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Supersymmetry and the geometry of Taub–NUT

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Abstract

The supersymmetric extension of Taub–NUT space admits four standard supersymmetries plus several additional non-standard ones. The geometrical origin of these symmetries is traced, and their algebraic structure is clarified. The result has applications to fermion modes in gravitational instantons as well as in long-range monopole dynamics.

The low-energy dynamics of spinning particles in a curved space with metric $g_{\mu\nu}(x)$ is described by the $d = 1$ supersymmetric σ-model [1–6]

$$L = \tfrac{1}{2} g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu + \tfrac{1}{2} i \eta_{ab} \psi^a \frac{D\psi^b}{D\tau}, \tag{1}$$

where η_{ab} is the flat metric of tangent space: $\eta_{ab} = \delta_{ab}$ for a Euclidean space, and $\mathrm{diag}(-,+,\ldots,+)$ for a Lorentzian space-time. The Grassmann variables ψ^a transform as a tangent space vector and describe the spin of the particle. More precisely, the antisymmetric tensor $S^{ab} = -i\psi^a\psi^b$ generates the internal part of the local tangent-space rotations; also, for charged particles its components are proportional to the tensor of electromagnetic dipole moments [7]. The classical equations of motion of the theory can be cast in the form

$$\frac{D^2 x^\mu}{D\tau^2} = \tfrac{1}{2} S^{ab} R_{ab}{}^{\mu}{}_{\nu} \dot{x}^\nu, \quad \frac{DS^{ab}}{D\tau} = 0. \tag{2}$$

All derivatives here are covariant with respect to general coordinate transformations and local tangent-space rotations. Eqs. (2) also hold in the quantum theory when interpreted as functional averages, and therefore have a classical meaning in the sense of Ehrenfest's theorem.

An interesting consequence of these equations is, that particle spin can be used as a probe of the geometry of space-time. Such information obtained from particle spin is complementary to that from geodesics defined by the worldlines of spinless test particles.

The constants of motion of a scalar particle in a curved space are determined by the symmetries of the manifold, and are expressible in terms of the Killing vectors and tensors. For a spinning particle a similar result holds, with two modifications [11,12]: first, the constants of motion related to a given Killing vector generally contain spin-dependent parts; second, there are Grassmann-odd constants of motion, related to supersymmetries, which do not have a counterpart in the scalar model, although they do have interesting geometrical origins. This is one of the ways in which spin can provide additional information about geometry.

In this Letter we study spinning particles in Taub–NUT space, which is a $D = 4$ self-dual Euclidean space. In a special choice of coordinates the metric takes the form

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$$\mathrm{d}s^2 = (1+2m/r)\left(\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\varphi^2\right) + \frac{4m^2}{1+2m/r}\left(\mathrm{d}\psi + \cos\theta\,\mathrm{d}\varphi\right)^2 . \tag{3}$$

The physical applications of the Lagrangian (1) with this metric are either to fermions in a gravitational instanton background for $m > 0$ [8], or to the long-range dynamics of interacting magnetic monopoles and their fermion modes [9,10] for $m < 0$ and $r > 2|m|$. However, from the analysis presented below it is clear that the supersymmetric extension also considerably clarifies the structure of the purely bosonic theory. As such the method and ideas seem to be of more general relevance. Some results on spinning Taub–NUT space of which we make use were obtained in Refs. [13–15].

For the purpose of describing symmetries and conservation laws, the covariant Hamiltonian formalism introduced in Refs. [12,16] is most useful. In this formalism the basic phase-space variables are (x^μ, Π_μ, ψ^a), with Π_μ the covariant rather than the canonical momentum:

$$\Pi_\mu = p_\mu - \tfrac{1}{2}\omega_{\mu ab}S^{ab}. \tag{4}$$

For spinless scalar particles, the two are identical. In terms of these variables the Hamiltonian takes the simple form

$$H = \tfrac{1}{2}g^{\mu\nu}\Pi_\mu\Pi_\nu. \tag{5}$$

In the covariant phase space formulation the Poisson–Dirac bracket of two scalar functions A, B is given by

$$\{A,B\} = \mathcal{D}_\mu A\frac{\partial B}{\partial \Pi_\mu} - \frac{\partial A}{\partial \Pi_\mu}\mathcal{D}_\mu B + R_{\mu\nu}\frac{\partial A}{\partial \Pi_\mu}\frac{\partial B}{\partial \Pi_\nu} + i(-1)^{a_A}\frac{\partial A}{\partial \psi^a}\frac{\partial B}{\partial \psi_a}, \tag{6}$$

with $R_{\mu\nu} = \frac{1}{2}S_{ab}R^{ab}_{\mu\nu}$ the spin-valued Riemann tensor, and

$$\mathcal{D}_\mu A = \partial_\mu A + \Gamma_{\mu\nu}{}^\lambda \Pi_\lambda \frac{\partial A}{\partial \Pi_\nu} + \omega_\mu{}^a{}_b\psi^b\frac{\partial A}{\partial \psi^a}. \tag{7}$$

Before applying the formalism to the model (3), we first observe that the elementary constants of motion – those linear in the momentum – for a scalar particle in Taub–NUT space consist of the 3-d total angular momentum $\boldsymbol{J}$ and a quantity which, for negative mass models, can be interpreted as the relative electric charge $q \equiv J_0$:

$$J_A = R_A \cdot \Pi, \tag{8}$$

where the R^μ_A with $A = (0,\ldots,3)$ and $\mu = (r,\theta,\varphi,\psi)$ are the components of the four corresponding Killing vectors:

$$\begin{aligned} R_0 &= (0,0,0,1)\,,\\ R_1 &= (0,-\sin\varphi,-\cot\theta\cos\varphi,\csc\theta\cos\varphi)\,,\\ R_2 &= (0,\cos\varphi,-\cot\theta\sin\varphi,\csc\theta\sin\varphi)\,,\\ R_3 &= (0,0,1,0)\,. \end{aligned} \tag{9}$$

Expressions for the extension of these constants of motion to spinning particles have been presented in Refs. [13,14]. A fast method to obtain them is by the following theorem:

Theorem. If on a manifold with metric $g_{\mu\nu}$ there exists a Killing vector R^μ, then the motion of a scalar particle on the manifold conserves the quantity

$$J = R^\mu\Pi_\mu, \tag{10}$$

and the motion of a spinning particle with Lagrangian (1.1) conserves the spin-dependent extension

$$\mathcal{J} = R^\mu\Pi_\mu + \tfrac{1}{2}B_{ab}S^{ab}, \tag{11}$$

where the base-space components of the antisymmetric tensor $B_{\mu\nu}$ are given in terms of the Killing vector by

$$B_{\mu\nu} = \tfrac{1}{2}\left(R_{\nu;\mu} - R_{\mu;\nu}\right). \tag{12}$$

All constants of motion of this form are superinvariant:

$$\{\mathcal{J}, Q\} = 0, \tag{13}$$

and the Lie algebra defined by the Killing vectors:

$$R^\mu_{A;\nu}R^\nu_B - R^\mu_{B;\nu}R^\nu_A = f_{AB}{}^C R^\mu_C, \tag{14}$$

is realized by the constants of motion $\mathcal{J}_A$ through the Poisson–Dirac brackets:

$$\{\mathcal{J}_A, \mathcal{J}_B\} = f_{AB}{}^C\mathcal{J}_C. \tag{15}$$

Note however, that in principle it is possible to add to the solution $B_{\mu\nu}$ in Eq. (12) an improvement term $\beta_{\mu\nu}$, provided it is covariantly constant:

$$\beta_{\mu\nu;\lambda} = 0. \tag{16}$$

If such a tensor exists, the quantity

$$\beta = \tfrac{1}{2}\beta_{ab}S^{ab} \tag{17}$$

is a constant of motion by itself; as a result there is some arbitrariness in the definition of $\mathcal{J}$, Eq. (11). In general, however, improvement terms will change the algebraic properties (13) and (15). Only specific choices of the improvement terms preserve the algebraic structure. The Taub–NUT geometry provides an explicit example of this phenomenon. In any case, without constraints on the algebra of Poisson–Dirac brackets the solutions of the generalized Killing equations, such as those found in Refs. [13,14], are not unique.

In addition to angular momentum and relative charge, the Taub–NUT geometry admits a constant of motion known as the Runge–Lenz vector [10,17], which is constructed out of a second-rank Killing tensor [18]. In the following we show that this constant of motion is closely related to a number of supersymmetries of the spinning Taub–NUT model.

The starting point of our analysis is the observation [12] that the spinning particle action defined by (1) or (5) has a conserved supercharge

$$Q_f = f^{\mu}{}_a \Pi_\mu \psi^a + \frac{i}{3!} c_{abc}\psi^a\psi^b\psi^c, \tag{18}$$

provided the tensors $(f^{\mu}{}_a, c_{abc})$ satisfy the differential constraints

$$D_\mu f_\nu{}^a + D_\nu f_\mu{}^a = 0,$$
$$D_\mu c_{abc} + R_{\mu\nu ab} f^\nu{}_c + R_{\mu\nu bc} f^\nu{}_a + R_{\mu\nu ca} f^\nu{}_b = 0. \tag{19}$$

The corresponding supersymmetry transformations of the bosonic coordinates are

$$\delta x^\mu = -i\epsilon f^{\mu}{}_a(x)\psi^a. \tag{20}$$

A particularly simple solution is provided by the vierbein field: $f^{\mu}{}_a = e^{\mu}{}_a$, $c_{abc} = 0$. In Eq. (18) this gives the standard supercharge $Q = \Pi \cdot \psi$, confirming that the theories defined by (1) always possess at least one supersymmetry.

However, depending on the geometry of the manifold, other solutions can exist. In particular, if the metric admits a tensor $f_{\mu\nu} = f_{\mu a} e_\nu{}^a$ of Killing–Yano type, it implies the presence of a new supersymmetry which anti-commutes with Q [12]. A tensor $f_{\mu\nu}$ is called Killing–Yano if it is anti-symmetric and its field strength satisfies

$$H_{\mu\nu\lambda} \equiv \tfrac{1}{3}\left(f_{\mu\nu;\lambda} + f_{\nu\lambda;\mu} + f_{\lambda\mu;\nu}\right) = f_{\mu\nu;\lambda}, \tag{21}$$

which follows in fact from the first eq.(19). Differentiation of $H_{\mu\nu\lambda}$ and use of the Ricci identity then shows, that $-2H_{abc}$ is a solution of the second equation (19) for c_{abc}. Hence the existence of a Killing–Yano tensor of the bosonic manifold is equivalent to the existence of a supersymmetry for the spinning particle with supercharge

$$Q_f = f^{\mu}{}_a \Pi_\mu \psi^a - \tfrac{1}{3} i H_{abc}\psi^a\psi^b\psi^c,$$
$$\{Q, Q_f\} = 0. \tag{22}$$

In the Taub–NUT geometry (3) four Killing–Yano tensors are known to exist [17]. Three of these, denoted by f_i, $i = (1,2,3)$, are special because they are covariantly constant; therefore they satisfy Eq. (21) trivially by having vanishing field strength. In two-form notation and in the coordinates (3) the explicit expressions for the f_i are

$$f_i = 4m\,(\mathrm{d}\psi + \cos\theta\,\mathrm{d}\varphi) \wedge \mathrm{d}x_i$$
$$- \varepsilon_{ijk}\,(1 + 2m/r)\,\mathrm{d}x_j \wedge \mathrm{d}x_k, \tag{23}$$

where the $\mathrm{d}x_i$ are standard expressions in terms of the three-dimensional spherical coordinates (r, θ, φ). The corresponding supercharges have the simple form

$$Q_i = f_i{}^{\mu}{}_a \Pi_\mu \psi^a, \tag{24}$$

and together with $Q_0 = Q$ they realize the $N = 4$ supersymmetry algebra

$$\{Q_A, Q_B\} = -2i\delta_{AB}H, \quad (A, B) = 0, \ldots, 3. \tag{25}$$

Indeed, the f_i define three anti-commuting complex structures on the Taub–NUT manifold, their components realizing the quaternion algebra

$$f_i f_j + f_j f_i = -2\delta_{ij}, \quad f_i f_j - f_j f_i = 2\varepsilon_{ijk} f_k. \tag{26}$$

Clearly, the existence of the Killing–Yano tensors is linked to the hyper-Kähler geometry of the manifold,

and our argument shows directly the relation between this geometry and the $N = 4$ supersymmetry of the theory. We can also give a *physical* interpretation of these particular Killing–Yano tensors: they correspond to components of the spin which are separately conserved. Namely, three constants of motion are obtained by rewriting the two-forms (23) in terms of the spin coordinates:

$$S_i = \tfrac{1}{4} i f_{iab} \psi^a \psi^b = -\tfrac{1}{4} f_{iab} S^{ab},$$
$$\{S_i, H\} = 0, \tag{27}$$

which realize an $SO(3)$ Lie-algebra:

$$\{S_i, S_j\} = \varepsilon_{ijk} S_k. \tag{28}$$

These constant two-forms transform as a vector under rotations generated by the total angular momentum $\mathcal{J}_i$. Obviously, they provide improvement terms of the type (17) for the components of total angular momentum. It follows, that we can define an improved form of the angular momentum

$$\mathcal{I}_i = \mathcal{J}_i - S_i, \tag{29}$$

with the property that it preserves the algebra:

$$\{\mathcal{I}_i, \mathcal{I}_j\} = \varepsilon_{ijk} \mathcal{I}_k, \tag{30}$$

and that it *commutes* with the $SO(3)$ algebra generated by the spins S_i. Equivalently, we can combine these two $SO(3)$ algebras to obtain the generators of a conserved $SO(4)$ symmetry among the constants of motion, a standard basis for which is spanned by $M_i^{\pm} = \mathcal{I}_i \pm S_i$. Under this $SO(4)$ the supercharges Q_A transform as a four-vector:

$$\{M_i^+, Q\} = 0, \quad \{M_i^-, Q\} = Q_i,$$
$$\{M_i^+, Q_j\} = \varepsilon_{ijk} Q_k, \quad \{M_i^-, Q_j\} = -\delta_{ij} Q, \tag{31}$$

Obviously these $SO(4)$ transformations preserve the $N = 4$ supersymmetry algebra (25). Note also, that the second equation above implies that the supercharges Q_i are Q-exact and hence trivially superinvariant. We conclude that the algebra of constants of motion $(M_i^{\pm}, Q_A, H)$ closes. It represents the universal part of the algebra of any model with $N = 4$ supersymmetry.

However, in the case of the Taub–NUT model, in addition to the four standard supersymmetries whose physical and geometrical origin has been clarified, the metric is known to possess a *fourth* Killing–Yano tensor, which is not trivial and leads to new constants of motion. Its components are contained in the two-form

$$Y = 4m\,(\,\mathrm{d}\psi + \cos\theta\,\mathrm{d}\varphi) \wedge \mathrm{d}r$$
$$+ 4r\,(r+m)\,(1 + r/2m)\,\sin\theta\,\mathrm{d}\theta \wedge \mathrm{d}\varphi. \tag{32}$$

The field strength has one independend non-vanishing component, given by

$$H_{r\theta\varphi} = 2\,(1 + r/2m)\,r\sin\theta. \tag{33}$$

The supercharge Q_Y constructed from Y is a scalar under rotations, but it is not invariant under the $SO(3)$ transformations generated by the S_i. Instead, by computing the brackets we identify another triplet of conserved supercharges

$$\Omega_i = \{M_i^-, Q_Y\} = -2\,\{S_i, Q_Y\}. \tag{34}$$

Although these supercharges provide solutions to eqs. (18), (19), they are not superinvariant, and therefore they do not imply the existence of any new Killing–Yano tensors. Of course, Q_Y itself is superinvariant by construction; but its brackets with the other supercharges Q_i do not vanish: they generate a triplet of new constants of motion

$$K_i = \{Q_Y, Q_i\}. \tag{35}$$

The explicit expressions for the components of K_i are constructed from (23), (32) and (33) according to the formulas given in Refs. [12,16]. The results can be summarized briefly as follows.

(i) The Killing tensors $K_{i\mu\nu}$ on the left-hand side of (34) are those first obtained in Ref. [18] and identified with a conserved vector of Runge–Lenz type for the monopole scattering problem in Ref. [17].

(ii) There is a contribution to K_i linear in the spin S^{ab}; however, there is no contribution quadratic in the spin, as observed in Ref. [15]. Comparing with the general expressions given in Ref. [12] this can be seen to follow from the fact that the complex structures f_i are covariantly constant and the self-duality of the Taub–NUT geometry, which together imply the identities

$$\epsilon^{abcd} R_{cd\mu\nu} f^{\mu}_{i\,a} f^{\nu}_{j\,b} = 0, \quad c_{iabc} = 0. \tag{36}$$

The Jacobi identities now imply, that the supercharges Ω_i are the lowest (fermionic) components of a superdoublet, the highest (bosonic) components of which form the Runge–Lenz vector, which is generated from them by a supersymmetry transformation:

$$K_i = -\{Q, \Omega_i\}. \tag{37}$$

This vector was used to obtain the spectrum of states of scalar monopole pairs in Ref. [19]. An analogous construction is now obviously possible for the scalar-fermion monopole pairs; this suggests the existence of a dynamical $SO(4,2)$ symmetry in which the new supercharges may find a natural interpretation..

The remaining brackets between supercharges give constants of motion which are polynomials in the ones already found. In particular, the bracket of Q_Y with itself gives

$$\{Q_Y, Q_Y\} = 2iH + i\left(\mathcal{J}^2 - \mathcal{J}_0^2\right), \tag{38}$$

where $\mathcal{J}$ and $\mathcal{J}_0$ are the total angular momentum and the relative electric charge of the monopoles (for $m < 0$) introduced before, Eqs. (9) and (11).

Finally we briefly discuss the quantum mechanics of the spinning particle in Taub–NUT space. Using the Lagrangian (1.1) in a path-integral, the symmetries investigated in this paper lead directly to relations between matrix elements in the coordinate representation. We can also investigate these relations at the level of operators by using the correspondence principle

$$\psi^a \to i\sqrt{\hbar/2}\gamma_5\gamma^a,$$
$$\Pi_\mu \to -i\hbar D_\mu = -i\hbar\left(\partial_\mu - \tfrac{1}{2}\omega_{\mu ab}\sigma^{ab}\right). \tag{39}$$

Then the supercharges correspond to generalizations of the Dirac operator:

$$Q_f \to i\sqrt{\hbar/2}\gamma_5 D_f$$
$$\equiv \frac{\hbar^{3/2}}{\sqrt{2}}\gamma_5\gamma^a\left(f^\mu{}_a D_\mu - \frac{1}{3!}c_{abc}\sigma^{bc}\right), \tag{40}$$

whilst their anticommutators, representing the operators for H and K_i, are given by the corresponding Laplacians. The $N = 4$ supersymmetry now implies the existence of 4 Dirac operators D_A with the property

$$\{D_A, D_B\} = -2\hbar^2\delta_{AB}\,\Box^{\rm cov}. \tag{41}$$

At least two operators from the set $(D_A, \gamma_5 D_A)$ can always be diagonalized simultaneously. However, a zero-mode of any of these Dirac operators is a zero-mode of the covariant Laplacian and vice versa, therefore the kernels of all four operators D_A coincide.

If a spinor transforms non-trivially under the (global) $SO(4)$, more possibilities arise. For example, a four-vector of spinors Ψ_A might satisfy one or both of the $SO(4)$-invariant conditions

$$D\cdot\Psi = 0, \quad D_A\Psi_B - D_B\Psi_A = 0. \tag{42}$$

If both conditions are satisfied, the Ψ_A are zero-modes of the Laplacian and hence of all Dirac operators D_A simultaneously. However, if the Ψ_A are non-zero modes of the Laplacian (finite mass particles), then they realize a partially broken $N = 4$ supersymmetry. Clearly the structure of the space of solutions may become quite intricate.

Finally, the zero-modes of the fifth Dirac-type operator D_Y coincide with those of the other Dirac operators if and only if they are extremal in the sense that the square of their relative charge equals the total angular momentum:

$$\mathcal{J}^2\Psi = \mathcal{J}_0^2\Psi, \tag{43}$$

or $j(j+1) - q^2 = 0$, where j is the quantum number of total angular momentum, and q that of the relative electric charge. If not, the zero-modes of the Laplacian correspond to doublets of the finite eigenvalue equations

$$D_Y\Psi = \lambda\Phi, \quad D_Y\Phi = -\lambda\Psi, \tag{44}$$

for $j(j+1) - q^2 > 0$. For $j(j+1) - q^2 < 0$ there are no normalizable solutions of eqs.(44) with real λ. These bounds correspond to those found to distinguish between monopole bound states and scattering states in Ref. [10].

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