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Filename	Appendix A: Some frequently used estimates

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Appendix A

Some frequently used estimates

We use the following well-known inequalities [Juk11].

$$
\begin{aligned}
e^t &\geq 1+t \\
(1+x)^n &\geq 1+xn && (n \in \mathbb{Z}, x \in \mathbb{R}, x \geq 1) \\
n! &= \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1+\Theta\left(\frac{1}{n}\right)\right) && \text{(Stirling's formula)} \\
\frac{2^n}{\sqrt{n}}\left(\sqrt{\frac{2}{\pi}} - O\left(\frac{1}{n}\right)\right) &\leq \binom{n}{\lfloor n/2 \rfloor} \leq \frac{2^n}{\sqrt{n}} \cdot \sqrt{\frac{2}{\pi}} \\
\binom{n}{k} &= \sqrt{\frac{n}{2\pi k(n-k)}}\left(\frac{n}{k}\right)^k \left(\frac{n}{n-k}\right)^{n-k} (1-O(k^{-1})) \\
& && (1 \leq k \leq n/2) \\
\left(\frac{n}{k}\right)^k &\leq \binom{n}{k} \leq \binom{n}{\leq k} \leq \left(\frac{en}{k}\right)^k && (1 \leq k) \\
\frac{e^{-k^2/n-1/(6k)}}{\sqrt{2\pi k}}\left(\frac{en}{k}\right)^k &\leq \binom{n}{k} && (1 \leq k \leq n/2) \\
\binom{n}{\leq k} &\leq \binom{n}{k}\left(1+\frac{k}{n-2k+1}\right) && (1 \leq k \leq n/2) \\
2^{n \cdot H(k/n) - O(\log n)} &\leq \binom{n}{\leq k} \leq 2^{n \cdot H(k/n)} && (1 \leq k \leq n/2) \\
\binom{n}{\alpha n} &= \frac{1+o(1)}{\sqrt{2\pi\alpha(1-\alpha)n}} \cdot 2^{n \cdot H(\alpha)} && (\alpha n \in \mathbb{N}, 0 < \alpha < 1)
\end{aligned}
$$

Here $H(x)$ stands for Shannon's entropy function:

$$H(x) = x \log\left(\frac{1}{x}\right) + (1-x)\log\left(\frac{1}{1-x}\right)$$

for $x \in (0,1)$, and $H(0) = H(1) = 0$.

Chernoff bounds

Let $X_1, \ldots, X_n$ be $[0,1]$-valued independent random variables with $\mathbb{E}[X_i] = p_i$. Define $X \triangleq \sum_{i=1}^n X_i$ and $p \triangleq \frac{1}{n}\mathbb{E}[X] = \frac{1}{n}\sum p_i$. Then for any $\delta > 0$ we have the following inequalities:

Multiplicative bounds

$$\begin{aligned}
\Pr[X \geq (1+\delta)pn] &\leq \left(\frac{e^\delta}{(1+\delta)^{1+\delta}}\right)^{pn} \\
&\leq e^{-pn\delta^2/3} && \text{if } \delta \leq 1 \\
\Pr[X \leq (1-\delta)pn] &\leq \left(\frac{e^\delta}{(1+\delta)^{1+\delta}}\right)^{pn} \leq e^{-pn\delta^2/2} \\
\Pr[|X - pn| \geq \delta pn] &\leq 2e^{-pn\delta^2/3} && \text{if } \delta \leq 1
\end{aligned}$$

Additive bounds

$$\begin{aligned}
\Pr[X \geq n(p+\delta)] &\leq e^{-2n\delta^2} \\
\Pr[X \leq n(p-\delta)] &\leq e^{-2n\delta^2}
\end{aligned}$$

The difference between the multiplicative and the additive forms of the bounds is a ratio of $O(p)$ in the exponent; multiplicative bounds are better for $p = o(1)$, and almost the same for constant p.

If we know $\mathbb{E}[X_i] \leq p_i$ instead, then the bound for $\Pr[X \geq (1+\delta)pn]$ continues to hold; similarly in the opposite direction. See [DP09, MU05, MR95] for in-depth treatments of concentration inequalities and their use in the analysis of randomized algorithms.