Does Eductive Stability imply Evolutionary Stability?

Note on R. Guesnerie,

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Abstract

This note presents a simple example of a model in which the unique rational expectations (RE) steady state equilibrium is eductively stable in the sense of Guesnerie (2002), but where evolutionary learning, as introduced in Brock and Hommes (1997), does not necessarily converge to the RE steady state price. The example is a Muthian cobweb model where producers have heterogeneous expectations and select forecasting strategies based upon recent realized profits. By means of a simple three types example we show that a locally stable RE fundamental steady state may co-exists with a locally stable two–cycle. We also study the Muthian model with a large number of different producer types, and investigate conditions under which an evolutionary adaptive learning process based upon recent realized profits enforces global convergence to the stable RE steady state and when persistent periodic price fluctuations can arise.

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1 Introduction

In a recent paper, Guesnerie (2002) presents a unified methodology to investigate expectational coordination. In particular, he discusses the concept of *strong rationality* of equilibria and how common knowledge (CK) may enforce expectational coordination on such equilibria. He argues that if a rational expectations (RE) equilibrium E^* is *not* locally strongly rational (LSR), after a small perturbation the equilibrium cannot be self-enforcing. We quote Guesnerie (2002, pp.446-447):

> *... the assertion: "it is CK that the equilibrium is exactly E^*" is always consistent. But a small "hesitation" or "perturbation" may transform the assertion into: "it is CK that the actual state is very close to (in a small neighbourhood of) E^*." Whenever the considered equilibrium is not LSR, the just made "trembling assertion" cannot be self enforcing, whatever the associated "tremble" neighbourhood –at least if it is of nonempty interior. In a sense, the assertion is inconsistent. On theoretical grounds, such an inconsistency seems to be a particularly undesirable property of an equilibrium prediction claiming to be grounded in "rationality".*

Guesnerie (1993), following the game theoretical terminology of Binmore (1987), called this type of reasoning an *eductive* justification of RE. Strong rationality or *eductive stability* may be seen as a necessary condition for an equilibrium to qualify as "rational".

An important related question is whether evolutive learning processes based upon past observations and past experience necessarily converge to a RE equilibrium. Guesnerie (2002) presents results that, for a certain class of models, eductive stability of a steady state RE implies that (Guesnerie (2002, p. 473):

> *"reasonable" adaptive learning processes are asymptotically stable."*

This class of learning processes includes for example frequently used adaptive learning schemes such as ordinary least squares (OLS) learning. See Evans and Honkapohja (2001) for a recent and extensive treatment of adaptive learning in economics; see also Milgrom and Roberts (1990), who made a general case for the correspondence between eductive stability and evolutive stability when learning has the "best response" property.

In this note, we address the following question: *does eductive stability always imply evolutive stability?* In fact, we show by means of simple examples that the answer is negative and that eductive stability is not a sufficient condition for adaptive learning processes to converge. In particular, we show that in a Muthian cobweb model which is eductively globally stable, the evolutionary learning processes, as proposed by Brock and Hommes (1997), need not converge to the RE steady state. In these adaptive evolutionary systems, the common knowledge of rationality ad infinitum assumption is not satisfied. Instead,

agents are boundedly rational and choose simple strategies according to their performance in the recent past, as measured by realized profits.

All our examples are Muthian cobweb models as discussed by Guesnerie (2002). Our first and simplest example of a RE steady state which is eductively stable but not evolutionary stable occurs when producers can choose between three strategies: optimistic, fundamentalist and pessimistic. Optimists predict prices to be at a certain level above their steady state value, pessimists predict them to be at another level below the steady state, whereas fundamentalists predict exactly the steady state value. Producers choose the forecasting strategy adaptively, according to its past realized profits.

Although this example is eductively stable in the sense of Guesnerie, the adaptive evolutionary learning process of Brock and Hommes may lock into a stable two–cycle. The intuition behind this example is quite simple. When the price is above its steady state value and optimists dominate the market, next period's realized market price will be below the steady state value. Pessimists will have earned higher profits than optimists. If the initial price is far enough away from the steady state, pessimists' earnings will be higher even than earnings of fundamentalists. In the adaptive evolutionary model, most producers will then switch to the pessimistic strategy, causing the next realized market price to be above the steady state value, with optimists earning highest profits, and the story repeats. When the intensity of choice for strategy selection is high, that is, when producers are sensitive to small differences in evolutionary fitness, the majority of traders switches quickly between optimistic and pessimistic beliefs, and prices will lock into a stable two–cycle at some positive distance from the fundamental steady state.

Notice that in this three type evolutionary model the fundamental steady state may be locally stable: with an initial price close enough to the fundamental, fundamentalists earn higher profits than optimists and pessimists. More producers will switch to the fundamentalists' strategy, enforcing prices to converge to the (locally stable) steady state. This implies that, although the model is globally eductively stable, the evolutionary system allows for two different long run outcomes: a steady state or a two–cycle.

In this example, eductive stability fails to ensure evolutionary stability of the system because of the following: though the strategies of both optimists and pessimists are non–rational in terms of common knowledge rationality, they represent the best available choice of the agents. It therefore seems to be the case that the number and distribution of available strategies is important; this leads to the question whether the co–existence of different stable evolutions is robust when the number of strategy types increases. For example, in the three type case discussed above adding more types might cause the amplitude of the two–cycle to become smaller, or even cause the two–cycle to disappear.

Brock, Hommes and Wagener (2002) recently introduced the notion of large type limit (LTL) to study evolutionary heterogeneous market systems with *many* different strategy types drawn from a certain initial distribution at the start of the economy. In this note, the LTL–framework will be applied to the Muthian cobweb model to study how the number of strategy types influences the global stability of the rational expectations equilibrium.

It turns out that decisive roles are played by the initial distribution of strategies as well as the intensity of choice.

More precisely, we show that two conditions have to be met. The first of these conditions requires the set of admissible strategies to be an open interval, and the initial density of strategy types to be strictly positive on the set of admissible strategies. The second requirement is that the intensity of choice to switch strategies is sufficiently large. Under these conditions, an evolutionary Muthian model with many producer types is likely to be globally stable, and eductive stability implies evolutive stability. However, we also show that if either of these conditions is violated, then there may be stable two–cycles coexisting with the stable rational expectations equilibrium.

This note is organised as follows. Section 2 recalls the concept of strong rationality or eductive stability as discussed in Guesnerie (2002). Section 3 presents the Muthian cobweb model with heterogeneous beliefs and evolutionary learning. In section 4 an example with three strategies is analysed and a numerical example with five strategies is presented. Evolutionary systems with many different strategies are studied in section 5, and sufficient conditions for global evolutive stability as well as examples with co-existing stable two–cycles are given. Section 6 concludes and all proofs are contained in the appendix.

2 Eductive stability

Guesnerie (2002) summarises the principles of his approach to expectational coordination as follows:

1. pick a rational expectations equilibrium and call it E^*;
2. introduce a Common Knowledge (CK) restriction that places an exogenous bound on the state space and that describes restricted, but CK, beliefs of the agents on the possible states of the system;
3. analyse the consequences of the combination of the CK assumption and the CK restriction on the states of the system.

In his paper, he illustrates this methodology with a simple Muthian cobweb model. Since our subsequent analysis is based on this model, we briefly sketch it.

Economic agents are farmers producing a certain crop. Based on his expectation $E(p)$ of the price p for next period's crop, farmer i chooses a supply level $S(E(\tilde{p}, i))$ that maximises his expected profit. The aggregate supply $\mathcal{S}(p)$ of the economy is then given by $\int S(p, i)\,\mathrm{d}i$; the corresponding aggregate consumer demand is denoted $\mathcal{D}(p)$. As usual, we will assume that D is decreasing and $\mathcal{S}$ is increasing.

In the absence of noise there is a unique rational expectations price E^*, which is equal to the perfect foresight equilibrium price p^*, and which satisfies

$$\int S(p^*, i)\,\mathrm{d}i = \mathcal{S}(p^*) = \mathcal{D}(p^*).$$

It is assumed to be common knowledge amongst the farmers that the price of the crop (which is here the state variable) has to take values in a neighbourhood $V(E^*) = [E^* - \varepsilon, E^* + \varepsilon]$ of the rational expectations equilibrium. Note that $\varepsilon > 0$ is not assumed to be small.

Farmer i expects that next period's price will be in $V(E^*)$. Moreover, he assumes that the other farmers have the same expectation; he infers that each of them will play a best response $S(p, j)$ to some price p in $V(E^*)$. The aggregate supply will then be within $[\mathcal{S}(p^* - \varepsilon), \mathcal{S}(p^* + \varepsilon)]$, and knowing the demand function, he obtains a set

$$\Gamma(V(E^*)) = \left[\mathcal{D}^{-1}(\mathcal{S}(p^* + \epsilon)), \mathcal{D}^{-1}(\mathcal{S}(p^* - \epsilon))\right]$$

of possible prices. If $\Gamma(V(E^*))$ is strictly contained in $V(E^*)$ this reasoning can be iterated, on the assumption that all other farmers reason in the same manner. Using in this way the knowledge of rationality, the knowledge of the knowledge of rationality and so on, farmer i constructs smaller and smaller intervals $\Gamma^n(V(E^*))$ of possible prices. If these intervals converge to E^*, then the rational expectations equilibrium E^* is called *strongly rational* or *eductively stable* with respect to the restriction $V(E^*)$.

Guesnerie (2002) shows that for the Muthian farmer model the rational expectations equilibrium p^* is (locally) eductively stable if the familiar cobweb stability condition $\mathcal{S}'(p^*)/\mathcal{D}'(p^*) > -1$ is satisfied. If in addition demand and supply are linear, say $\mathcal{S}(p) = sp$ and $\mathcal{D}(p) = A - dp$, with $(0 <) < s/d < 1$, then the p^* is even globally eductively stable.

3 The Muth model with evolutionary learning

In this section we recall the Muth farmer model with heterogeneous beliefs and evolutionary learning, as introduced in Brock and Hommes (1997).

Producers are expected profit maximisers; they solve

$$\mathrm{Max}_q \;\; \Pi = \mathrm{Max}_q \;\; p^e q - c(q), \tag{1}$$

where p^e is the expected price and $c(q)$ are costs from producing quantity q. Assuming a quadratic cost function $c(q) = q^2/(2s)$, the first order condition $\Pi' = 0$ yields the linear supply curve

$$S(p^e) = sp^e. \tag{2}$$

Consumer demand is assumed to be linearly decreasing in the market price, that is

$$D(p) = A - dp. \tag{3}$$

The rational expectations steady state price p^*, for which demand equals supply, is given by

$$p^* = \frac{A}{s+d}. \tag{4}$$

We will refer to p^* as the *fundamental* price.

Producers can choose from H different forecasting rules. Let n_{ht} denote the fraction of producers using rule h at date t; this rule gives the forecast p^e_{ht} for p_t. Note that every forecast rule is equivalent to a production strategy. Heterogeneous market equilibrium is given by

$$D(p_t) = \sum_{h=1}^{H} n_{ht} S(p^e_{ht}). \tag{5}$$

The market equilibrium equation (5) represents the first part of the model. We now turn to the evolutionary part of the model describing how the fractions of the different producer types change over time. The basic idea of evolution is that fractions are updated according to past performance, given by realized profits in the recent past. In period t producer type h realises the net profit

$$\begin{aligned} \pi_{ht} &= p_t S(p^e_{ht}) - c(S(p^e_{ht})) = p_t s p^e_{ht} - \frac{(s p^e_{ht})^2}{2s} \\ &= \frac{s}{2} p^e_{ht}(2p_t - p^e_{ht}). \end{aligned} \tag{6}$$

A natural choice for the fitness or performance measure is a weighted sum of realized profits

$$U_{ht} = \pi_{ht} + w U_{h,t-1}, \tag{7}$$

where the weight parameter w measures the memory strength. According to this fitness measure, realized profits further in the past contribute with exponentially declining weights. In the case of infinite memory, $w = 1$, fitness equals accumulated wealth. In this note, to keep the model tractable, we focus on the other extreme case $w = 0$, with fitness equal to the most recently realized net profit.

The fractions n_{ht} of belief types are updated according to a discrete choice model:

$$n_{ht} = \exp(\beta U_{h,t-1}) \ / \ Z_{t-1}, \tag{8}$$

where $Z_{t-1} = \sum_{h=1}^{H} \exp(\beta U_{h,t-1})$ is a normalisation factor. This evolutionary mechanism for prediction rules has been proposed by Brock and Hommes (1997). It can be derived from a random utility model, where the fitness of all strategies is publically known, but subject to noise or error. If the noise terms are IID across agents and across types and

drawn from a double exponential distribution, as the number of agents tends to infinity, the probability of selecting strategy h converges to the discrete choice fraction n_{ht}. The parameter β is called the intensity of choice. It is inversely proportional to the noise level and $(1/\beta)$ is sometimes referred to as the propensity to err. In the extreme case $\beta = 0$, corresponding to noise of infinite variance, agents do not switch strategies at all and all fractions are fixed and equal to $1/H$. The other extreme case $\beta = \infty$ corresponds to the case of no noise, where agents do not make errors and all agents use the best predictor each period. Of course, in this evolutionary setting the 'best' predictor may change over time.

The beliefs of the producers are assumed to be of the form

$$p^e_{ht} = p^* + b_h, \tag{9}$$

where b_h does not depend on t or on past realised prices. The forecasting rule (9) reflects type h's "model of the market", that is, the belief of strategy type h how prices will deviate from the fundamental price; note that this fundamental is not necessarily known to the agents. Types with positive bias are called optimistic, those with negative bias pessimistic.

We rewrite the model in terms of the *deviation* x_t from the fundamental price, given by

$$x_t = p_t - p^*. \tag{10}$$

The market equilibrium equation (5) takes then the form

$$x_t = -\frac{s}{d}\sum_{h=1}^{H} n_{ht} b_h. \tag{11}$$

The evolutionary fitness measure $U_{ht} = \pi_{ht}$ can also be written in deviations form. Indeed, observe that the discrete choice fractions n_{ht} are independent of the profit level, that is, they remain the same when subtracting the same term from all profits π_{ht}. In particular, subtracting the profit $\pi_{Rt} = (s/2)p_t^2$ that would be earned by a rational agent, that is an agent with perfect foresight, yields

$$\begin{aligned} \pi_{ht} - \pi_{Rt} &= \frac{s}{2}p^e_{ht}(2p_t - p^e_{ht}) - \frac{s}{2}p_t^2 = -\frac{s}{2}(p_t - p^e_{ht})^2 \\ &= -\frac{s}{2}(x_t - b_h)^2. \end{aligned} \tag{12}$$

Hence, for this model fitness as measured by the most recent realized profits is equal to negative squared forecast errors, up to a constant factor. The discrete choice probabilities based upon last period's realized profit reduce to

$$n_{ht} = \frac{\exp(\beta U_{h,t-1})}{Z_{t-1}} = \frac{\exp(-\frac{\beta s}{2}(x_{t-1} - b_h)^2)}{Z_{t-1}}, \tag{13}$$

where as before $Z_{t-1} = \sum_{h=1}^{H} \exp(-\frac{\beta s}{2}(x_{t-1} - b_h)^2)$ is a normalisation factor. The Muthian model with evolutionary learning and H competing forecasting rules $f_h = p^* + b_h$, written in deviations from the RE fundamental price, is thus given by (11) and (13). Substituting (13) into (11) yields

$$x_t = \left(-\frac{s}{d}\right) \sum_{h=1}^{H} n_{ht} b_h = \left(-\frac{s}{d}\right) \frac{\sum_{h=1}^{H} b_h \, \mathrm{e}^{-\frac{\beta s}{2}(x_{t-1}-b_h)^2}}{\sum_{h=1}^{H} \mathrm{e}^{-\frac{\beta s}{2}(x_{t-1}-b_h)^2}} = g_H(x_{t-1}). \tag{14}$$

The evolutionary dynamics with H belief types b_h is thus described by a one-dimensional map g_H. This dynamics depends upon the number of types H, the initial distribution of types b_h, the market (in)stability ratio s/d and the intensity of choice β, measuring the sensitivity to differences in evolutionary fitness.

4 Examples with few belief types

Brock and Hommes (1997) considered a Muthian cobweb model where producers could choose between a cheap, simple forecasting rule and a more sophisticated but costly forecasting rule. In particular, they considered a two type example where producers either use a freely available naive forecasting rule or the rational expectations forecasting rule at positive per period information costs. They showed that if the cobweb dynamics are *unstable* under naive expectations, then increasing the intensity of choice also destabilises the evolutionary learning model, and chaotic price fluctuations may arise with producers switching between the simple, cheap, destabilising naive forecasting strategy and the sophisticated, costly and stabilising rational strategy. However, if the Muthian model is *stable* under the naive rule, evolutionary learning enforces prices to convergence to the RE fundamental price. Hence, if the Muthian model is eductively stable in the sense of Guesnerie, the Brock and Hommes (1997) two type example of naive expectations versus costly rational expectations will be (globally) stable under evolutionary learning.

From now on we restrict our attention to the case that the Muthian model is stable under naive expectations, that is, the case that the slopes of demand and supply satisfy the familiar 'cobweb theorem'(Ezekiel (1938)):

$$(0 <)\frac{s}{d} < 1. \tag{15}$$

Under this assumption the Muthian model is eductively stable in the sense of Guesnerie (2002). In this section, we show that for three producer types evolutionary learning does not always converge to the RE fundamental steady state price, but may "lock into" a stable two-cycle.

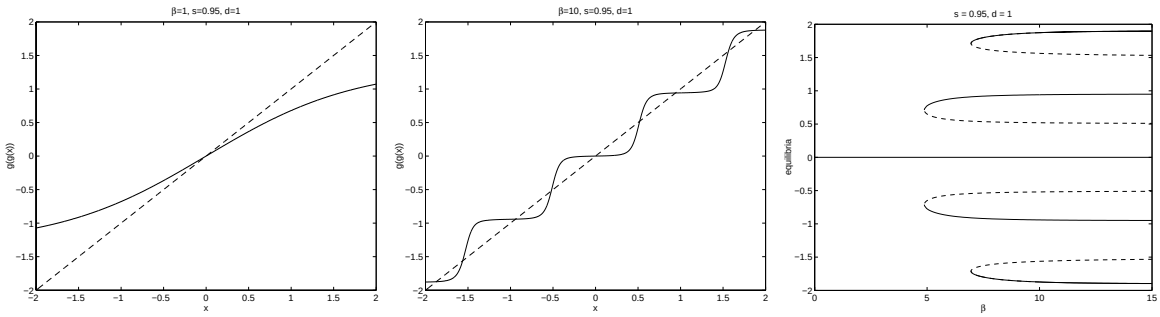

Figure 1: *Dynamics of a Muthian cobweb model under evolutionary dynamics with five strategy types. Shown is the graph of the second iterate $g(g(x_{t-1}))$; intersections (other than at the origin) of this graph with the diagonal are period two cycles. Left: low intensity of choice; middle: high intensity of choice. The rightmost graph shows the (β, x)–bifurcation diagram: solid lines indicate branches of stable equilibria, dashed lines unstable equilibria.*

Let $b > 0$ be a given positive bias. Producers can choose from three different forecasting rules:

$$p^e_{1t} = p^*, \tag{16}$$
$$p^e_{2t} = p^* + b, \tag{17}$$
$$p^e_{3t} = p^* - b, \tag{18}$$

Type 1 are fundamentalists, believing that prices will always be at their fundamental value (or equivalently, expected deviations x^e_{ht} from the fundamental will always be zero). Type 2 are optimists, expecting that the price of the good will always be above the fundamental price, whereas type 3 agents are pessimists, always expecting prices below the fundamental price. Notice that this example is symmetric in the sense that the optimistic and the pessimistic strategy are exactly balanced around the fundamental price, but this is not essential in what follows. We have the following:

Theorem A. Let $0.5 < \dfrac{s}{d} < 1$. The Muthian model with evolutionary learning given by (11) and (13), with three producer types given by (16-18) has the following properties:

(i) for all $\beta > 0$, the fundamental steady state is the unique steady state and it is locally stable;

(ii) for β sufficiently large, there exists a locally stable two–cycle;

(iii) for $\beta = \infty$ the locally stable two–cycle is given by $\{x_1, x_2\} = \{-bs/d, +bs/d\}$, with corresponding fractions of optimists and pessimists switching from 0 to 1 along the two–cycle.

The proof of the theorem is given in the appendix. In fact, a similar result holds for evolutionary systems with more than three belief types; these can be chosen in such a way that multiple stable two–cycles co-exist. Figure 1 illustrates this for an evolutionary Muthian model with *five* belief types $b_h \in \{-2, -1, 0, +1, +2\}$, where b_h is the belief concerning the deviation from the fundamental steady state as before. In this example, for small intensity of choice, $0 < \beta < \beta_1 \approx 5$ the fundamental steady state is globally stable (Figure 1, left plot). As β increases two stable two–cycles are created by saddle-node bifurcations, the first for $\beta = \beta_1 \approx 5$ and the second one for $\beta = \beta_2 \approx 7$ (Figure 1, right plot). For $\beta > \beta_2$ two (locally) stable two–cycles co-exist with the locally stable fundamental steady states, separated by two unstable two–cycles (Figure 1, middle plot). Obviously, since a stable steady state and a stable two–cycle are structurally stable, the results also hold for slightly asymmetric cases.

5 Evolutionary dynamics with many belief types

In the previous section we have seen that, in general, for the Muthian model global eductive stability does *not* imply evolutionary stability. In the simple three type evolutionary learning example considered, price fluctuations may settle down to a stable two–cycle with supplier types switching between optimistic and pessimistic strategies. As indicated in the introduction, this behaviour might be the consequence of the fact that there are not "enough" strategies available.

In this section we therefore analyse evolutionary dynamics in the Muthian model with *many* strategy types. Recall that the evolutionary dynamics with H belief types b_h is given by (14). At the beginning of the market, at date 0, a large number H stochastic belief parameters $b = b_h \in \mathbb{R}$ are drawn from a common initial distribution with density function $\varphi(b)$. Thereafter these H beliefs compete against each other according to the evolutionary dynamics specified in (14).

Evolutionary systems with many different belief types are difficult to handle analytically. Brock, Hommes and Wagener (2002) have recently introduced the notion of *large type limit (LTL)* to approximate evolutionary systems with many belief types. The LTL of the Muth model can be derived as follows. Divide both numerator and denominator of (14) by H to get

$$x_t = \left(-\frac{s}{d}\right) \frac{\dfrac{1}{H}\displaystyle\sum_{h=1}^{H} b_h \, \mathrm{e}^{-\frac{\beta s}{2}(x_{t-1}-b_h)^2}}{\dfrac{1}{H}\displaystyle\sum_{h=1}^{H} \mathrm{e}^{-\frac{\beta s}{2}(x_{t-1}-b_h)^2}}. \tag{19}$$

The LTL is obtained by replacing sample averages by sample means in both the numerator

and the denominator, yielding

$$x_t = \left(-\frac{s}{d}\right) \frac{\displaystyle\int b\, \mathrm{e}^{-\frac{\beta s}{2}(x_{t-1}-b)^2} \varphi(b)\, \mathrm{d}b}{\displaystyle\int \mathrm{e}^{-\frac{\beta s}{2}(x_{t-1}-b)^2} \varphi(b)\, \mathrm{d}b} = g(x_{t-1}). \tag{20}$$

Applying a uniform law of large numbers, Brock, Hommes and Wagener (2002) have shown that if the number of strategies H is sufficiently large, the LTL dynamical system is a good approximation of the evolutionary dynamical system with H belief types. More precisely, they proved that as the number of strategies H tends to infinity, the H-type map g_H (given by (14)) and its derivatives converge almost surely to the LTL-map g (given by (20)) and its derivatives, respectively. An important corollary of the LTL-theorem is that all generic and persistent dynamical properties (such as steady states, periodic cycles, local stability of steady states and periodic points, bifurcations, and even chaos and strange attractors) of the LTL also occur with probability arbitrarily close to one in the system with H belief types if the number of types H is sufficiently large. The evolutionary dynamics with many trader types can thus be studied using the LTL-dynamics (20). The next lemma describes the LTL map g for large intensity of choice:

Lemma. Let J be the interior of the support of the distribution φ from which the belief parameters b_h are drawn at date 0, that is, $J = \mathrm{int}\{b \,|\, \varphi(b) \geq 0\}$.

For all $x \in J$:

$$\lim_{\beta\to\infty} g(x) = \left(-\frac{s}{d}\right) x \quad \text{and} \quad \lim_{\beta\to\infty} g'(x) = -\frac{s}{d},$$

uniformly on all compact subsets K of J.

This lemma states that, in the interior of the support of the distribution φ from which the belief parameters b_h are drawn, the LTL-map and its derivative approach a linear map with slope $-s/d$ as β becomes large. The following theorem is an immediate consequence of this lemma:

Theorem B. (Global stability of evolutionary systems with many trader types.)
Let φ be a strictly positive probability density function, that is, let $\varphi(b) > 0$ for all b and $\int \varphi(b)\, \mathrm{d}b = 1$. Let the belief parameters $b = b_h \in \mathbb{R}$ be stochastically drawn at date 0 from a common distribution with probability density $\varphi(b)$. For β sufficiently large we have the following:

B1 if $-1 < -s/d < 0$, then the fundamental steady state $x^* = 0$ is the globally stable fixed point of the LTL (20);

B2 if $-s/d < -1$, then the fundamental steady state $x^* = 0$ is (globally) unstable and all solution orbits of the LTL (20) (except for the steady state) oscillate or diverge.

Theorem B is a statement about stability of the LTL-system, but immediate consequences for evolutionary systems with H trader types when H is large are obtained by applying the LTL theorem of Brock, Hommes and Wagener (2002) mentioned above. Let us discuss its implications in the case B1: if there are sufficiently many strategies (that is, if H is large), if the probability of finding a strategy in any open interval is positive ($\varphi(b) > 0$), and if the intensity of choice β to switch strategies is large enough, then the fundamental steady state is globally stable with a probability that goes to 1 as H increases beyond all bounds. In that case, eductive stability implies global stability in a heterogeneous market with evolutionary learning.

Recall that the Muthian model is eductively stable if and only if $-1 < -s/d < 0$. Theorem B therefore essentially states that, if every strategy types has a positive probability density of occurring, and if agents are highly sensitive to differences in evolutionary fitness, then eductive stability and evolutionary stability are equivalent. Since the intensity of choice β is the inverse of the propensity to err, we can also reformulate this by stating that eductive and evolutionary stability are equivalent if all types occur with positive probability density and if agents only make small errors in evaluating evolutionary fitness.

Many trader types examples with two–cycles

Our final theorem shows that both conditions in theorem B ($\varphi(b) > 0$ for all b and β sufficiently large) are also necessary for equivalence of eductive and evolutionary stability. For the statement of the theorem, recall that a proper locally stable two–cycle is a point $x_* \in \mathbb{R}$ such that $g(g(x_*)) = x_*$ and $g(x_*) \neq x_*$.

Theorem C. If either $\varphi(b)$ fails to be strictly positive for all $b \in \mathbb{R}$, or if φ is chosen depending on β, then there are examples of LTL's with at least one (locally) stable proper two-cycle. More concretely:

C1 There is an open and bounded set $D \subset \mathbb{R}$, containing 0, and a density function φ, strictly positive on D and zero in the complement of D, such that the following holds.

There is an open set $P \subset [1/2, 1]$ such that if $s/d \in P$, and if β sufficiently large, then the LTL (20) has a (locally) stable proper two–cycle $x_* \neq 0$.

C2 For every $n \geq 1$, for any sufficiently large, fixed β, and for $s/d < 1$ and sufficiently close to 1, there exists a strictly positive density function φ_β, such that the corresponding LTL (20) has n locally stable proper two–cycles.

Theorem C implies that, even though the Muthian model is eductively stable (i.e. $s/d < 1$), there are evolutionary systems with many trader types where a locally stable two–cycle occurs with probability arbitrarily close to 1. Theorem C1 shows that if the strategies are not spread entirely over the strategy space, that is, if the density function $\varphi(b)$ is

zero on certain intervals, global stability of the fundamental steady state may fail, even when the intensity of choice is arbitrarily large. Theorem C2 reaches the same conclusion, asserting that for any given β (large enough) there is an initial distribution of strategies with everywhere positive density such that there are many coexisting stable two–cycles. Having everywhere positive distributions is therefore in itself not sufficient for global stability when β is large but finite[1].

6 Concluding Remarks

If a RE steady state is not eductively stable in the sense of Guesnerie (2002), expectational coordination on this RE steady state seems unlikely, since the belief of a small deviation from the steady state may trigger an even larger realized deviation. Eductive stability can thus be seen as a necessary condition for expectational coordination. In this note we have however argued that global eductive stability is not a sufficient condition for evolutionary learning to enforce convergence to the RE steady state. We have presented simple examples of Muthian cobweb models with heterogeneous beliefs where the RE fundamental steady state is globally eductively stable in the sense of Guesnerie (2002), but evolutionary learning in the sense of Brock and Hommes (1997) does not enforce convergence to the unique RE steady state. In particular, we have presented examples of evolutionary systems with three or five belief types which can lock into a stable two-cycle, co-existing with the stable steady state, with up an down price fluctuations and the majority of agents switching between optimistic and pessimistic strategies.

An important issue related to the stability of an evolutionary system is the number of belief types. As shown in this note, when the number of belief types is small, "evolutionary cycles" can occur where the majority of traders switches constantly from one biased belief to another. An increase in the number of types, for instance by an "invasion" of new types, may destroy these "evolutionary cycles", and can possibly enforce converge to a stable RE steady state. In particular, we have shown that if the initial distribution function of beliefs is strictly positive everywhere and the intensity of choice to switch strategies is large enough, an evolutionary system with many trader types is likely to be globally stable. In that case eductive stability and evolutionary stability coincide. Both conditions are necessary, that is, if either the initial distribution of strategies is not strictly positive everywhere or if the intensity of choice is finite (so that agents are only boundedly rational and make errors), then there are examples of globally eductive stable Muthian models for which the corresponding evolutionary system with many trader types may lock into a locally stable two–cycle.

[1] As can be seen from the proof in the appendix, in this case the density function φ_β depends on the parameter β. From theorem B we infer the existence of $\beta' \gg \beta$ such that the system with density φ_β and intensity of choice β' has a globally stable fundamental state.

Appendix: Proofs of the theorems

Proof of Theorem A.

We follow a strategy of proof similar to Brock and Hommes (1998; section 4.2, Lemma 8) for the asset pricing model with H purely biased beliefs b_h. When there is no memory in the fitness measure, that is when $w = 0$, the fraction n_{ht} of type h is given by

$$n_{ht} = \frac{\exp(\beta U_{h,t-1})}{Z_{t-1}} = \frac{\exp(-\frac{\beta s}{2}(x_{t-1} - b_h)^2)}{Z_{t-1}}, \tag{21}$$

with Z_{t-1} a normalisation factor. Since the fractions n_{ht} are independent of the fitness level, we may add the common term $(\beta s/2)x_{t-1}^2$ to all $U_{h,t-1}$ to obtain

$$n_{ht} = \frac{\exp(-\frac{\beta s}{2}(-2x_{t-1}b_h + b_h^2))}{Z_{t-1}}, \tag{22}$$

where the normalisation factor has been redefined as

$$Z_{t-1} = \sum_{h=1}^{H} \exp(-\frac{\beta s}{2}(-2x_{t-1}b_h + b_h^2)).$$

If $w = 0$, past fitnesses $U_{h,t-1}$, and consequently the fractions n_{ht}, only depend on x_{t-1}. Then equation (11) implies that the deviation x_t from the fundamental price depends only on x_{t-1}, as it is of the form

$$x_t = -\frac{s}{d}\sum_{h=1}^{H} n_{ht}b_h = g(x_{t-1}). \tag{23}$$

We claim that the one-dimensional map g is decreasing. Writing $\mathrm{e}^y = \exp(-\frac{\beta s}{2}(-2x_{t-1}b_h + b_h^2))$, the derivative g' is given by

$$\begin{aligned}
g'(x_{t-1}) &= -\frac{s}{d}\sum_{h=1}^{H} b_h \frac{\mathrm{d}n_{ht}}{\mathrm{d}x_{t-1}} \\
&= -\frac{s}{d}\sum_{h=1}^{H}\{b_h \frac{Z_{t-1}\,\mathrm{e}^y \cdot \beta s b_h - \mathrm{e}^y \cdot (\sum_h \mathrm{e}^y \beta s b_h)}{Z_{t-1}^2}\} \\
&= -\frac{\beta s^2}{d}\sum_{h=1}^{H}\{b_h \frac{\mathrm{e}^y \cdot b_h - n_{ht}(\sum_h \mathrm{e}^y b_h)}{Z_{t-1}}\} \qquad (24) \\
&= -\frac{\beta s^2}{d}\sum_{h=1}^{H}\{n_{ht}b_h^2 - n_{ht}b_h(\sum_h n_{ht}b_h)\} \\
&= -\frac{\beta s^2}{d}\left[\sum_{h=1}^{H} n_{ht}b_h^2 - \left(\sum_{h=1}^{H} n_{ht}b_h\right)\left(\sum_{h=1}^{H} n_{ht}b_h\right)\right]
\end{aligned}$$

The last inequality follows because the last term between square brackets can be interpreted as the *variance* of a stochastic process b_t, taking values b_h with probability n_{ht}. We conclude that the map g is decreasing; hence the system (23) has a unique steady state.

Note that if the beliefs are exactly balanced, that is if for each belief b_h its opposite belief $-b_h$ is also present in the market, then the unique steady state x^* coincides with the fundamental: $x^* = 0$.

We now restrict our attention to the system with three belief types (16)-(18). It has exactly balanced beliefs, and hence the fundamental is the unique steady state.

To investigate the stability of the steady state, observe that the steady state fractions are given by

$$n_1^* = \frac{1}{1 + 2\exp(-\frac{\beta s}{2}b^2)}, \qquad n_2^* = n_3^* = \frac{\exp(-\frac{\beta s}{2}b^2)}{1 + 2\exp(-\frac{\beta s}{2}b^2)}.$$

The local stability of the fundamental steady state is governed by $g'(0)$. Applying (24) yields that

$$g'(0) = -\frac{\beta s^2}{d} 2b^2 \frac{\exp(-\frac{\beta s}{2}b^2)}{1 + 2\exp(-\frac{\beta s}{2}b^2)} = -\frac{2s}{d} h(z),$$

where $z = \beta s b^2/2$ and $h(z) = 2z\,\mathrm{e}^{-z}/(1 + 2\,\mathrm{e}^{-z})$. A straightforward computation shows that $h(z) = p(z)n(z)$, where $p(z) > 0$ for all z and $n(z) = 1 - z + 2\,\mathrm{e}^{-z}$. Using that $\mathrm{e}^{-z} \leq 1 - z + z^2/2$, it follows that $\mathrm{e}^{-1/2} \leq 1 - 1/2 + 1/8 = 5/8$, and hence that

$$n\left(\frac{3}{2}\right) = 1 - \frac{3}{2} + 2\,\mathrm{e}^{-3/2} \leq \frac{125}{256} - \frac{1}{2} < 0.$$

As $n(1) = 2/\mathrm{e} > 0$ and as $n(z)$ is continuous and strictly decreasing, it has a unique zero z_m in the interval $(1, 3/2)$, and the function h takes its global maximum at this point. We have $h(z_m) = z_m - 1 < 1/2$. Since $1/2 < s/d < 1$ by assumption, it follows that $-1 < g'(0)(< 0)$, implying local stability of the steady state $x^* = 0$. This proves assertion (i) of theorem A.

Now turn to assertion (ii), the existence of a stable two–cycle for sufficiently large β. Fix s and d such that $s/d > 1/2$, and let $s/d - 1/2 = \varepsilon$; note that $0 < \varepsilon < 1/2$. Moreover, let $\delta = \frac{b}{2}\varepsilon/(1+\varepsilon)$ and let $U = \{x \,|\, x \geq b/2 + \delta\}$. For $x \in U$, take $\beta > 0$ so large that $\mathrm{e}^{-2\beta s b x}$, $\mathrm{e}^{-2\beta s(2x-b)} < \varepsilon/4$. Then

$$\begin{aligned} g(x) &= -\frac{s}{d}\sum n_{ht} b_h = -\frac{s}{d}\frac{b - b\,\mathrm{e}^{-2\beta s b x}}{1 + \mathrm{e}^{-\frac{\beta s}{2}b(2x-b)} + \mathrm{e}^{-2\beta s b x}} \\ &< -b\frac{s}{d}\frac{1+\varepsilon}{1+2\varepsilon} = -b\left(\frac{1}{2} + \varepsilon\right)\frac{1 - \varepsilon/4}{1 + \varepsilon/2} = -\frac{b}{2}\left(1 + \frac{\varepsilon(5 - 2\varepsilon)}{4(1+\varepsilon)}\right) = -\frac{b}{2} - \delta \end{aligned}$$

uniformly in x. Since g is monotonous, this implies that $g(U) \subset -U$ and $g^2(U) = g(g(U)) \subset U$. Since g^2 monotonically increasing and bounded, all points in U converge

to a fixed point of g^2 in U. Since the only fixed point of g is the origin, the fixed points of g^2 are two–cycles. If no fixed point of g^2 restricted to U were stable, then $g^2(x) \geq x$ for all $x \in U$; but this is impossible since g is bounded.

In the special case $\beta = +\infty$, the system reads as

$$g(x) = \begin{cases} bs/d & \text{if} \quad x < -\frac{1}{2}b, \\ 0 & \quad\quad -\frac{1}{2}b < x < \frac{1}{2}b, \\ -bs/d & \quad\quad \frac{1}{2}b < x. \end{cases}$$

It can be verified directly that that the system has a locally stable two–cycle $\{x_1, x_2\} = \{-bs/d, bs/d\}$, with corresponding fractions of optimists (and pessimists) switching from 0 to 1 along the two–cycle. This proves (iii) and completes the proof of the proposition.

Proof of Theorem B

Let $\varphi(b)$ be a fixed continuous density function, that is, let $\varphi(b) \geq 0$ for all b, $\int \varphi \, \mathrm{d}b = 1$. Introduce the probability density function

$$\psi = \psi_{x,\beta,s}(b) = \frac{\mathrm{e}^{-\frac{\beta s}{2}(x-b)^2}\varphi(b)}{\displaystyle\int \mathrm{e}^{-\frac{\beta s}{2}(x-\tilde{b})^2}\varphi(\tilde{b}) \, \mathrm{d}\tilde{b}}.$$

Let

$$h(x) = \int b\psi_{x,\beta,s}(b) \, \mathrm{d}b.$$

Let J be the interior of the support of φ, that is, $J = \mathrm{int}\{b \,|\, \varphi(b) \geq 0\}$. We will show that

Lemma 1 *For all $x \in J$:*

$$\lim_{\beta\to\infty} h(x) = x \quad \textit{and} \quad \lim_{\beta\to\infty} h'(x) = 1,$$

uniformly on all compact subsets K of J.

The LTL-map g in (20) satisfies $g(x) = -(s/d)h(x)$, and Lemma 1 is thus equivalent to the lemma in Section 5. Hence, if $J = \mathbb{R}$ and $-1 < -s/d < 0$, then for every compact interval K enclosing 0, there is a $\beta_0 > 0$ such that all points in K tends to the globally stable fixed point 0 under iteration of g. If $J = \mathbb{R}$ and $-s/d < -1$, then the lemma implies that g has an unstable fixed point 0 and all other solutions either oscillate or diverge. Therefore the lemma implies the statements of theorem B.

We proceed to prove the lemma. We shall first show that as $\beta \to \infty$, the expectation $\mathrm{E}_\psi b = h(x)$ tends to x.

By the coordinate transformation $b = x + y/\sqrt{\beta s}$, we find that

$$\Delta(x) = \mathrm{E}_\psi b - x = \frac{\int (b-x)\,\mathrm{e}^{-\frac{\beta s}{2}(b-x)^2}\varphi(b)\,\mathrm{d}b}{\int \mathrm{e}^{-\frac{\beta s}{2}(b-x)^2}\varphi(b)\,\mathrm{d}b} = \frac{1}{\sqrt{\beta s}}\frac{\int y\,\mathrm{e}^{-y^2/2}\varphi\Big(x + y/\sqrt{\beta s}\Big)\,\mathrm{d}y}{\int \mathrm{e}^{-y^2/2}\varphi\Big(x + y/\sqrt{\beta s}\Big)\,\mathrm{d}y}.$$

Let $M = \sup_{b\in\mathbb{R}}\varphi(b)$. Since φ is continuous and $\varphi(x) > 0$, there are $\delta > 0$, $0 < \varepsilon < \varphi(x)/2$ such that $|\varphi(b) - \varphi(x)| < \varepsilon$ for all $b \in (x-\delta, x+\delta)$. Set $A = \delta\sqrt{\beta s}$. Then

$$|\Delta(x)| \le \frac{1}{\sqrt{\beta s}}\frac{M}{\varphi(x)-\varepsilon}\frac{\int |y|\,\mathrm{e}^{-y^2/2}\,\mathrm{d}y}{\int_A^A \mathrm{e}^{-y^2/2}\,\mathrm{d}y} \to 0 \qquad \text{as} \quad \beta \to \infty;$$

note that the convergence is uniform if x is restricted to a compact subset K of $\mathbb{R}$. This implies the first part of the lemma.

The derivative of h is obtained by differentiating under the integral (compare equation (24)):

$$\begin{aligned} h'(x) &= \beta s \int b(b-x)\psi\,\mathrm{d}b - \beta s \int b\psi\,\mathrm{d}b \cdot \int (b-x)\psi\,\mathrm{d}b \\ &= \beta s \int b^2\psi\,\mathrm{d}b - \beta s\left(\int b\psi\,\mathrm{d}b\right)^2 = \beta s\,\mathrm{Var}_\psi b = \beta s\,\mathrm{Var}_\psi(b-x). \end{aligned}$$

Here $\mathrm{Var}_\psi b$ is the variance of a stochastic variable b distributed according to the probability distribution ψ. The last equality holds since b and $b - x$ have the same variance.

As above, we have that

$$h'(x) = \frac{\int y^2\,\mathrm{e}^{-y^2/2}\varphi\Big(x + y/\sqrt{\beta s}\Big)\,\mathrm{d}y}{\int \mathrm{e}^{-y^2/2}\varphi\Big(x + y/\sqrt{\beta s}\Big)\,\mathrm{d}y} - \left(\frac{\int y\,\mathrm{e}^{-y^2/2}\varphi\Big(x + y/\sqrt{\beta s}\Big)\,\mathrm{d}y}{\int \mathrm{e}^{-y^2/2}\varphi\Big(x + y/\sqrt{\beta s}\Big)\,\mathrm{d}y}\right)^2.$$

Let $A, M, \delta, \varepsilon > 0$ be as above. Then

$$h'(x) \le \frac{\varphi(x)+\varepsilon}{\varphi(x)-\varepsilon}\frac{\int_{-A}^{A} y^2\,\mathrm{e}^{-y^2/2}\,\mathrm{d}y + \dfrac{M}{\varphi(x)+\varepsilon}\int_{[-A,A]^c} y^2\,\mathrm{e}^{-y^2/2}\,\mathrm{d}y}{\int_{-A}^{A}\mathrm{e}^{-y^2/2}\,\mathrm{d}y}$$

Fix $\alpha > 0$ arbitrarily; then there exists β_1 such that if $\beta > \beta_1$, it follows that

$$h'(x) \le \frac{\varphi(x)+\varepsilon}{\varphi(x)-\varepsilon}(1+\alpha).$$

Likewise, for $\alpha > 0$ there exists $\beta_2 > 0$ such that for $\beta > \beta_2$:

$$\begin{aligned} h'(x) &\geq \frac{\varphi(x)-\varepsilon}{\varphi(x)+\varepsilon} \frac{\displaystyle\int_{-A}^{A} y^2 \,\mathrm{e}^{-y^2/2}\,\mathrm{d}y - \frac{M}{\varphi(x)-\varepsilon}\int_{[-A,A]^c} \mathrm{e}^{-y^2/2}\,\mathrm{d}y}{\displaystyle\int \mathrm{e}^{-y^2/2}\,\mathrm{d}y} \\ &\quad - \left(\frac{\varepsilon \displaystyle\int_{-A}^{A} |y|\,\mathrm{e}^{-y^2/2}\,\mathrm{d}y + M\int_{[-A,A]^c} |y|\,\mathrm{e}^{-y^2/2}\,\mathrm{d}y}{(\varphi(x)-\varepsilon)\displaystyle\int_{-A}^{A} \mathrm{e}^{-y^2/2}\,\mathrm{d}y} \right)^2 \\ &\geq \frac{\varphi(x)-\varepsilon}{\varphi(x)+\varepsilon}(1-\alpha) - C\varepsilon^2, \end{aligned}$$

where C does only depend on β_2. Since α and ε were arbitrary, it follows that

$$h'(x) \to 1$$

as $\beta \to \infty$. Note that, as before, the convergence is uniform if x is restricted to a compact subset K of $\mathbb{R}$. This implies the second part of the lemma, and completes the proof of theorem B.

Proof of theorem C.

Proof of C1. Let $0 < \delta < 1/8$ be a positive constant; and let D_1, D_2 and D_3 be intervals

$$D_1 = (-1-\delta, -1+\delta), \qquad D_2 = (-\delta, \delta), \qquad D_3 = (1-\delta, 1+\delta).$$

Moreover, let $D = D_1 \cup D_2 \cup D_3$. Densities φ and ψ are defined as follows:

$$\begin{aligned} \varphi(b) &= \begin{cases} (6\delta)^{-1} & \text{if } t \in D \\ 0 & \text{otherwise,} \end{cases} \\ \psi(b) &= \mathrm{e}^{-\frac{\beta s}{2}(b-x)^2}\varphi(b)/Z, \end{aligned}$$

where the normalisation factor Z is determined by $\int_D \psi(b)\,\mathrm{d}b = 1$. As in the proof of Theorem B, a function $h(x)$ is introduced by setting $h(x) = -(d/s)g(x)$. Recall that

$$h(x) = \mathrm{E}_\psi b \qquad \text{and} \qquad h'(x) = \beta s\,\mathrm{Var}_\psi b.$$

The relevant properties of h are given in the following lemma.

Lemma 2 *Take x in the compact interval $K = [\frac{1}{2}+\delta, 1-2\delta]$. Then for every $\varepsilon > 0$ there is a $\beta_0 > 0$, such that for $\beta > \beta_0$ we have that*

$$|h(x) - (1-\delta)| < \varepsilon \qquad \textit{and} \qquad h'(x) < \varepsilon, \tag{25}$$

uniformly in x.

Assuming the truth of the lemma, the theorem is proved as follows. Let s and d be fixed, such that

$$\frac{\frac{1}{2}+\delta}{1-2\delta} < \frac{s}{d} < 1-2\delta.$$

Choose $\varepsilon = \delta$. If $\beta > \beta_0$, with β_0 obtained from the lemma, it follows for all $x \in K$ that

$$-(1-2\delta) < g(x) < -\left(\frac{1}{2}+\delta\right).$$

As g is continuous, by construction of K there is a point $x_* \in K$ such that $g(x_*) = -x_*$; symmetry of g yields consequently $g(g(x_*)) = x_*$, so that x_* is a period 2 point.

Moreover, we have for $x \in K$ that

$$0 \geq g'(x) = \left(-\frac{s}{d}\right) h'(x) > -\frac{s}{d}\delta > -\delta > -1.$$

We conclude that the period–2 point x_* is attracting.

Proof of the lemma. Introduce ℓ_j and u_j as the respective lower and upper endpoints of the interval D_j. By partial integration, we obtain

$$h(x) = \frac{1}{Z}\sum_j \int_{\ell_j}^{u_j} b\,\mathrm{e}^{-\frac{\beta s}{2}(b-x)^2}\,\mathrm{d}bx - \frac{1}{Z\beta s}\sum_j \mathrm{e}^{-\frac{\beta s}{2}(b-x)^2}\Big|_{\ell_j}^{u_j} \tag{26}$$

In order to obtain information on Z, the following well–known asymptotic expansion (see for instance [11], I.**152**) is used:

$$\mathrm{e}^{\frac{1}{2}a^2}\int_a^\infty \mathrm{e}^{-\frac{1}{2}b^2}\,\mathrm{d}b = \frac{1}{a} - \frac{1}{a^3} + \mathcal{O}\left(\frac{1}{a^5}\right) \qquad \text{as } a \to \infty \tag{27}$$

As usual the notation $f(x) = \mathcal{O}(g(x))$ is taken to mean that there is a constant $C > 0$ such that $|f(x)| \leq Cg(x)$.

Recall that $Z = \sum_j \int_{\ell_j}^{u_j} \exp(-(\beta s/2)(b-x)^2)\,\mathrm{d}b$. If c_j denotes the element of D_j closest to x, then the expansion (27) yields for x in the complement of D:

$$Z = \sum_j \left(\frac{1}{\beta s|x-c_j|} - \frac{1}{(\beta s)^2|x-c_j|^3} + \mathcal{O}\left((\beta s)^{-3}\right)\right)\mathrm{e}^{-\frac{\beta s}{2}(x-c_j)^2} \tag{28}$$

Let x be restricted to the open interval $I = (\frac{1}{2}+\delta, 1-\delta)$, then the point in D closest to x is ℓ_3, and any point y in $D_1 \cup D_2$ satisfies $|y-x| \geq |\ell_3 - x| + 2\delta$. To simplify the expansion of Z, we multiply equation (28) by $\exp(\beta s(x-\ell_3)^2/2)$, and we note that the other exponentials can be bounded from above by $\exp(-\beta s\delta)$. This yields

$$Z = \left(\frac{1}{\beta s|x-\ell_3|} - \frac{1}{(\beta s)^2|x-\ell_3|^3} + \mathcal{O}\left((\beta s)^{-3}\right)\right)\mathrm{e}^{-\frac{\beta s}{2}(x-\ell_3)^2}.$$

Substitution in equation (26), and recalling that $|x-\ell_3| = \ell_3 - x$, yields

$$h(x) = \ell_3 + \frac{1}{\beta s(\ell_3 - x)} + \mathcal{O}\left((\beta s)^{-2}\right)$$

for $x \in I$. Now the first half of (25) follows.

Recall that $h'(x) = \beta s \operatorname{Var}_\psi T$. We have

$$\begin{aligned}
\operatorname{Var}_\psi b &= -h(x)^2 + \frac{1}{Z}\sum_j \int_{\ell_j}^{u_j} b^2 \,\mathrm{e}^{-\frac{\beta s}{2}(b-x)^2}\,\mathrm{d}b \\
&= -h(x)^2 + \frac{1}{Z}\sum_j \int_{\ell_j}^{u_j} \frac{1}{\beta s}\left(b\cdot\beta s(b-x) + x\cdot\beta s(b-x) + \beta s x^2\right)\,\mathrm{e}^{-\frac{\beta s}{2}(b-x)^2}\,\mathrm{d}b \\
&= -h(x)^2 + x^2 + \frac{1}{\beta s} - \frac{1}{Z\beta s}\sum_j (b+x)\,\mathrm{e}^{-\frac{\beta s}{2}(b-x)^2}\Big|_{\ell_j}^{u_j}.
\end{aligned}$$

Using the same kind of asymptotic arguments as before, this expression can be expanded to

$$\begin{aligned}
\operatorname{Var}_\psi b &= -h(x)^2 + x^2 + \frac{1}{\beta s} + \frac{\ell_3 + x}{Z\beta s}\,\mathrm{e}^{-\frac{\beta s}{2}(\ell_3 - x)^2} + \mathcal{O}\left((\beta s)^{-2}\right) \\
&= -h(x)^2 + x^2 + \frac{1}{\beta s} + (\ell_3 + x)(\ell_3 - x)\left(1 + \frac{1}{\beta s(\ell_3 - x)^2}\right) + \mathcal{O}\left((\beta s)^{-2}\right) \\
&= \mathcal{O}\left((\beta s)^{-2}\right).
\end{aligned}$$

Hence

$$h'(x) = \beta s \operatorname{Var}_\psi b = \mathcal{O}\left(\frac{1}{\beta s}\right).$$

The second half of (25) follows; this concludes the proof of the lemma.

Proof of C2. The previous result shows that there exist LTL systems which have for some fixed $\beta > 0$ the same behaviour as an evolutionary system with finitely many, n, types. However, in constructing this example, the underlying type distribution had compact support. To construct an example of an LTL support of the type distribution equal $\mathbb{R}$ having a stable two–cycle, the distribution necessarily has to depend on β because of theorem B.

Take for instance the following $2n+1$-modal distribution:

$$\varphi(b) = \frac{1}{2n+1}\sum_{h=-n}^{n} \frac{1}{\sqrt{2\pi}\sigma_h}\,\mathrm{e}^{-\frac{(b-b_h)^2}{2\sigma_h^2}}.$$

We make a couple of specifications. First, the b_h are assumed to be distributed symmetrically around 0: $b_{-h} = -b_h$. Moreover, they are assumed to be arranged as follows:

$$b_h = 4^h b_1, \quad \text{if } h > 0.$$

Finally, all σ_h are taken equal to $\sigma_h = \sqrt{2/\beta s}$.

The LTL can be computed in this case. It reads as:

$$x_t = g(x_{t-1}) = \left(-\frac{s}{d}\right)\left(\frac{1}{2}x_{t-1} + \frac{1}{2}\frac{\sum_{h=-n}^{n} \mathrm{e}^{-\frac{\beta s}{4}(b_h - x_{t-1})^2} b_h}{\sum_{h=-n}^{n} \mathrm{e}^{-\frac{\beta s}{4}(b_h - x_{t-1})^2}}\right).$$

Note that $g(x)$ is an odd function.

Fix $i > 0$: then $b_i \geq 0$ (the case $b_i < 0$ necessitates only minor modifications of the following). We may write

$$x_t = g(x_{t-1}) = \left(-\frac{s}{d}\right)\left(\frac{1}{2}\left(x_{t-1} + b_i\right) + r\right),$$

with

$$r = \frac{1}{2}\frac{\sum_{h=-n}^{n} \mathrm{e}^{-\frac{\beta s}{4}(b_h - x_{t-1})^2}(b_h - b_i)}{\sum_{h=-n}^{n} \mathrm{e}^{-\frac{\beta s}{4}(b_h - x_{t-1})^2}}.$$

Let δ and Δ denote, respectively,

$$\delta = \min_{h \neq i} |b_h - b_i|, \quad \text{and} \quad \Delta = \max_{h \neq i} |b_h - b_i|.$$

Let U denote the open interval $U = (b_i - \delta/3, b_i + \delta/3)$.

We claim that if $\frac{6}{7} < s/d \leq 1$ then for β sufficiently large there is a point $x_* \in U$ such that

$$g(x_*) = -x_*. \tag{29}$$

It then follows by the oddness of g,

$$g(g(x_*)) = g(-x_*) = -g(x_*) = x_*,$$

that is, the point x_* is a periodic point of period 2. The claim is first shown for

$$g_0(x) = \left(-\frac{s}{d}\right)\left(\frac{1}{2}x + \frac{1}{2}b_i\right);$$

afterwards it is extended to $g = g_0 + r$ by a perturbation argument.

Note that the equation $g_0(x) = -x$ has solution

$$x_0 = \frac{s/d}{2 - s/d} b_i.$$

Hence, since $\frac{6}{7} < s/d \leq 1$, it follows that $\frac{3}{4}b_i < x_0 \leq b_i$, and consequently that

$$|x_0 - b_i| < \frac{1}{4}b_i \leq \frac{\delta}{3}.$$

where we used that $b_{i-1} = \frac{1}{4} b_i$. The inequality implies that $x_0 \in U$.

To apply a perturbation argument to equation (29), the magnitudes $|r(x)|$ and $|r'(x)|$ have to be estimated. First

$$\sup_{x \in U} |r(x)| \leq \frac{1}{2} \frac{\sum_{h=-n}^{n} \mathrm{e}^{-\frac{\beta s}{4}(b_h - x)^2} |b_h - b_i|}{\sum_{h=-n}^{n} \mathrm{e}^{-\frac{\beta s}{4}(b_h - x)^2}} \leq \frac{2n\Delta}{2} \frac{\mathrm{e}^{-\frac{\beta s}{4}\frac{4\delta^2}{9}}}{\mathrm{e}^{-\frac{\beta s}{4}\frac{\delta^2}{9}}} \leq n\Delta\, \mathrm{e}^{-\frac{\beta s}{12}\delta^2}.$$

The derivative $|r'(x)|$ is estimated along the same lines; a little computation yields that

$$\sup_{x \in U} |r'(x)| \leq 2n^2 \Delta^2 \beta s\, \mathrm{e}^{-\frac{\beta s}{12}\delta^2}.$$

Hence, by choosing β sufficiently large, both $|r(x)|$ and $|r'(x)|$ can be made arbitrarily small, uniformly in x.

By the intermediate value theorem, it follows that if β is sufficiently large, equation (29) has a solution x_* in U. Taking β even larger if necessary, it follows that $-1 < f' < 0$ in U. Hence the solution x_* of equation (29) is unique in U. As in the proof of B2, it follows that x_* is an attracting period two cycle.

Hence, we constructed an LTL system with n attracting periodic 2–cycles, together with an attracting fixed point at 0.

Note that as these cycles are hyperbolic, the distribution φ might be perturbed slightly to a distribution that is not symmetric around $b = 0$, while all the 2–cycles persist. The existence of multiple periodic 2–cycles is an 'open' property, enjoyed by an open set of systems (open with respect to some specific function topology).

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