Defect-mediated melting and the breaking of quantum double symmetries

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In this paper, we apply the method of breaking quantum double symmetries to some cases of defect-mediated melting. The formalism allows for a systematic classification of possible defect condensates and the subsequent confinement and/or liberation of other degrees of freedom. We also show that the breaking of a double symmetry may well involve a (partial) restoration of an original symmetry. A detailed analysis of a number of simple but representative examples is given, where we focus on systems with global internal and external (space) symmetries. We start by rephrasing some of the well-known cases involving an Abelian defect condensate, such as the Kosterlitz-Thouless transition and one-dimensional melting, in our language. Then we proceed to the non-Abelian case of a hexagonal crystal, where the *hexatic phase* is realized if translational defects condense in a particular rotationally invariant state. Other conceivable phases are also described in our framework.

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I. INTRODUCTION

In two-dimensional (quantum) physics certain quantum groups and Hopf algebras play an important role because these extended symmetries allow for a treatment of ordinary and topological quantum numbers on equal footing, whereas the natural appearance of an R-matrix accounts for the topological interactions in the system.[1] This means that both topological defects and ordinary excitations (such as Goldstone modes) may appear in the same representation of the underlying—often hidden—Hopf algebra $\mathcal{A}$, even in non-Abelian situations where the mutual dependence of these dual quantum numbers would otherwise be untractable. For a brief, tailor made introduction to Hopf algebras and the notational conventions we adopt in this paper, we refer to Appendix A of a related paper,[2] or to a more general review.[3] These extended symmetry concepts have found interesting applications in the domain of quantum Hall liquids,[4] crystals and liquid crystals,[5] $(2+1)$-dimensional gravity,[6] and potentially also to rotating Bose-Einstein condensates[7,8] and other systems described by conformal field theory.

Once this type of (hidden) symmetry was identified, a formalism for the breaking of quantum double symmetries was proposed,[9,10] by assuming the formation of condensates of ordinary (electric), defect (magnetic), or mixed (dyonic) type. As was expected, the ordinary condensates reproduce the conventional theory of symmetry breaking, though the systematic analysis of the subsequent confinement of topological degrees of freedom using the braid group is not standard. In the papers just mentioned it was shown that the usual picture of symmetry breaking must be augmented with significant novel ingredients. In the case of Hopf double symmetry breaking we assume the condensate to be represented by a fixed vector $|\phi_0\rangle$ in some nontrivial representation Π_α^A of the Hopf algebra $\mathcal{A}$. This leads to the definition of an intermediate algebra $\mathcal{T}$ as the suitably defined stabilizer subalgebra of the vacuum state: $\mathcal{T}$ was originally defined as the largest Hopf subalgebra of $\mathcal{A}$ whose elements a satisfy

$$\Pi_\alpha^A(a)|\phi_0\rangle = \varepsilon(a)|\phi_0\rangle \quad \forall\ a \in \mathcal{T}, \tag{1}$$

where $\varepsilon(a)$ denotes the counit or trivial representation of the Hopf algebra in question. This analysis is further generalized in a related paper,[2] where it is argued that sometimes more general (non-Hopf) algebras play an essential role. We then must distinguish between a right or a left intermediate algebra, $\mathcal{T}_r$ and $\mathcal{T}_l$, respectively, which are all related in the sense that $\mathcal{T} \subseteq \mathcal{T}_r \cap \mathcal{T}_l$. The complication that arises at this "intermediate" level is that certain representations of $\mathcal{T}$ may have nontrivial braiding properties with the vacuum state. This in turn means that the vacuum cannot be single valued around a particle carrying that representation. If this happens to be the case, it has the physical implication that such particles (representations) must be confined. The low energy theory of nonconfined degrees of freedom is then characterized by yet a different algebra called $\mathcal{U}$. So the breaking of Hopf symmetries involves three algebras: the unbroken algebra $\mathcal{A}$, the intermediate algebra (be it $\mathcal{T}$, $\mathcal{T}_r$ or $\mathcal{T}_l$), and the unconfined algebra $\mathcal{U}$. The generic breaking scheme is depicted in Fig. 1. With thisformalism of Hopf symmetry breaking available it is important to take a closer look at suitable physical situations to check whether the theory produces correct and useful results. This is the subject of the remainder of this paper. Some applications to non-Abelian nematic crystals are the subject of another paper appearing in parallel.[5]

II. ABELIAN DEFECT CONDENSATES

In this section we will treat some examples of "Abelian" condensates. The rather well known examples of Abelian defect condensates are interesting because they cause confinement but at the same time an analysis of the unconfined particles reveals that a so-called symmetry restoration takes place. In other words Hopf symmetry breaking by defects may lead to the restoration of ordinary symmetries.

A. Kosterlitz-Thouless revisited

The most well-known example of a defect condensate is the one that mediates the Kosterlitz-Thouless transition in

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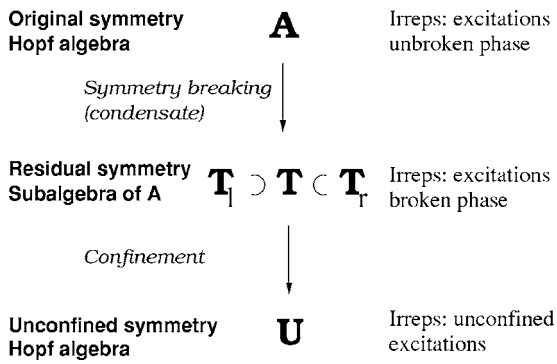

FIG. 1. The symmetry breaking scheme. The original symmetry algebra $\mathcal{A}$ is broken to a residual symmetry algebra. There are several possible definitions of the residual symmetry, in this paper we will use $\mathcal{T}_r$. Excitations in the broken phase correspond to irreps of $\mathcal{T}_r$, and some of these excitations are confined. The unconfined excitations are irreps of the unconfined symmetry algebra $\mathcal{U}$, where $\mathcal{U}$ is a Hopf algebra. This analysis is richer than the usual group symmetry breaking, where one only has an unbroken group G and a residual symmetry group $H \subset G$.

the two-dimensional XY model.[11] In the isotropic phase, the internal symmetry group is SO(2). Representations of SO(2) are labelled by an integer: $n \in \mathbb{Z}$. For reasons that will become clear shortly, we also introduce particles that transform under projective unitary representations of SO(2), which is equivalent to saying that they transform under faithful representations of the universal covering group $\mathbb{R}$ of SO(2). These $\mathbb{R}$ representations are labelled by a real number $\lambda \in \mathbb{R}$, giving the irrep ρ_λ,

$$\rho_\lambda(x) = e^{i\lambda x}, \quad x \in \mathbb{R}. \tag{2}$$

Thinking of λ as labelling a spin degree of freedom one has that the particle is bosonic if $\lambda \in \mathbb{Z}$, because $\rho_\lambda(2\pi)=1$. If $\lambda \in \mathbb{Z}+\frac{1}{2}$, the irrep is fermionic, because $\rho_\lambda(2\pi)=-1$. For any other value of λ, the irrep would be called an Abelian anyon.

In the low temperature phase of our system, the SO(2) symmetry is completely broken by a bosonic ($\lambda=1$) condensate implying that the covering group $\mathbb{R}$ is broken to $\mathbb{Z}$. The residual symmetry transformations, labelled by $n \in \mathbb{Z}$, correspond to rotations by $2\pi n$. The Hopf symmetry in this phase is the *quantum double* of $\mathbb{Z}$ denoted by $D(\mathbb{Z})$, because the defects are labelled by an integer n, as a consequence of the fact that $\Pi_1(\mathbb{R}/\mathbb{Z})=\Pi_0(\mathbb{Z})=\mathbb{Z}$. The electric irreps in this phase are labelled by $\mu \in [0,2\pi]$, corresponding to the irreps of $\mathbb{Z}$. However, we can still include all the electric irreps of the isotropic theory. In other words, we can restrict the irreps of $\mathbb{R}$ to $\mathbb{Z}$, giving representations of $\mathbb{Z}$, some of which are equivalent. We like to take all the electric irreps along, because we want to see which irreps of the isotropic phase survive unconfined in the various phases. Note that the Goldstone boson transforms trivially under $\mathbb{Z}$, because 2π rotations do not affect it. So in this phase we have the representations Π_μ^n of $D(\mathbb{Z})$, where

$$D(\mathbb{Z}) = F(\mathbb{Z}) \times \mathbb{C}\mathbb{Z}. \tag{3}$$

The notation employed here is explained elsewhere.[2,9] We will briefly explain the meaning of $D(H)=F(H)\times\mathbb{C}H$, for general group H. $\mathbb{C}H$ is the group algebra of H, i.e., its elements are of the form $\lambda_i h_i$, where $\lambda_i \in \mathbb{C}$ and $h_i \in H$, and they perform global symmetry transformations when acting on states. $F(H)$ is the function algebra of H, its elements are of the form $\lambda_i P_{h_i}$. The P_{h_i} are projection operators, when they act on a state they project out states that have topological charge h_i. Now the elements of $D(H)$ are a combination of the two, i.e., a basis of $D(H)$ is elements of the form $P_{h_i}h_j$, with $h_i, h_j \in H$. This element first performs the global symmetry transformation h_j, and then projects out states with topological charge h_i.The multiplication in $D(H)$ can easily be inferred by physical considerations, and is set by

$$P_{h_i}h_j \times P_{h_k}h_m = \delta_{h_i,h_jh_kh_j^{-1}} P_{h_i}h_jh_m. \tag{4}$$

If we heat the system in this $D(\mathbb{Z})$ phase, the Kosterlitz-Thouless transition takes place corresponding to the formation of a condensate of the defect $|\phi_0\rangle=|n=1\rangle$. To see what happens one first must determine the intermediate algebra $\mathcal{T}$ and then use the braid relations to determine which representations are confined to obtain the effective unconfined algebra $\mathcal{U}$. The present case corresponds to a single defect condensate and the general structure of $\mathcal{T}$ and $\mathcal{U}$ have been given elsewhere.[2] The outcome is

$$\mathcal{T}_r = F[H/(g_A)] \times \mathbb{C}N_A, \tag{5}$$

$$\mathcal{U} = F[N_A/(g_A)] \times \mathbb{C}N_A/(g_A), \tag{6}$$

where (g_A) is the group generated by the group element g_A corresponding to the defect in the condensate, and this group (g_A) is a normal subgroup of N_A, which is the normalizer in H of the element g_A. In the case we are considering, $(g_A)=N_A$, so for $\mathcal{U}$ we obtain the rather trivial result,

$$\mathcal{U} = D(\mathbb{Z}/\mathbb{Z}) = D(e),$$

from which we conclude that there are no surviving defects in the unconfined phase and also that from the set electric reps of $H=\mathbb{Z}$ (i.e., $\mu \in [0,2\pi]$), none are left.

But what about the original set of spin representations (2)? Considering the braiding of a representation with spin λ with a defect $|n\rangle$, we obtain that[12]

$$\mathcal{R}^2 \cdot |n\rangle|\lambda\rangle = e^{i2\pi n\lambda}|n\rangle|\lambda\rangle. \tag{7}$$

The only irreps of $\mathbb{R}$ that braid trivially with the condensate $|1\rangle$ are those labelled by an integer $m \in \mathbb{Z}$,

$$\rho_m(x) = e^{i2\pi mx}, \quad x \in \mathbb{R}.$$

So the residual theory has no defects, and the unconfined electric irreps are labelled by $n \in \mathbb{Z}$, in other words the fractional charges are confined. A theory with no defects and charges labelled by $n \in \mathbb{Z}$ is an SO(2) theory. This argument suggests that *the symmetry has been restored to a full* SO(2), i.e., the original $\mathbb{R}$ theory has been compactified, and the "spin" is now quantized.

In general, condensing the $|n\rangle$ defect gives a theory with $\mathbb{Z}_n$ defects (defects defined modulo n), and the electric irreps are labelled by $\frac{k}{n}$, $k \in \mathbb{Z}$, meaning that the symmetry is restored to $SO(2)/\mathbb{Z}_n$.

We would like to note that, since it is impossible to have long range order in two dimensions after breaking of continuous symmetries (due to the Mermin-Wagner theorem), the phases we are considering may not have long range order, but quasi-long-range order. This does not alter the analysis above, because we work out the residual quantum numbers after condensation, which is independent of the behavior of the correlation functions.

B. One-dimensional melting

In the previous example we studied a phase with internal symmetry, we now turn to a similar example with a spatial translational symmetry. Consider an effectively one-dimensional system, say defined along the x axis. In the high energy phase, the external (space) symmetry group of the system is the translation group $\mathbb{R}$. As in the previous example, the irreducible unitary representations (irreps) ρ_λ of the external symmetry group $\mathbb{R}$ are again labelled by $\lambda \in \mathbb{R}$ [see (2)]. We now break this $\mathbb{R}$ symmetry to $\mathbb{Z}$, by the formation of a one-dimensional crystal, i.e., a one-dimensional lattice. The lattice sites are at positions $x=\ldots,-2a, -a,0,a,2a,\ldots$. We can label the lattice sites by elements of $\mathbb{Z}$, where $m \in \mathbb{Z}$ corresponds to the lattice site at position $x=ma$.

The irreps Ω_μ of the residual external symmetry group $\mathbb{Z}$ are labelled by $\mu \in [0,2\pi/a)$,

$$\Omega_\mu(m) = e^{i\mu ma}, \quad m \in \mathbb{Z}. \tag{8}$$

μ is the *momentum* of the irrep Ω_μ. In the broken phase with broken continuous translation invariance *phonons* appear as Goldstone modes. These phonons have momenta $k \in \mathbb{R}$. Furthermore the broken phase is also characterized by possible defects (dislocations) labelled by an integer n because $\Pi_1(\mathbb{R}/\mathbb{Z})=\Pi_0(\mathbb{Z})=\mathbb{Z}$.

Now imagine that we condense the defect $|n\rangle$. If we now consider the braiding of a phonon with momentum k with this defect, we obtain again that

$$\mathcal{R}^2 \cdot |n\rangle|k\rangle = e^{ikna}|n\rangle|k\rangle. \tag{9}$$

This braiding is trivial if and only if $e^{ikna}=1$, which is the case if k is an integer multiple of $\frac{2\pi}{na}$. Since $k \in B$, we have n phonons that braid trivially with the condensate, namely the phonons $|k\rangle$ with momentum $k=0,\frac{2\pi}{na},2\frac{2\pi}{na},\ldots,(n-1)\frac{2\pi}{na}$. The defects $|m\rangle$, $m \in \mathbb{Z}$ are unconfined, but they are now defined modulo $|n\rangle$, since $|n\rangle$ is condensed.

Thus the condensation of $|n\rangle$ has led to a theory with $\mathbb{Z}_n$ defects, and $\mathbb{Z}_n$ phonons and we end up with an unconfined algebra $\mathcal{U}=D(\mathbb{Z}_n)$.

In particular, if we take $n=1$, i.e., we condense the $|1\rangle$ defect, we obtain a phase with trivial symmetry, in which there are no defects, and all phonons are confined.

Let us now discuss melting. If we condense the defect $|1\rangle$, then the unconfined modes are those with momentum $k=0,\frac{2\pi}{a},2\frac{2\pi}{a},3\frac{2\pi}{a},\ldots$. So all phonons are confined (since a phonon cannot have exactly zero crystal momentum), and the unconfined translational irreps have momentum $k \in \{\frac{2\pi}{a}n,\ n \in \mathbb{Z}\}$. Thus we have a theory with momentum modes labelled by $n \in \mathbb{Z}$, and no defects. This is obviously a theory with a periodic translational U(1) symmetry. This is exactly the same argument as the one for the Kosterlitz-Thouless phase, except that there the Goldstone modes were not confined, yet the symmetry was restored to SO(2). In this theory one could of course also consider situations in which various mutually compatible condensates (that trivially braid with each other) occur simultaneously, leading to yet other residual symmetries and excitations.

Note that the discussion above is unphysical, because there are no phase transitions in one-dimensional systems. We merely gave the simplest system for which we can carry through our analysis. However, if the results are only slightly modified they apply to higher dimensional systems that are effectively one dimensional. Examples of effective one-dimensional (1D) crystals are arrays of liquid lines or planes, i.e., respectively, two-dimensional (2D) and three-dimensional (3D) smectics. We will now perform the analysis of a defect condensate in a 2D smectic.

C. A defect condensate in a two-dimensional smectic

We may extend the previous reasoning to a particular two-dimensional system, where we consider only translational degrees of freedom. In the high energy phase, the (translational) symmetry of the system is $\mathbb{R}^2$. The irreps $\rho_{\vec{k}}$ of $\mathbb{R}^2$ are labelled by $\vec{k}=(k_x,k_y) \in \mathbb{R}^2$,

$$\rho_{\vec{k}}(x,y) = e^{i(k_x x+k_y y)}. \tag{10}$$

We break this symmetry to $\mathbb{R}\times\mathbb{Z}$, by condensing the mode $\rho_{(0,a)}$. The system is then composed of a set of lines parallel to the x axis. In the y direction, we have the same situation as above, i.e., the symmetry is broken from $\mathbb{R}$ to $\mathbb{Z}$ in the y direction. In the x direction, the symmetry is still unbroken, so we can neglect the x direction. Let us be more precise about this. The phonons, as Goldstone modes of the broken translations in the y direction have a momentum that points in the y direction so they have $k_x=0$.

Now condense the basic dislocation in the y direction, the $|1\rangle$ defect. The phonons all braid nontrivially with the condensate. Just as above, all the phonons are confined.

Now that we have seen how the symmetry is restored, we can also discuss what happens in the Abrikosov lattice in type II superconductors, for example, in our terminology. The symmetry is first fully restored by the defect condensate, and then it is broken to a lattice, as it were by the condensation of two defect density modes, similar to the formation of the smectic. It is a dual version of what we just discussed.

III. NON-ABELIAN DEFECT CONDENSATES

We now want to move on to defect condensates in a non-Abelian setting, and it turns out that we then must distinguish different types of defect condensates. Consider a phase described by a quantum double $D(H)$, or a generalized quan-

tum double $F(H_m)\otimes \mathbb{C}H_{el}$,[13] and pick a purely magnetic representation Π_1^A (1 is the trivial representation of the centralizer N_A). A basis of the vector space on which this irrep acts is given by $\{|g_i^A\rangle\}$, where the g_i^A are the different defects in the conjugacy class[14] C_A whose representative element we denote by g_A. In that defect class C_A we may consider the following inequivalent types of condensates:

(i) Single defect condensate,

$$\phi = |g_i^A\rangle. \tag{11}$$

(ii) Class sum defect condensate,

$$\phi = \sum_{g_i^A \in A} |g_i^A\rangle = :|C_A\rangle. \tag{12}$$

We denote the condensate by $|C_A\rangle$, where g_A is the preferred element of $\mathcal{A}$.[15]

(iii) Combined defect condensate,

$$\phi = \sum_{g_i \in E} |g_i\rangle, \tag{13}$$

where E is some subset of the defects in one class. We need only take the elements to be within one class because it turns out that we need only study the cases where the condensate is the sum of vectors in the same irrep.

The single defect and class sum defect condensates are a special case of the combined defect condensate. In the following we will encounter both class sum and combined defect condensates.

A. The hexagonal crystal and its regular modes

We start with the planar, achiral, hexagonal crystal which has a space symmetry group $\mathbb{Z}^2 \rtimes \mathbb{Z}_6$. Strictly speaking in this case the defect classes (i.e., the sets of defects transformed into each other under residual global symmetry transformations) are larger than the conjugacy classes of the magnetic group, but for our present purposes it suffices to restrict our considerations to the conjugacy classes.

The first Brillouin zone (rescaled) is shown in Fig. 2, it is a hexagon. Points in the Brillouin zone correspond to irreps of the translation group. Namely, a point in the Brillouin zone is a momentum vector $\vec{k}=(k_x,k_y)$, and the associated irrep of $\mathbb{Z}^2$ is

$$(n,m) \mapsto e^{i(k_x n + k_y m)}. \tag{14}$$

B. Topological defects in the hexagonal phase

The homotopy groups which define the defects are

$$\Pi_1(G/H) = \tilde{H}, \tag{15}$$

$$\Pi_2(G/H) = 0, \tag{16}$$

where $\tilde{H}$ is the lift of H in the universal covering group $\tilde{G}$ of G. In three dimensions, there are no monopoles because the space group H is discrete. The line defects are characterized by elements of $\tilde{H}$.

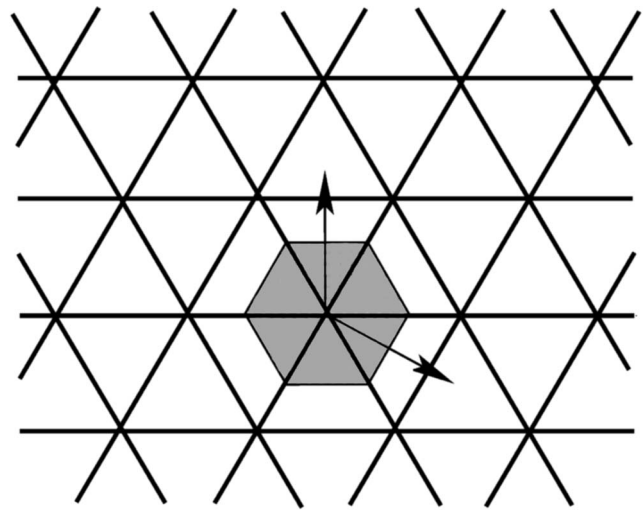

FIG. 2. A triangular lattice, representing a crystal with hexagonal symmetry. The dashed vectors are vectors in the reciprocal lattice, and the shaded hexagon represents the first Brillouin zone.

First we consider a two-dimensional hexagonal crystal. To determine $\tilde{H}$ we need to lift H to the covering group of $G = \mathbb{R}^2 \rtimes \mathrm{SO}(2)$. The translational part $\mathbb{R}^2$ is already simply connected. The covering group of SO(2) is $\mathbb{R}$. An element θ of $\mathbb{R}$ corresponds to a rotation over an angle θ. In the covering group, however, a rotation over 2π is no longer equivalent to the identity. Similarly, the covering of $\mathbb{Z}_6$ is

$$\widetilde{\mathbb{Z}_6} = \mathbb{Z} \times \mathbb{Z}_6 \simeq \mathbb{Z}. \tag{17}$$

Therefore $\widetilde{\mathbb{Z}^2 \rtimes \mathbb{Z}_6} \simeq \mathbb{Z}^2 \rtimes \mathbb{Z}$, and

$$\Pi_1(G/H) \simeq \mathbb{Z}^2 \rtimes \mathbb{Z}. \tag{18}$$

Therefore a defect is an element of $\mathbb{Z}^2 \rtimes \mathbb{Z}$. We denote it by $(n,m)r^p$, with $n,m,p \in \mathbb{Z}$. r is a $\frac{2\pi}{6}$ rotation, (1,0) a translation by one lattice vector in the x direction, and (0,1) a translation by one lattice vector in the y direction. A defect of the form (n,m) (i.e., with $p=0$) is a dislocation. An r^p defect $(n=m=0)$ is a disclination. Note that the structure of defects described by (18) is indeed non-Abelian, which means that we must think of them forming multiplets corresponding to the conjugacy classes of the first homotopy group. We return to this question in the following section when we construct the Hopf algebra describing this phase.

Note that although both rotational and translational Goldstone modes are present in a two-dimensional crystal, it is known that the rotational Goldstone mode, called the *roton*, is much more massive[16] than the translational modes, because it costs a lot of energy to rotate different parts of the crystal relative to each other. For the same reason, *disclinations are much more massive than dislocations*, and at low temperatures we only have dislocations and translational Goldstone modes.

For a three-dimensional crystal, the analysis is very similar. For example, in a crystal made up of layers of hexagonal crystal, $G=\mathbb{R}^3 \rtimes \mathrm{SO}(3)$ is broken to $H=\mathbb{Z}^3 \rtimes \mathbb{Z}_6$, so the de-

TABLE I. The classification of defects for the two-dimensional hexagonal crystal in classes of $\overline{\mathbb{Z}^2 \rtimes \mathbb{Z}_6} = \mathbb{Z}^2 \rtimes \mathbb{Z}$.

Representative g_A	Conjugacy class C_A
r^{6k}, $k \in \mathbb{Z}$	$C_{r^{6k}} = \{r^{6k}\}$
$(a,b)r^{6k}$	$C_{(a,b)r^{6k}} =$
$k \in \mathbb{Z}$, $a,b \geq 0$	$\{(a,b)r^{6k}, (-b,a+b)r^{6k},$
	$(-a-b,a)r^{6k}, (-a,-b)r^{6k},$
	$(b,-a-b)r^{6k}, (a+b,-a)r^{6k}\}$
r^{1+6k}, $k \in \mathbb{Z}$	$C_{r^{1+6k}} =$
	$\{(m,n)r^{1+6k} : (m,n) \in \mathbb{Z}^2\}$
r^{-1+6k}, $k \in \mathbb{Z}$	$C_{r^{-1+6k}} =$
	$\{(m,n)r^{-1+6k} : (m,n) \in \mathbb{Z}^2\}$
r^{2+6k}, $k \in \mathbb{Z}$	$C_{r^{2+6k}} =$
	$\{(m,n)r^{2+6k} : m-n \in 3\mathbb{Z}\}$
$(1,0)r^{2+6k}$, $k \in \mathbb{Z}$	$C_{(1,0)r^{2+6k}} =$
	$\{(m,n)r^{2+6k} : m-n \notin 3\mathbb{Z}\}$
r^{3+6k}, $k \in \mathbb{Z}$	$C_{r^{3+6k}} =$
	$\{(m,n)r^{3+6k} : (m,n) \in 2\mathbb{Z} \times 2\mathbb{Z}\}$
r^{3+6k}, $k \in \mathbb{Z}$	$C_{(1,0)r^{3+6k}} =$
	$\{(m,n)r^{3+6k} : (m,n) \notin 2\mathbb{Z} \times 2\mathbb{Z}\}$
r^{4+6k}, $k \in \mathbb{Z}$	$C_{r^{4+6k}} =$
	$\{(m,n)r^{4+6k} : m-n \in 3\mathbb{Z}\}$
$(1,0)r^{4+6k}$, $k \in \mathbb{Z}$	$C_{(1,0)r^{4+6k}} =$
	$\{(m,n)r^{4+6k} : m-n \notin 3\mathbb{Z}\}$

fects have an extra label, which represents translational defects in the direction perpendicular to the layers,

$$\Pi_1(G/H) = \mathbb{Z}^3 \rtimes \mathbb{Z}. \tag{19}$$

For a dislocation characterized by (n,m,k), the vector (n,m,k) is the Burgers vector of the defect. A general dislocation has both screw and edge character, and the Burgers vector is given by the integral of the order parameter field $\vec{u}(\vec{x})$ along a loop L encircling the core.[17] The fusion rules of line defects have some interesting features, which are crucial for the story to come. We illustrate these features for the two-dimensional hexagonal crystal. To determine the fusion rules, we need the conjugacy classes of the defects. This requires knowledge of the multiplication in $\mathbb{Z}^2 \rtimes \mathbb{Z}_6$, which is set by

$$r(a,b)r^{-1} = (-b,a+b). \tag{20}$$

The conjugacy classes are listed in Table I. From this we can, for example, prove that two disclinations can fuse to form any dislocation in this crystal. Namely, the fusion rules are given by multiplying entire conjugacy classes. If we fuse C_r and $C_{r^{-1}}$, we can get any dislocation (m,n) because $(m,n)r$ is an element of C_r. Thus, from a topological point of view a dislocation can "dissociate" into two disclinations.

C. The double symmetry of the hexagonal crystal

In the preceding section we saw that the magnetic group of a hexagonal crystal with large symmetry transformations is $\mathbb{Z}^2 \rtimes \mathbb{Z}$. Let us now consider a hexagonal crystal and include the reflections, then the magnetic group is the same,[18] $H_m = \mathbb{Z}^2 \rtimes \mathbb{Z}$. The electric group is $H_{el} = \widetilde{\mathbb{Z}^2 \rtimes D_6} = (\mathbb{Z}^2 \rtimes D_6) \times \mathbb{Z}$. The last $\mathbb{Z}$ part corresponds to rotations of $2\pi n$, $n \in \mathbb{Z}$, which we denote by r^{6n}. These rotations are in the center of the group. Thus the Hopf symmetry is[19]

$$\mathcal{A} = F(\mathbb{Z}^2 \rtimes \mathbb{Z}) \otimes \mathbb{C}[(\mathbb{Z}^2 \rtimes D_6) \times \mathbb{Z}]. \tag{21}$$

We write an element of $(\mathbb{Z}^2 \rtimes D_6) \times \mathbb{Z}$ as $(k,l)s^j r^{k+6n}$, with $k,l,n \in \mathbb{Z}$, $j \in \mathbb{Z}_2$, $n \in \mathbb{Z}$. The reflection in D_6 is denoted by s. We take s to be the reflection with respect to the x axis (see Table II).

To analyze the quantum numbers of the excitation sectors of our theory, we proceed as usual and must construct the representations of the algebra given above.[2,9,10] The representations of the generalized quantum doubles have a generic structure. There is an orbit or defect class of the group in the

TABLE II. The orbits of $\mathbb{Z}^2 \times \mathbb{Z}$ under the action of $(\mathbb{Z}^2 \rtimes D_6) \times \mathbb{Z}$. The conjugacy classes are defined in Table I.

Orbits
$\mathcal{O}_{r^{6k}} = C_{r^{6k}}$
$\mathcal{O}_{(a,b)r^{6k}} = \{C_{(a,b)r^{6k}}, C_{(a+b,-b)r^{6k}}\}$
$\mathcal{O}_{r^{1+6k}} = \{C_{r^{1+6k}}, C_{r^{-1-6k}}\}$
$\mathcal{O}_{r^{2+6k}} = \{C_{r^{2+6k}}, C_{r^{-2-6k}}\}$
$\mathcal{O}_{(1,0)r^{2+6k}} = \{C_{(1,0)r^{2+6k}}, C_{(1,0)r^{-2-6k}}\}$
$\mathcal{O}_{r^{3+6k}} = \{C_{r^{3+6k}}, C_{r^{-3-6k}}\}$
$\mathcal{O}_{(1,0)r^{3+6k}} = \{C_{(1,0)r^{3+6k}}, C_{(1,0)r^{-3-6k}}\}$

TABLE III. The irreps of $F((\mathbb{Z}^2 \rtimes \mathbb{Z}) \otimes \mathbb{C}(\mathbb{Z}^2 \rtimes D_6) \times \mathbb{Z}$. The index $\lambda \in [0,2\pi)$ corresponds to the irreps of the $\mathbb{Z}$ part of the centralizers, which represents the rotations by an angle $2\pi n$. The index ρ in the first representation listed, refers to the irreps of $\mathbb{Z}^2 \rtimes D_6$ defined in Table IV.

Conjugacy classes	N	Irreps Π_α^A
$C_{r^{6k}}$	$(\mathbb{Z}^2 \rtimes D_6) \times \mathbb{Z}$	$\Pi_{\rho,\lambda}^{r^{6k}}$
$k \in \mathbb{Z}$		
$C_{(a,b)r^{6k}}, C_{(a+b,-b)r^{6k}}$	$(\mathbb{Z}^2) \times \mathbb{Z}$	$\Pi_{\vec{k},\lambda}^{(a,b)r^{6k}}$
$a,b \geq 0$; $b \geq a$; $k \in \mathbb{Z}$		$\vec{k} \in \mathbb{R}^2$
$C_{r^{1+6k}}, C_{r^{-1-6k}}$	$(\mathbb{Z}_6) \times \mathbb{Z}$	$\Pi_{m,\lambda}^{r^{1+6k}}$
$k \in \mathbb{Z}$		$m \in \mathbb{Z}_6$
$C_{r^{2+6k}}, C_{r^{-2-6k}}$	$(\mathbb{Z}_6) \times \mathbb{Z}$	$\Pi_{m,\lambda}^{r^{2+6k}}$
$k \in \mathbb{Z}$		$m \in \mathbb{Z}_6$
$C_{(1,0)r^{2+6k}}, C_{(1,0)r^{-2-6k}}$	$(\mathbb{Z}_3) \times \mathbb{Z}$	$\Pi_{n,\lambda}^{(1,0)r^{2+6k}}$
$k \in \mathbb{Z}$		$n \in \mathbb{Z}_3$
$C_{r^{3+6k}}, C_{r^{-3-6k}}$	$(\mathbb{Z}_6) \times \mathbb{Z}$	$\Pi_m^{r^{3+6k}}$
$k \in \mathbb{N}$		$m \in \mathbb{Z}_6$
$C_{(1,0)r^{3+6k}}, C_{(1,0)r^{-3-6k}}$	$(\mathbb{Z}_2) \times \mathbb{Z}$	$\Pi_{l,\lambda}^{(1,0)r^{3+6k}}$
$k \in \mathbb{N}$		$l \in \mathbb{Z}_2$

TABLE IV. The irreps of $\mathbb{Z}^2 \rtimes D_6$. The number of an orbit corresponds to the number in Fig. 3. $(D_6)_{\vec{k}}$ is the little group of the orbit.

Orbit number	$(D_6)_{\vec{k}}$	Number of irreps	Dim. of irreps
1	D_6	6	1, 1, 1, 1, 2, 2
2	$\mathbb{Z}_2$	2	6, 6
3	$\mathbb{Z}_2$	2	6, 6
4	e	1	12
5	$\mathbb{Z}_2 \rtimes \mathbb{Z}_2$	4	3, 3, 3, 3
6	$\mathbb{Z}_2$	2	6, 6
7	D_3	3	2, 2, 4

function part labelling the defects, paired with a representation of the centralizer N_A for the ordinary and/or Goldstone modes. These representations will be constructed in this section and the results are listed in Table III. We need the orbits $\mathcal{O}$ of $[(\mathbb{Z}^2 \rtimes \mathbb{Z}_6) \times \mathbb{Z}]$ under the action of $(\mathbb{Z}^2 \rtimes D_6) \times \mathbb{Z}$, the centralizer N of a chosen preferred element of each orbit, and the irreps of the centralizer. The action of $(\mathbb{Z}^2 \rtimes D_6) \times \mathbb{Z}$ on $\mathbb{Z}^2 \rtimes \mathbb{Z}_6$ is given by

$$r(a,b)r^{-1} = (-b, a+b),$$

$$s(a,b)s^{-1} = (a+b, -b).$$

$$srs^{-1} = r^{-1}$$

There is one class of orbits, $\mathcal{O}_{r6k}$, for which determining the irreps of the centralizer $(\mathbb{Z}^2 \rtimes D_6) \times \mathbb{Z}$ is a little complicated. We can determine the irreps of $\mathbb{Z}^2 \rtimes D_6$ and those of $\mathbb{Z}$ separately, because the irreps of a direct product of groups are the tensor products of the irreps of the separate groups. The irreps of $\mathbb{Z}^2 \rtimes D_6$ can be constructed by the method of induced representations described in the Appendix . We must pick a momentum vector $\vec{k}$ which corresponds to an irrep of the translation group $\mathbb{Z}^2$, and act on it with D_6. The elements that transform $\vec{k}$ into a vector that corresponds to the same irrep form a subgroup of D_6, which we denote by $(D_6)_{\vec{k}}$. Thus we are considering orbits in the first Brillouin zone (note that these classes have nothing to do with the orbits listed in TableI). We must find all possible cases of vectors in the first Brillouin zone, with different $(D_6)_{\vec{k}}$. For the hexagonal crystal, there are seven different types of orbits, and a representative element of each type is given in Fig. 3.

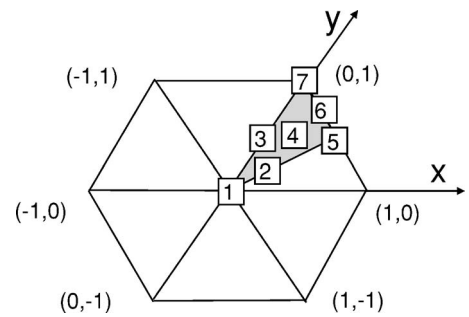

FIG. 3. The first Brillouin zone of the planar hexagonal crystal. The gray area is a set of representatives of orbits under rotations and reflections. The numbered squares are representatives of the different types of orbits. The x and y axes have been rescaled. The unit distance on these rescaled axes corresponds to a distance of $\frac{\pi}{a}$, where a is the lattice spacing in the crystal.

We then call $\mathbb{Z}^2 \rtimes (D_6)_{\vec{k}}$ the *little group* of $\vec{k}$. We must determine the irreps of the little group, and then *induce* irreps of the whole group. This procedure gives all the irreps. which are given in Table IV.

As an example, we explicitly give the irrep corresponding to orbit 5 in Fig. 3. In the Appendix it is illustrated how to derive this. The matrices of the irrep are

$$(a,b) \mapsto \begin{pmatrix} e^{i(\pi/2)a}e^{i(\pi/2)b} & 0 & 0 \\ 0 & e^{i(\pi/2)(a+b)}e^{i(\pi/2)(-a)} & 0 \\ 0 & 0 & e^{i(\pi/2)b}e^{i(\pi/2)(-a-b)} \end{pmatrix},$$

$$r \mapsto \begin{pmatrix} 0 & 0 & e^{i\pi m} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad s \mapsto \begin{pmatrix} 0 & & e^{i\pi n} \\ 0 & e^{i\pi n} & 0 \\ e^{i\pi n} & 0 & 0 \end{pmatrix}.$$

D. Non-abelian defect condensates in hexagonal crystals

Let us now turn to the *hexatic* phase. Before turning to the Hopf symmetries involved in the next section, we first recall some of the standard analysis. The phase is two dimensional. It is translationally invariant, but the rotational group is broken to $\mathbb{Z}_6$. Such a phase looks strange at first sight, because it seems that to break rotational symmetry we need to put the atoms on a lattice, thereby also breaking translational symmetry. We know that this is not the case because such a hexatic phase is actually realized in nature.[20,21]

That it is *not* impossible to break translational symmetry without breaking rotational symmetry, can be understood from the representation theory[22] of $\mathbb{R}^2 \rtimes \mathrm{SO}(2)$. A suitable

order parameter field is provided by $\phi = e^{i6\theta}$, where θ measures the angle of the bonds between neighboring molecules makes with the x axis. In the high-temperature isotropic phase, $\langle e^{i6\theta}\rangle = 0$. In the ordered phase, $\langle e^{i6\theta}\rangle \neq 0$ because the angle of the bonds is on average a multiple of $\frac{2\pi}{6}$.

The hexatic phase can arise via spontaneous symmetry breaking from the isotropic phase, but it can also be the outcome of the *melting* of a two-dimensional hexagonal crystal. This is what happens in nature: as we increase the temperature of the crystal, at some point the translational symmetry is restored, but locally the molecules are still arranged hexagonally. Of course, locally there is some translational symmetry, but the correlations of translations decay exponentially.

We noted before that in a two-dimensional crystal the disclinations and the roton are very massive, and are not excited at low temperatures. As the temperature is increased and the translational symmetry is *restored* (signalling the transition to the hexatic phase), the translational Goldstone modes and the dislocations disappear from the spectrum. At the same time, the roton becomes massless, while the disclinations are the allowed defects in this phase.

The traditional phase diagram of a two-dimensional crystal contains a line separating the crystal phase and the liquid phase. The phase transition is first order. However, for the hexagonal crystal the transition to the liquid phase can occur via two second-order phase transitions: first a phase transition to the translationally invariant phase with broken rotational symmetry, and then a transition to the liquid.

The mechanism behind the phase transition from crystal to hexatic is well known, and quite remarkable: it is the formation of a *dislocation condensate*.[16] Defects in a phase with global symmetry breaking cost an infinite amount of energy, because their energy diverges logarithmically with the size of the system.[17] However, defect-antidefect pairs cost a finite amount of energy, and at finite temperatures they have a finite density, and an average separation between the defects in a pair, which we call a. At the phase transition, a diverges, so that the single defects fill the ground state. This *restores translational symmetry*. In the next section we will discuss why it restores symmetries. Here we merely note what happens next if the ground state is now filled with single dislocations. We have seen that these dislocations in principle can decay into a disclination-antidisclination pair. They also have an average separation, b. At the second phase transition, the disclinations blow out, thereby *restoring rotational symmetry*.

The hexatic phase has been experimentally observed in numerous three-dimensional phases, such as smectics, dense solutions of DNA, flux arrays of high-T_c superconductors, superfluids,[20] etc. All these phases have some kind of layered order. The hexatic smectic is also called the smectic B phase. The bond-orientational order is within the layers of the smectic. For references on hexatic phases, we refer to the book by Nelson.[21]

E. The hexatic phase

Let us now apply our methods to this situation of the hexatic phase as an example of defect-mediated melting. We start with the Hopf symmetry of the hexagonal crystal,

$$\mathcal{A} = F[\mathbb{Z}^2 \rtimes (\mathbb{Z}_6 \times \mathbb{Z})] \otimes \mathbb{C}[(\mathbb{Z}^2 \rtimes D_6) \times \mathbb{Z}], \qquad (22)$$

and its representations which were given in Table III.

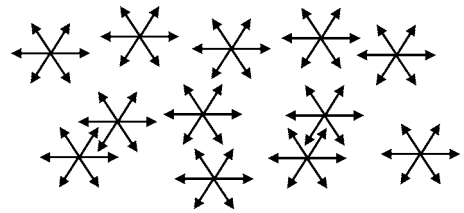

FIG. 4. An artist impression of the class sum dislocation defect condensate $|\phi_0\rangle = |(1,0)\rangle + |(0,1)\rangle + |(-1,1)\rangle + |(-1,0)\rangle + |(0,-1)\rangle + |(1,-1)\rangle$ with hexagonal symmetry.

Now we consider a class sum defect condensate of the basic dislocation $(1,0)$ belonging to the class $C_{(a,b)r^{6k}}$ with $k=0$ (see Fig. 4),

$$|\phi_0\rangle = |(1,0)\rangle + |(0,1)\rangle + |(-1,1)\rangle + |(-1,0)\rangle + |(0,-1)\rangle + |(1,-1)\rangle. \qquad (23)$$

It is not hard to determine what the intermediate $\mathcal{T}_r$ and unconfined $\mathcal{U}$ symmetries are in this case (the derivation is carried out in a related paper[2]):

$$\mathcal{T}_r = F[\mathbb{Z}^2 \rtimes (\mathbb{Z}_6 \times \mathbb{Z})/\mathbb{Z}^2] \otimes \mathbb{C}[(\mathbb{Z}^2 \rtimes D_6) \times \mathbb{Z}],$$
$$\mathcal{U} = F(\mathbb{Z}_6 \times \mathbb{Z}) \otimes \mathbb{C}(D_6 \times \mathbb{Z}). \qquad (24)$$

$\mathbb{Z}_6 \times \mathbb{Z}$ corresponds to disclinations, and $D_6 \times \mathbb{Z}$ corresponds to rotations and inversions. The $\mathbb{Z}$ part of $D_6 \times \mathbb{Z}$ corresponds to multiples of the 2π rotation.

The particular condensate of dislocations is invariant under rotations, hence disclinations are not confined. Basically the conclusion of our analysis is that the translational modes of the lattice have completely disappeared. There are no more dislocations (because they are condensed), and the translational phonons are confined. As we discussed earlier, this signals the restoration of translational symmetries, so the unconfined and residual symmetry algebras of the hexatic phase are actually

$$\mathcal{T}_r = F[\mathbb{Z}^2 \rtimes (\mathbb{Z}_6 \times \mathbb{Z})/\mathbb{Z}^2] \otimes \mathbb{C}[(\mathbb{R}^2 \rtimes D_6) \times \mathbb{Z}],$$
$$\mathcal{U} = F(\mathbb{Z}_6 \times \mathbb{Z}) \otimes \mathbb{C}[\mathbb{R}^2 \rtimes (D_6 \times \mathbb{Z})]. \qquad (25)$$

We have thus succeeded in interpreting the phase transition from the hexagonal crystal to the hexatic phase (including also the translational symmetry restoration), as a Hopf symmetry breaking phenomenon through a particular defect class condensate.

F. A topological nematic

We now wish to comment on a phase called the *topological nematic*.[23] This is a two-dimensional phase obtained from an isotropic crystal, where dislocations have condensed, but rotational symmetry is completely unbroken. In the defect

condensate (23) that led to the hexatic phase, the residual rotational symmetry group was $\mathbb{Z}_6$, but in the topological nematic the rotational symmetry group is U(1). Clearly, an isotropic crystal only makes sense in the continuum limit, because by definition a crystal is not isotropic. However, in the continuum limit one can definitely have a Lagrangian of a crystal in terms of the displacement field $\vec{u}(\vec{x})$, that is invariant under $\mathrm{ISO}(2)=\mathbb{R}^2 \rtimes \mathrm{SO}(2)$. One can show that the two-dimensional hexagonal and isotropic crystals have exactly the same Lagrangians[24] (they only have two independent elastic constants), so we could consider this isotropic crystal to be the continuum limit of a hexagonal crystal.

Now the topological nematic is a phase obtained by a defect condensate in the isotropic crystal. To obtain a defect condensate that fully restores the rotational symmetry group SO(2), we must first realize that in the continuum limit the dislocations carry a continuous label. The Burgers vector of a dislocation in the continuum limit is characterized by a two-dimensional vector $|(a,b)\rangle$, with $a,b \in \mathbb{R}^2$. Thus a dislocation condensate that leads to restoration of the full rotational symmetry group is given by

$$|\phi_r\rangle = \int_0^{2\pi} d\theta |(\cos\theta, \sin\theta)\rangle. \tag{26}$$

Strictly speaking, this condensate is the continuum limit of the class sum defect condensate we considered in the hexatic phase. It is indeed a class sum defect condensate of the isotropic crystal. It is in that sense that our analysis naturally incorporates the topological nematic.

Furthermore, our analysis demonstrates how the partial restoration of the symmetry group can be understood by considering defect condensates that are not a class sum, as we will show next. Let us return to the hexatic phase and analyze the transition to the isotropic phase. The hexatic phase is a phase described by the Hopf algebra (25)

$$\mathcal{A} = F(\mathbb{Z}_6 \times \mathbb{Z}) \otimes \mathbb{C}[\mathbb{R}^2 \rtimes (D_6 \times \mathbb{Z})].$$

Now assume a disclination condensate of $|r\rangle$, the $\frac{2\pi}{6}$ rotation, then the residual symmetries are successively

$$\begin{aligned} \mathcal{T}_r &= F(e) \otimes \mathbb{C}[\mathbb{R}^2 \rtimes (\mathbb{Z}_6 \times \mathbb{Z})], \\ \mathcal{U} &= \mathbb{C}\mathbb{R}^2. \end{aligned} \tag{27}$$

G. A quantum smectic?

We mentioned that with non-Abelian defects there are in general quite a few possibilities for the condensate. Our formalism allows us to determine what the physical implications are. We conclude by considering, instead of the class condensate (23), the defect-antidefect condensate,

$$|\phi_0\rangle = |(1,0)\rangle + |(-1,0)\rangle.$$

The residual and unconfined symmetry algebras turn out to be

$$\mathcal{T}_r = F[(\mathbb{Z}^2 \rtimes \mathbb{Z}_6) \times \mathbb{Z}]/0 \times \mathbb{Z} \otimes \mathbb{C}[\mathbb{Z}^2 \rtimes (\mathbb{Z}_2 \rtimes \mathbb{Z}_2)], \tag{28}$$

$$\mathcal{U} = F[(\mathbb{Z} \rtimes \mathbb{Z}_2) \times \mathbb{Z}] \otimes \mathbb{C}[\mathbb{Z} \rtimes (\mathbb{Z}_2 \rtimes \mathbb{Z}_2)].$$

The dislocations along the $(1,0)$ direction are condensed, and the phonons with momentum in that direction are confined. Consequently the symmetry is restored in that direction, so that

$$\begin{aligned} \mathcal{T}_r &= F[(\mathbb{Z}^2 \rtimes \mathbb{Z}_6) \times \mathbb{Z}]/0 \times \mathbb{Z} \otimes \mathbb{C}[\mathbb{Z}^2 \rtimes (\mathbb{Z}_2 \rtimes \mathbb{Z}_2)], \\ \mathcal{U} &= F[(\mathbb{Z} \rtimes \mathbb{Z}_2) \times \mathbb{Z}] \otimes \mathbb{C}[(\mathbb{R} \times \mathbb{Z}) \rtimes (\mathbb{Z}_2 \rtimes \mathbb{Z}_2)]. \end{aligned} \tag{29}$$

This is a 2D smectic. Note that in a conventional classical smectic, the Goldstone modes corresponding to oscillations of the rods within a plane are massive (because of the analog of the Higgs phenomenon[17]). Determination of whether the modes are massive or not requires a dynamical analysis. What we do know from our topological analysis is that these Goldstone modes are not confined.

The mass term for the Goldstone modes in classical smectics arises because the internal symmetry of the rods is coupled to the external symmetries (the Lagrangian is only invariant under coupled internal and external rotations). In the 2D smectic we have uncovered now, there are no internal symmetries. However, the defect-antidefect pairs in our condensate behave like rods in a conventional smectic. Note that they are coupled to external symmetries, in fact they arise due to the breaking of external symmetries. Thus we also expect the Goldstone modes associated with the oscillation of these "rods" to be massive, and only the longitudinal oscillation of the planes to be massless. And so our analysis has naturally led us to the quantum smectic discussed by Zaanen and Nussinov.[23]

IV. CONCLUSIONS

In this paper we have applied the formalism of quantum double symmetry breaking to a number of physical phenomena linked to the formation defect condensates. We showed that both for cases with internal and external symmetries, nontrivial known and conjectured results can be recovered. It is especially fruitful to apply the method in cases where one is dealing with non-Abelian symmetries and consequently with non-Abelian defects, because in such situations many inequivalent condensates can be considered, which lead to effective theories with very different low energy degrees of freedom. It is also nice that the criterion for confinement determines to what extent a restoration of the original gauge symmetry takes place. The method enables one also to systematically analyze dyonic condensates, but in this paper we have restricted ourselves to pure defect condensates.

It is clear that there are many two-dimensional (quantum) systems that can be systematically studied from this general point of view. In a paper appearing in parallel we treat non-Abelian nematic crystals with nontrivial point group symmetries of the tetrahedral, octahedral and icosahedral type.[5] All systems in which conformal field theory plays a role, such as quantum Hall systems, or Bose-Einstein condensates, fall naturally in the category that should be analyzed. For example, it would be interesting to investigate whether the plateau hierarchy proposed by Haldane,[25] where the plateaus are

linked by the successive formation of quasiparticle condensates, could possibly fit in our scheme. It may also be interesting to investigate the conceivable phases of $(2+1)$-dimensional gravity, along these lines.

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APPENDIX: INDUCING REPRESENTATIONS OF $\mathbb{Z}^2 \rtimes D_6$

Here, we illustrate the derivation of an irrep of $\mathbb{Z}^2 \rtimes D_6$ discussed in the main text, using Mackey's induction theorem [see any good book on representation theory (Ref. 26)]. Take orbit number 5 in Fig. 3, with momentum $\vec{k}=\left(\frac{\pi}{2},\frac{\pi}{2}\right)$ [i.e., $\left(\frac{\pi}{2a},\frac{\pi}{2a}\right)$ with $a=1$]. A rotation of 180 degrees (r^3) gives an equivalent vector, i.e., a momentum that represents the same irrep of $\mathbb{Z}^2$. As a matter of fact, so do e, sr^2, sr^{-1}, so $H_{\Pi_{\vec{k}}}$ is isomorphic to $\mathbb{Z}^2 \rtimes \mathbb{Z}_2 = (r^3) \times (sr^2)$, where in general (g) is the group generated by g. It has four one-dimensional irreps. Now $[D_6 : \mathbb{Z}^2 \rtimes \mathbb{Z}_2]=3$, so we get four three-dimensional irreps. Denote an irrep of $\mathbb{Z}^2 \rtimes \mathbb{Z}_2$ by $\rho_{m,n}$, with $m,n \in \mathbb{Z}_2$. Explicitly, a basis of the irreps is given by a set of representatives of left $\mathbb{Z}^2 \rtimes \mathbb{Z}_2$ cosets, which we take to be $\{|e\rangle, |r\rangle, |r^2\rangle\}$ (the order is important when we write down the matrices explicitly). The matrices of the irrep are

$$(a,b) \mapsto \begin{pmatrix} e^{i(\pi/2)a}e^{i(\pi/2)b} & 0 & 0 \\ 0 & e^{i(\pi/2)(a+b)}e^{i(\pi/2)(-a)} & 0 \\ 0 & 0 & e^{i(\pi/2)b}e^{i(\pi/2)(-a-b)} \end{pmatrix},$$

$$r \mapsto \begin{pmatrix} 0 & 0 & e^{i\pi m} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad s \mapsto \begin{pmatrix} 0 & & e^{i\pi n} \\ 0 & e^{i\pi n} & 0 \\ e^{i\pi n} & 0 & 0 \end{pmatrix}.$$

As a representative example, let us illustrate the derivation of the second column of the matrix corresponding to s (the derivations of the other columns is analogous),

$$s|r\rangle = |sr\rangle = |r(sr^2)\rangle = |r\rangle e^{i\pi n}. \tag{A1}$$

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[12] Note that $\mathcal{R}$ involves a counterclockwise "half braid," completely moving around is achieved by acting with $\mathcal{R}^2$. Braiding of a defect and a representation results in just applying the group element corresponding to the defect on the given representation.

[13] In this paper we adopt a notation with a tensor product symbol "$\otimes$" between the magnetic and electric part of the double, to keep the notation unambiguous in the cases we are to investigate. The notation used in other papers we refer to is the ordinary product, but in this paper we must deal with electric and magnetic symmetries which themselves involve a number of products.

[14] C_A is a conjugacy class in the $D(H)$ case, and an orbit in H_m under the action of H_{el} in the $F(H_m) \otimes \mathbb{C}H_{el}$ case.

[15] We note that this condensate is in fact gauge invariant (under H transformations). This means that if we are discussing a local theory these condensates would also be acceptable ground states. For the case of global symmetries such restrictions are not necessary.

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[18] The inversions are not connected to the identity, therefore adding them does not influence $\Pi_1(G/H)$.

[19] There is a subtlety due to the infinity of the groups. Drawing on the analogy with generalized quantum doubles with finite groups, as a vector space this Hopf algebra is $F\{[(\mathbb{Z}^2 \rtimes \mathbb{Z}_6) \times \mathbb{Z}]\} \otimes \mathbb{C}(\mathbb{Z}^2 \rtimes D_6)$. However, to make the representation theory tractable we only take functions with compact support in the

second variable. One can check that this is an algebra (i.e., the finite sum and product of such elements also has compact support in the second variable). We do not take compact support in the first variable because of our analysis of defect condensates, to come. If we condense the dislocation $|(1,0)\rangle$, for example, then we need functions that are constant on left $(1,0)$ cosets, and these left cosets have an infinite number of elements.

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