Analysis on the stability of Josephson vortices at tricrystal boundaries: A $3\phi_0/2$-flux case

H. Susanto* and S. A. van Gils

Department of Applied Mathematics, University of Twente, P.O. Box 217, 7500 AE Enschede, The Netherlands

A. Doelman

Korteweg-de Vries Institute, Faculty of Sciences, University of Amsterdam, Plantage Muidergracht 24, 1018 TV Amsterdam, The Netherlands

G. Derks

Department of Mathematics and Statistics, University of Surrey, Guilford, Surrey GU2 7XH, United Kingdom

We consider Josephson vortices at tricrystal boundaries. We discuss the specific case of a tricrystal boundary with a π junction as one of the three arms. It is recently shown that the static system admits an $(n+1/2)\phi_0$ flux, $n=0,1,2$ [Phys. Rev. B **61**, 9122 (2000)]. Here we present an analysis to calculate the linear stability of the admitted states. In particular, we calculate the stability of a $3\phi_0/2$ flux. This state is of interest, since energetically this state is preferable for some combinations of Josephson lengths, but we show that in general it is linearly unstable. Finally, we propose a system that can have a stable $(n+1/2)\phi_0$ state.

Half-integer flux quantization is a tool to probe the symmetry of unconventional superconductors.[1–3] Half-flux quanta will be naturally created at the intersection of three grain boundaries if one of the boundaries has a phase shift of π.[4] In a recent paper, Kogan, Clem, and Kirtley[4] consider theoretically Josephson vortices at tricrystal boundaries. When one of the three Josephson junctions is a π junction, a half-integer flux is spontenously generated and attached to the joint. For experimental reports on the observation of Josephson vortices in grain boundaries with a π junction we refer to Refs. 1–3. Kogan, Clem, and Kirtley[4] also consider a general case where the Josephson lengths of the junctions λ_J's are not the same. Besides the $\phi_0/2$ state, they also notice the existence of multiple half-flux states—i.e., $(n+1/2)\phi_0$, $n=1,2$.

The system of three Josephson junctions meeting at one end point has been considered first by Nakajima, Onodera, and Ogawa.[5] The derivation of this system coupled via the boundary conditions using an electrical analog is given by Nakajima and Onodera.[6] The dynamic behavior of integer fluxes in this system with three junctions of the same type and with the same Josephson length has been discussed in Refs. 7 and 8.

Knowing the eigenvalues of a state is of importance, also for experimentalists, since one can then predict whether a particular state can be observed in experiments or not. In this paper we will calculate analytically the linear stability of static (multiple) semifluxons sitting at or near the meeting point of a tricrystal junction with one π junction. We will consider the general case where the Josephson lengths are not the same. As an example we will calculate the stability of a $3\phi_0/2$ state. We use this state as a particular example since energetically this state is preferable for some combinations of Josephson lengths, compared to $\phi_0/2$ at the branch point plus ϕ_0 at infinity.[4] Nonetheless, calculation of the energy of this state does not establish the stability of it. In this report, we will show that this state is in general unstable.

The time-dependent governing equation of the phase difference along the junctions is described by the perturbed sine-Gordon equation

$$\lambda_i^2 \phi_{xx}^i - \phi_{tt}^i = \theta^i \sin \phi^i + \alpha \phi_t^i, \tag{1}$$

with $i=1,2,3$, $x>0$, $t>0$, and α is a positive damping coefficient. The damping coefficient is not necessarily the same for all the junctions. The subscript J of the Josephson length is omitted for brevity. The index i numbers the junction. The constant θ^i represents the type of ith junction. Without loss of generality, we consider the case $\theta^1=-\theta^2=-\theta^3=-1$. This models a tricrystal boundary with one π junction. The overall coupling boundary conditions at the intersection are[4–6]

$$\phi^1 + \phi^2 + \phi^3 = 0,$$

$$\phi_x^1 = \phi_x^2 = \phi_x^3, \tag{2}$$

all evaluated at $x=0$.

The total Hamiltonian energy of Eq. (1) is given by[4]

$$H = \sum_i \int_0^\infty \frac{1}{2}(\lambda_i \phi_x^i)^2 + \theta^i(1 - \cos \phi^i)dx. \tag{3}$$

A time-independent solution of Eq. (1) representing a $3\phi_0/2$ flux is given by[4]

$$\phi_0^1 = 4 \tan^{-1}(e^{(x-x_1)/\lambda_1}) - \pi,$$

$$\phi_0^2 = 4 \tan^{-1}(e^{(x-x_2)/\lambda_2}),$$

$$\phi_0^3 = 4 \tan^{-1}(e^{(x-x_3)/\lambda_3}) - 2\pi, \tag{4}$$

where the x_i are determined by Eq. (2). For simplicity we scale λ_1 to 1 such that in the calculation we need to consider only λ_2 and λ_3. We will discuss the linear stability of the state given by Eq. (4), but the method can be applied to other

soliton solutions admitted by Eqs. (1) and (2). Combining Eqs. (4) and (2) gives[4]

$$2\gamma_2\gamma_3\eta^3 - (1 + \gamma_2^2 + \gamma_3^2)\eta^2 + 1 = 0,$$

$$\gamma_i = \lambda_i/\lambda_1, \quad \eta = \sin[2\,\tan^{-1}(e^{-x_1/\lambda_1})],$$

$$e^{-x_i/\lambda_i} = \frac{1 \pm \sqrt{1 - \gamma_i^2 \eta^2}}{\gamma_i \eta}, \quad i = 2,3. \tag{5}$$

The first case we consider is that $\lambda_i=1$ for all i. Consequently we have $x_1=x_2=x_3=0$.[4] In this case the system has $\mathbf{S}_3$ symmetry.[9] We linearize about the solution ϕ_0^i. We write $\phi^i(x,t) = \phi_0^i + u^i(x,t)$ and substitute the spectral ansatz $u^i = e^{\omega t}v^i(x)$. Retaining the terms linear in u^i gives the eigenvalue problem

$$v_{xx}^i - (\omega^2 + \alpha\omega + \theta^i \cos\phi_0^i)v^i = 0, \tag{6}$$

with boundary conditions at $x=0$ given by

$$v^1 + v^2 + v^3 = 0, \quad v_x^1 = v_x^2 = v_x^3. \tag{7}$$

The spectrum ω consists of the essential spectrum and the point spectrum (isolated eigenvalues). The essential spectrum is given by those ω for which there exist a solution to

$$v_{xx}^i - [\omega^2 + \alpha\omega + (\lim_{x\to\infty}\theta^i \cos\phi_0^i)]v^i = 0,$$

i.e.,

$$v_{xx}^i - (\omega^2 + \alpha\omega + 1)v^i = 0 \tag{8}$$

of the form $v^i = e^{i\kappa x}$, with κ real.

It follows that

$$\omega = \frac{-\alpha \pm \sqrt{\alpha^2 - 4(1 + \kappa^2)}}{2}. \tag{9}$$

It is easy to see that $\mathrm{Re}(\omega) < 0$. The right-hand side of Eq. (9) is plotted in Fig. 1, with κ as parameter.

The above stability analysis shows that solution (4) can be stable. We cannot conclude whether the solution is linearly stable or not before analyzing the point spectrum.

To complete the analysis, our next task is to find the point spectrum ω. The point spectrum consists of those values of ω for which there exist solutions v^i to Eq. (6) with boundary conditions (7) that converge to 0 at ∞.

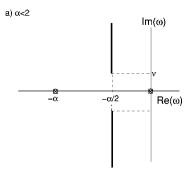The eigenfunction v^i that corresponds to the eigenvalue is of the form[10,11]

$$v^i(x) = c_i e^{\mu(x-x_i)/\lambda_i}\left(\tanh\frac{x - x_i}{\lambda_i} - \mu\right), \quad \mu^2 = \omega^2 + \alpha\omega + 1, \tag{10}$$

where $\mathrm{Re}(\mu) < 0$ and c_i needs to be determined from Eq. (7). Hence, we obtain

$$\mu(c_1 + c_2 + c_3) = 0,$$

$$c_1(1 - \mu^2) = c_2(1 - \mu^2) = c_3(1 - \mu^2).$$

The fact that v^i cannot be zero for all i implies that $\mu=0$ or $\mu=\pm 1$. From the condition that $\mathrm{Re}(\mu) < 0$, we obtain μ
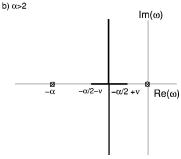

FIG. 1. A sketch of the point spectra (crosses) and the essential spectrum Eq. (9) (thick lines) for two cases of α: (a) $0 < \alpha < 2$ and (b) $\alpha > 2$. When $\alpha \geq 2$, there is a part of the boundary lines that is at the negative real line from point $(-\alpha/2 - \sqrt{\alpha^2/4 - 1}, 0)$ to point $(-\alpha/2 + \sqrt{\alpha^2/4 - 1}, 0)$. There is no spectrum with positive real part implying the linear stability of solution (4).

$=-1$ or $\omega=0,-\alpha$ with the corresponding eigenfunctions given by

$$[v^1, v^2, v^3] = [1, 0, -1]e^{-x}(\tanh x + 1) = [1, 0, -1]\mathrm{sech}\,x,$$

$$[v^1, v^2, v^3] = [1, -1, 0]e^{-x}(\tanh x + 1) = [1, -1, 0]\mathrm{sech}\,x.$$

This result shows that there are quadruple eigenvalues at zero when the damping term is absent. A double-zero eigenvalue bifurcates to the left half-plane when α is nonzero. The value $\mu=0$ gives bounded but not decaying eigenfunctions from which we obtain the edge of the essential spectrum [$\kappa=0$ in Eq. (9)]. A sketch of the locations and the bifurcation of the point spectra is presented in Fig. 1.

Hence, we conclude that the solution given in Eq. (4) is linearly stable.

Next we will consider the general case of the $3\phi_0/2$ state for any given combinations of Josepshon lengths. It is clear that x_i [see Eqs. (5)] can be either positive or negative, but not all combinations of x_i's satisfy the governing equation. In Fig. 2, we show the sign-set diagram showing combinations of signs of x_i that are needed for a solution to satisfy the governing equations.[12] A solution with two +'s has a higher Hamiltonian energy [see Eq. (3)] than a solution with one + for given values of λ_i.[4]

To search for an asymptotically stable $3\phi_0/2$ state, it is suggested to look at tricrystals with the Josephson length of the π arm being larger than those of the 0 arms.[4]

We have obtained an expression for the eigenvalues of the $3\phi_0/2$ state. Combining Eqs. (7) with (10) yields a polyno-
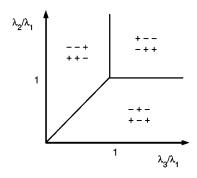
FIG. 2. The sign-set diagram for $(\mathrm{sgn}(x_1),\mathrm{sgn}(x_2),\mathrm{sgn}(x_3))$. This diagram corresponds to Fig. 4 of Ref. 4

mial of order 5 in μ, with coefficients that depend on λ_2/λ_1 and λ_3/λ_1. Asymptotic analysis shows that one root is less than -1 if $|\lambda_2/\lambda_1|\ll 1$ and $|\lambda_3/\lambda_1|\ll 1$. In Fig. 3 we show numerically that this result extends to general values of λ_2/λ_1 and λ_3/λ_1. Remembering that $\mu^2=\omega^2+\alpha\omega+1$, the two
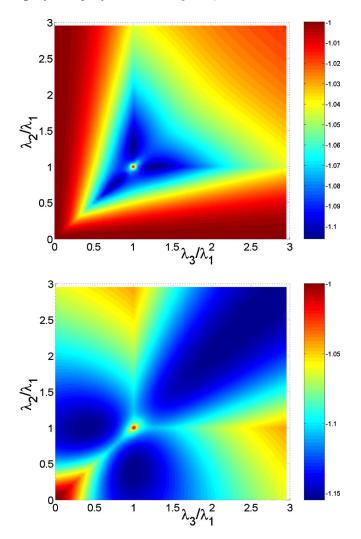
FIG. 3. (Color online) The contourplot of the smallest μ as a function of λ_2/λ_1 and λ_3/λ_1 for the $3\phi_0/2$ solution with (upper) one $+$ and (lower) two $+$'s in the $(\mathrm{sgn}(x_1),\mathrm{sgn}(x_2),\mathrm{sgn}(x_3))$. A $3\phi_0/2$ state is marginally stable if $\mu=-1$ which is attained only at unphysical combinations.
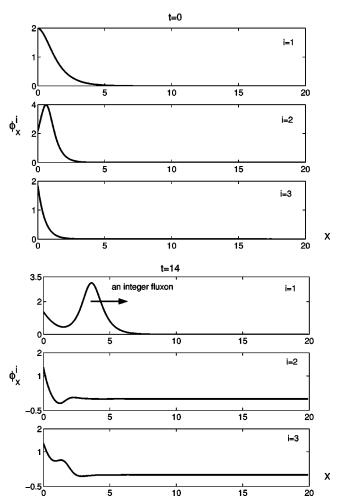

FIG. 4. The evolution of a $3\phi_0/2$ state (4) with $\alpha=0.05$ and $\lambda_2=\lambda_3=0.5$; i.e., the Josephson length of the π junction is larger than the Josephson lengths of the ordinary junctions. As an initial state, we choose the solution with one positive x_i. The release of an integer fluxon moving away from the branch point shows the instability of the state.

plots inform us that when the Josephson lengths differ, a pair of eigenvalues at the real line bifurcates from the quadruple zero. This implies instability if there is a Josephson length different from the others. Further numerical analysis shows that in this case the other zero eigenvalues move along the imaginary axis (with negative real part when $\alpha\neq 0$). Solutions with one $+$ have another pair of eigenvalues at the imaginary axis bifurcating from the edges of the continuous spectrum.

The calculation we have done shows that there is no stable $3\phi_0/2$ state in tricrystal junctions with one π arm, except at some unphysical combinations of the Josephson lengths. We have used numerical simulations of Eq. (1) to confirm the result of our linear stability analysis. In the scheme we take $\phi_t^i(x,0)=0$ and $\phi^i(x,0)=\phi_0^i$ as the initial conditions. Indeed we observed the same result for the stability or instability. In Fig. 4 we present the evolution of two $3\phi_0/2$ states for a given value of λ_i's.

With the above analysis, it can be easily shown that the $\phi_0/2$ state is stable and the $5\phi_0/2$ state is unconditionally unstable.

One can also show, using the same analysis, that the $3\phi_0/2$ state will be unconditionally stable in the tetracrystals with one π arm.[3] One can also calculate that the $5\phi_0/2$ state will be marginally stable in pentacrystals with one π arm. We conjecture that a stable $(1/2+n)\phi_0$ state exists in $2(n+1)$ or more junctions connected to a joint with one of the arms is a π junction. All the stable states require the maximum field to be at the joint (see Fig. 1 in Ref. 4).

To summarize, we have described an analysis to study the (in)stability of a state in a tricrystal junction. We have considered a special case—i.e., $3\phi_0/2$ flux—and shown that the state is linearly unstable. According to the theory presented in Ref. 4 and combining the result with the stability analysis we present here gives a clear explanation why a $3\phi_0/2$ state is never observed in experiments, especially in film geometry.[4] The stability analysis can be applied to discuss the stability of solutions of other Josephson junction systems. We also have written systems that can presumably have a stable $(1/2+n)\phi_0$ state.

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*Electronic address: h.susanto@math.utwente.nl

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