th real money balances by the representa-

tive agent of country i. The expenditure share parameters should satisfy the following condition $1 = \theta_{ih} + \theta_{if} + \delta_{ih} + \delta_{if}$, where the subscripts ij, for $j \in [h, f]$, denote the demand for the j-th good or money by the i-th representative agent.

In our setting agents store up their claim on future consumption in three types of assets. Thus the portfolio decision of the financial agents in our setting concern the optimal allocation of their wealth among the various types of assets in the economy, including money. From Merton's (1971) analysis we know that the stock of real wealth of the representative agent of country i at time t, $W_i(t)$, when there are no non-capital gains, can be defined as:

$$W_i(t) = \sum_{q=1}^{n} N_{q,i}(t-h) P_{q,i}(t) \quad i \in \{h, f\}, \tag{3.13}$$

where $N_{q,i}(t)$ is the number of the q-th security purchased and held at time t by the representative agent of country i and $P_{q,i}(t)$ is the current value of these securities express in currency i. The stock of real wealth of the representative agent of country i can be allocated to consumption, money holdings and investments. As mentioned above the agent's holdings of real money balances for transaction purposes implies the existence of opportunity cost. In this regard, the representative agent is "consuming", c.q. "paying", the liquidity services of money. The cost of holding money (i.e. making use of the liquidity services of real cash balances) over one period, $R_i m_i^d \triangle$, is included in the agent's budget constraint as an outlay. The allocation problem of the agents in this economy is solved simultaneously during the period and can be formulated as follows:

$$-\left[\widehat{c}_i(t) + \widehat{R}_i(t)\widehat{m}_i^d(t)\right]\triangle = \sum_{q=1}^{n}\left[N_{q,i}(t) - N_{q,i}(t-\triangle)\right] P_{q,i}(t), \tag{3.14}$$

where

$\widehat{c}_i(t) = c_{ii}(t) + \widetilde{c}_{ij}(t),$

$\widehat{R}_i(t)\,\widehat{m}_i^d(t) = R_i(t)\,m_{ii}^d(t) + R_j(t)\,\widetilde{m}_{ij}^d(t)$,
$\widetilde{c}_{ij}(t) = \epsilon_{ij}(t)c_{ij}(t)$, and $\widetilde{m}_{ij}^d(t) = \epsilon_{ij}(t)m_{ij}^d(t)$.

As in Merton (1971), equations (3.13) and (3.14) can, by taking the limits ($\lim_{\triangle\to 0}$), be rewritten as:

$$W_i(t) = \sum_{q=1}^{n} N_{q,i}(t)\,P_{q,i}(t) \tag{3.15}$$

and

$$-\left[\widehat{c}_i(t) + \widehat{R}_i(t)\,\widehat{m}_i^d(t)\right]dt = \sum_{q=1}^{n} dN_{q,i}(t)\,dP_{q,i}(t) + \sum_{q=1}^{n} dN_{q,i}(t)\,P_{q,i}(t)\,. \tag{3.16}$$

By applying Itô's lemma on the stock of real wealth, $W_i(t)$, in equation (3.15) and substituting equation (3.16) for the term $\sum_1^n dN_{q,i}(t)\,dP_{q,i}(t) + \sum_1^n dN_{q,i}(t)\,P_{q,i}(t)$, i.e. the net value of additions to wealth, we obtain the following dynamic wealth equation:

$$dW_i(t) = \sum_{q=1}^{n} N_{q,i}(t)\,dP_{q,i}(t) - \left[\widehat{c}_i(t) + \widehat{R}_i(t)\,\widehat{m}_i^d(t)\right]dt. \tag{3.17}$$

Following Merton (1971) we define $w_{q,i}(t) \equiv N_{q,i}(t)\,P_{q,i}(t)\,/W_i(t)$ as the fraction of agent's i real wealth allocated to q-th security at time t. The representative agent can at each point in time allocate its wealth among investments in the production technologies, nominal assets, and the N contingent claims. Let a_i, b_i, and f_i be the fraction of wealth allocated to these financial instruments, then $w_{q,i}(t)$ can be defined as $w_{q,i} = \begin{bmatrix} a_i & b_i & f_i \end{bmatrix}$, where f_i is a N-dimensional vector of the fraction of wealth allocated to the N contingent claim in the economy. The two-dimensional vector $a_i' = \begin{bmatrix} a_{ih} & a_{if} \end{bmatrix}$, consist of the portfolio demand for equity investment in country h and f, respectively, by the i-th representative agent. The portfolio demand of the i-th investor for nominal bonds in country h and f is captured by the (2×1)-vector $b_i' = \begin{bmatrix} b_{ih} & b_{if} \end{bmatrix}$. These portfolio weights must by definition sum up to one, i.e. $a_i'\mathbf{1} + b_i'\mathbf{1} + f_i'\mathbf{1} = 1$, where $\mathbf{1}$ is an identity vector. Consequently, after substituting the portfolio weights and the dynamics for the value of

each security, the dynamic budget constraint of the representative agent i, for $i \in \{h, f\}$, can be formulated as follows:

$$\begin{aligned} dW_i(t) &= [a_i'(\alpha - \mathbf{1}r_i)W_i + b_i'(\beta - \mathbf{1}r_i)W_i + f_i'(\zeta - \mathbf{1}r_i)W_i + r_iW_i \\ &\quad -c_{ih}(s) - \widetilde{c}_{if}(s) - R_h m_{d_{ih}} - R_f \widetilde{m}_{d_{if}}]\,dt + W_i a_i' S_\eta(X,Y,t)\,dw_\eta(t) \\ &\quad + W_i b_i' S_B(X,Y,t) dw_B(t) + W_i f_i' S_F(X,Y,t) dw_F(t), \end{aligned} \tag{3.18}$$

where

$$\alpha' = \begin{bmatrix} \alpha_i(X_i,Y,t) & \widetilde{\alpha}_j(X,Y,t) \end{bmatrix},$$
$$\beta' = \begin{bmatrix} \beta_i(X_i,Y,t) & \widetilde{\beta}_j(X,Y,t) \end{bmatrix},$$
$$S_\eta(X,Y,t)\,dw_\eta(t) = \begin{bmatrix} S_{\eta_i}(X_i,Y,t) & 0 \\ 0 & S_{\widetilde{\eta}_j}(X,Y,t) \end{bmatrix} \begin{bmatrix} dw_{\eta_i}(t) \\ dw_{\widetilde{\eta}_j}(t) \end{bmatrix},$$
$$S_B(X,Y,t) dw_B(t) = \begin{bmatrix} S_{\pi_i}(X_i,Y,t) & 0 \\ 0 & S_{\widetilde{B}_j}(X,Y,t) \end{bmatrix} \begin{bmatrix} dw_{\pi_i}(t) \\ dw_{\widetilde{B}_j}(t) \end{bmatrix}.$$

Notice that $S_F(X,Y,t)$ is a N-dimensional matrix of diffusion terms of the N contingent claims. In line with Stulz (1986), the fraction of wealth allocated to local and foreign nominal assets is defined as $b_{ii} = \frac{m_{ii}^d + B_i}{W_i}$ and $b_{ij} = \frac{m_{ij}^d + \widetilde{B}_j}{W_i}$, respectively. This definition reflects the arguments presented above, namely that nominal bonds and money holdings are not distinguishable for portfolio selection purposes because of their similar risk exposures. In the next chapter we derive the equilibrium conditions for the endogenous variables in this economy.