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Renormalisation group flow in an exactly solvable model with fluctuating geometry

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Abstract

A recently proposed renormalization group technique, based on the hierarchical structures present in theories with fluctuating geometry, is implemented in the model of branched polymers. The renormalization group equations can be solved analytically, and the flow in coupling constant space can be determined.

1. Introduction

The renormalization group is one of the most important concepts in field theory and in the theory of critical phenomena. It uses two important concepts: "blocking" of regions of space and "blocked fields" which are defined in terms of the fields in the original regions.

In theories like quantum gravity where the fields themselves determine space the blocking of regions of space becomes a non-trivial notion. A variety of methods have been proposed [1–6]. Here we will study a model for which the methods proposed in [2] and further developed in [3] can be implemented and the corresponding renormalization group equations solved. The model is the so-called branched polymer model [7–9]. While this model indeed is a toy model, it is an appropriately chosen toy model: Whenever models of fluctuating geometry are studied, branched polymers appear naturally unless coupling constants are chosen with care. In two-dimensional quantum gravity, viewed as a statistical model via dynamical triangulations, it is generally believed that the theory degenerates to a theory of branched polymers if the central charge c of the matter fields coupled to two-dimensional gravity is sufficiently large. Maybe it even happens for all values of $c > 1$. In three- and four-dimensional quantum gravity, again studied via dynamical triangulations, it is known that the weak coupling phase, i.e. the phase with a small bare gravitational coupling constant, is a phase of branched polymers [10,11]. Since two- and four-dimensional quantum gravity, implemented via dynamical triangulations, are precisely the models where renormalization group techniques have been applied, it seems ap-

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propriate to study the method used in [3] for branched polymers.

2. The BP-model

Let us define the model of branched polymers (the BP-model).

We consider the ensemble of *planar rooted planted trees*. A *tree* is a graph without closed loops. A *rooted* tree is a tree with one marked vertex and *planted* tree is a tree where the *degree* of the root (i.e. number of branches) is one. We call the marked vertex and the link emerging from it *the root* of the planted tree. Two planted trees are considered as distinct if they cannot be mapped on each other by a continuous deformation of the plane such that the root of one tree is mapped to the root of the other tree.

The grand canonical partition function is defined as

$$Z(t) = \sum_{T \in \mathcal{T}} \rho(T), \tag{1}$$

where $\mathcal{T}$ denotes the ensemble of rooted, planted trees and

$$\rho(T) = t_0^{n_0} t_1^{n_1} \dots t_k^{n_k} \dots \tag{2}$$

The t_i's can be viewed as the weights given to vertices of order $i+1$ relative to the root vertex which by definition has weight 1, i.e. we have

$$Z(t) = t_0 + t_0 t_1 + t_0 t_1^2 + \dots + t_0^2 t_2 + \dots \tag{3}$$

We will consider the t_i's as our coupling constants and further assume $t_i \geq 0$, although there exist so-called multicritical BP-models where some weights can be negative [8].

The partition function Z satisfies the known equation (see [8]):

$$Z = \mathcal{F}(Z), \tag{4}$$

where

$$\mathcal{F}(Z) = \sum_{i=0}^{\infty} t_i Z^i. \tag{5}$$

Sometimes it will be convenient to emphasize that $\mathcal{F}(Z)$ can be viewed as a function of both t and Z. We will in these situations write $\mathcal{F}(t; Z)$. We note the following properties. The average number of vertices of type i above the root is given by

$$\langle N_i \rangle = \frac{t_i}{Z} \frac{dZ}{dt_i}, \tag{6}$$

and if we denote by N the total number of vertices above the root we get from (5)

$$\langle N \rangle = \frac{1}{1 - \mathcal{F}'(Z)}, \quad \langle N_i \rangle = t_i Z^{i-1} \langle N \rangle. \tag{7}$$

It is seen that the critical behavior is governed by the function

$$R(x) = 1 - \mathcal{F}'(x) = 1 - t_1 - 2t_2 x - 3t_3 x^2 - \dots \tag{8}$$

The equations

$$R(Z(t)) = 0, \quad Z(t) = \mathcal{F}(Z(t)), \tag{9}$$

where $Z(t) \geq 0$ and $t_i \geq 0$ will determine a critical surface in the space of coupling constants t_i in the sense that the "volume" $\langle N \rangle$ diverges when $R(Z(t)) \to 0$ for certain values of the coupling constant.

It is easily shown that $\mathcal{F}'(Z)$ has the interpretation as the generating function for branched polymers with a marked vertex of order 1 a link distance one from the root, and by iteration that $(\mathcal{F}'(Z))^n$ is the generating function for branched polymers with one marked vertex of order 1 separated a link distance n from the root. This function can be viewed as the two-point function $G(n)$ in the fractal geometry generated by the branched polymers [8] and we have

$$G(n) = e^{n \log \mathcal{F}'(Z)} \to e^{-R(Z)n} \tag{10}$$

for $R(Z) \to 0$. It is seen that $R(Z)$ acts like a mass, and the scaling limit is again controlled by $R(Z) \to 0$.

A generic point on the critical surface will correspond to branched polymers. In fact, let us consider a critical point t_i^c away from the $t_1 = 1$ boundary of the critical surface and introduce the notation $Z_c = Z(t^c)$. It follows that $Z_c > 0$ and $R'(Z_c) < 0$ for a solution to $R(Z_c) = 0$. Let us (for simplicity) approach the critical point from a point t away from the critical surface by a deformation $t_1 = t_1^c + \Delta t_1$, and $t_n = t_n^c$ for $n \neq 1$. To lowest order we have

$$\frac{1}{2} R'(Z_c)(\Delta Z)^2 + \Delta t_1 Z_c^i = 0, \tag{11}$$

which shows that ΔZ goes like $\sqrt{\Delta t_1}$, the critical behavior which characterizes branched polymers.

The structure of the critical surface becomes more transparent if we note that it follows from the definition of $Z(t)$ and (4), (5) that we can write

$$Z(t) = \frac{t_0}{1-t_1} H(\tilde{t}_2, \tilde{t}_3, \ldots,), \tag{12}$$

$$R(Z) = (1-t_1) K(\tilde{t}_2, \tilde{t}_3, \ldots), \tag{13}$$

where

$$\tilde{t}_n = \frac{t_n t_0^{n-1}}{(1-t_1)^n}, \quad n \geq 2, \tag{14}$$

$$K(\tilde{t}) = \left(1 - 2\tilde{t}_2 H(\tilde{t}) - 3\tilde{t}_3 H^2(\tilde{t}) - \cdots\right) \tag{15}$$

and where $H(\tilde{t}_2, \tilde{t}_3, \ldots)$ has a convergent power expansion in $\tilde{t}_i$'s around $\tilde{t}_i = 0$, with $H(0,0,\ldots) = 1$.

Eq. (13) shows that critical behavior (i.e. $R(Z) \to 0$) can be obtained if either $K(\tilde{t}) \to 0$ or $t_1 \to 1$. In the first case we have the branched polymer situation shown in (11). If $t_1 \to 1$ while the $\tilde{t}_n$'s are constant we have a new kind of critical behavior, which we denote *weakly branched polymers*. In the simplest case where all $t_n = 0$ for $n > 1$

$$Z(t) = \frac{t_0}{1-t_1}, \tag{16}$$

i.e. the critical behavior of a linear chain (a polymer) with a chemical potential t_1 per chain link (the vertices of order two). When $t_1 \to 1$ for constant $\tilde{t}_n$'s we have

$$\langle N \rangle = \frac{1}{K(\tilde{t})\,(1-t_1)}, \tag{17}$$

while $\langle N_n \rangle$ stay finite in the limit $t_1 \to 1$ for $n \neq 1$:

$$\langle N_0 \rangle = \frac{1}{H(\tilde{t}) K(\tilde{t})}, \quad \langle N_n \rangle = \frac{\tilde{t}_n H^{n-1}(\tilde{t})}{K(\tilde{t})}. \tag{18}$$

This limit has an interpretation in terms of polymers which are allowed to branch and break. Let p_0 be the probability per unit polymer length of breaking, $p_2, p_3, \ldots$ the probability per unit length of branching in $3, 4, \ldots$ linear pieces. If the polymers are made of pieces of length a we have

$$t_0 = a p_0, \quad t_2 = a p_2, \ldots, \quad 1 - t_1 = \sum_{i \neq 1} t_i. \tag{19}$$

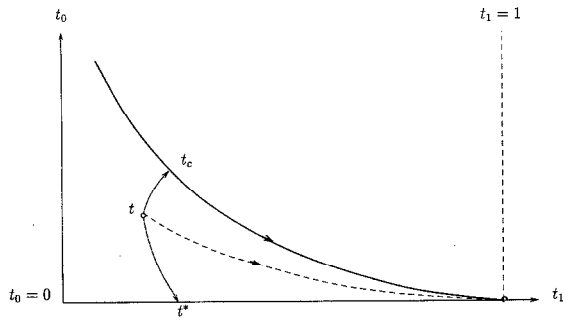

Fig. 1. The phase t_0, t_1 part of the phase diagram. The thick line symbolizes the critical surface for fixed t_n, $n > 1$. t is the initial choice of coupling constants. The flow to t^* is obtained by the renormalization group transformation and corresponds to a flow to a finite linear chain. The approach to t_c at the critical surface leads to the generic BP while the approach to the critical surface along the dashed line leads to the weakly branched polymers. Finally, the arrow at the critical surface indicates the flow *on* the critical surface towards $t_1 = 1$ under the action of the renormalization group transformations.

It is seen that the limit $a \to 0$ corresponds to weakly branched polymers since the $\tilde{t}_n$'s are only functions of the p_i's.

Eq. (19) describes the approach to the point $t = (0,1,0,0,\ldots)$ but the structure (13) is more general. Assume that the t_k's, $k > 1$ are non-negative constants and let $n > 1$ be the smallest integer for which $t_n > 0$, assuming it exists. In this case only $\tilde{t}_n$ survives in the limit $t_1 \to 1$ and

$$t_0 = c(1-t_1)^{\frac{n}{n-1}}. \tag{20}$$

This implies that $Z(t) \to 0$ and

$$\langle N_0 \rangle \to \text{const}, \quad \langle N_n \rangle \to \text{const}$$
$$\langle N_1 \rangle \to \frac{1}{1-t_1}, \tag{21}$$

while all other $\langle N_i \rangle$'s are either identically zero or approach zero for $t_1 \to 1$.

The different ways of approaching the critical surface are illustrated in Fig. 1. It is of course possible to fine-tune the approach to the critical surface such that both $t_1 \to 1$ and $K(\tilde{t}) \to 0$ and obtain a hybrid between weakly branched polymers and ordinary branched polymers. Note the different role of the coupling constants t_i, $i \geq 2$ at the boundary $t_1 = 1$ and for $t_1 < 1$ at the critical surface.

3. RG equations

In [2,3] the following procedure was suggested as a replacement for the conventional blocking of cells in real space renormalization group transformations in the case of fluctuating geometry: cut away the last generation of baby universe outgrowths, i.e. baby universes which are not themselves contained in any baby universes. This was viewed as a way to cut away fine structure details and in a general model with many coupling constants the new fractal structure should be obtained by a change in coupling constants. In this way successive cutting should induce a renormalization group flow in the coupling constant space.

Let us apply the construction to the BP-model. The vertices can be classified as *outer vertices*, i.e. the vertices of order one, and *inner vertices*. The marked vertex at the root has a special status and is not allowed to be touched. The renormalization group transformation consists of cutting away all outer vertices and their associated links. Every rooted tree can be obtained from another tree by this procedure, but they will have a new weight

$$\rho^{(1)}(\mathcal{T}) = \sum_{\mathcal{T}':\mathrm{RG}(\mathcal{T}')=\mathcal{T}} \rho^{(0)}(\mathcal{T}'), \tag{22}$$

where the sum is over all trees $\mathcal{T}'$ which after the transformation coincide with $\mathcal{T}$. The trees $\mathcal{T}'$ are generated by taking a tree $\mathcal{T}$ and i) adding one or more outer vertices (and associated links) to all outer vertices in T (which then cease to be outer vertices in T') and ii) adding zero or more outer vertices (and associated links) as neighbors to the inner vertices. Notice that the trees T' will always be different from the *root term* consisting of a single link with two vertices, the marked one and an outer vertex. This means that $\rho^{(1)}$ will not contain the term t_0.

It is not too difficult to convince oneself that

$$\rho^{(1)}(\mathcal{T}) = (t_0^{(1)}/t_0)^{n_0}(t_1^{(1)}/t_1)^{n_1}\ldots\rho^{(0)}(\mathcal{T}), \tag{23}$$

which corresponds to the following redefinition of the coupling constants t_i

$$t_0^{(1)} = \mathcal{F}(t;x)|_{x=t_0} - t_0,$$
$$t_l^{(1)} = \frac{1}{l!}\frac{\partial^l \mathcal{F}(t;x)}{\partial x^l}|_{x=t_0}, \quad l > 0. \tag{24}$$

These redefinitions can be put in a RG equation:

$$Z(\{t_i\}) - t_0 = Z(\{t_i^{(1)}\}). \tag{25}$$

Observe that the transformation (24) corresponds to the following redefinition of Z:

$$Z^{(1)} = Z - t_0,$$
$$Z^{(1)} = \mathcal{F}(t;x)|_{x=t_0+Z^{(1)}} - \mathcal{F}(t;x)|_{x=t_0}$$
$$= \mathcal{F}^{(1)}(Z^{(1)}), \tag{26}$$

where $F^{(1)}(Z^{(1)})$ is defined as $\mathcal{F}(Z)$, just using $t^{(1)}$ and $Z^{(1)}$ variables. It follows by definition that

$$\mathcal{F}'(t;x)|_{x=Z} = \mathcal{F}^{(1)\prime}(t;x)|_{x=Z^{(1)}}, \tag{27}$$

where differentiation is with respect to x. According to (7) $\langle N\rangle$ depends only on $\mathcal{F}'(x)$ and is consequently an invariant under the renormalization group transformation.

Eqs. (26) will be the central point of this discussion. They can be used to study the flow of the coupling constants t_i under the action of RG. We can iterate (26) and obtain the following recursion relation

$$\mathcal{F}^{(k+1)}(x) = \mathcal{F}^{(k)}(\mathcal{F}^{(k)}(0) + x) - \mathcal{F}^{(k)}(0),$$
$$\mathcal{F}^{(0)}(x) = \mathcal{F}(x). \tag{28}$$

The corresponding changes in Z are

$$Z^{(k+1)} = Z^{(k)} - \mathcal{F}^{(k)}(0),$$
$$Z^{(0)} = Z. \tag{29}$$

where $Z^{(k)}$ satisfies

$$Z^{(k)} = \mathcal{F}^{(k)}(Z^{(k)}). \tag{30}$$

Eqs. (29) and (30) can be solved

$$\mathcal{F}^{(k)}(x) = \mathcal{F}(\alpha_k + x) - \alpha_k,$$
$$\alpha_{k+1} = \mathcal{F}(\alpha_k),$$
$$\alpha_0 = 0. \tag{31}$$

Similarly

$$Z^{(k)} = Z - \alpha_k. \tag{32}$$

The fixed point of these equations is

$$\mathcal{F}^{(k)}(x) \to \mathcal{F}^*(x) = \mathcal{F}(\alpha^* + x) - \alpha^*,$$

$$Z^{(k)} \to Z^* = Z - \alpha^*, \tag{33}$$

where α^* satisfies

$$\alpha^* = \mathcal{F}(\alpha^*). \tag{34}$$

If we start out with an arbitrary set of (non-negative) coupling constants t_i the partition function will either be divergent ($Z = \mathcal{F}(Z)$ has no solution) or it will have one or two solutions. In case it has one solution $R(Z) = 0$ and we are at the critical surface, but generically it will have two solutions, of which we should choose the one with the smallest value of Z. Applying the transformation group the coupling constants will flow to t^*, characterized by α^*. However, we know from (34) and (4) that

$$\alpha^* = Z(t). \tag{35}$$

In other words

$$Z^* = 0,$$
$$\mathcal{F}^*(x) = \sum_{l=1}^{\infty} t_l^* x^l,$$
$$t_0^* = 0, \tag{36}$$

where the coefficients t_l^* are the Taylor expansion coefficients of $\mathcal{F}(x)$ around a point $x = \alpha^* = Z(t)$

$$t_l^* = \frac{1}{l!}\frac{\partial^l \mathcal{F}(x)}{\partial x^l}|_{x=\alpha*}. \tag{37}$$

The $k \to \infty$ limit of (30) and (31) can be described by putting

$$\alpha_k = \alpha^* - \varepsilon_k. \tag{38}$$

Close to the fixed point we get from $\alpha_{k+1} = \mathcal{F}(\alpha_k)$

$$\varepsilon_{k+1} \approx \mathcal{F}'(\alpha^*)\varepsilon_k = \left(1 - \frac{1}{\langle N\rangle}\right)\varepsilon_k. \tag{39}$$

In addition we have

$$Z^{(k)} = \varepsilon_k$$
$$t_0^{(k)} = \mathcal{F}(\alpha_k) - \alpha_k \approx \frac{\varepsilon_k}{\langle N\rangle}, \tag{40}$$

and in effect

$$\langle N_0^{(k)}\rangle = \frac{t_0^{(k)}}{Z^{(k)}}\langle N\rangle \to 1, \tag{41}$$

for $k \to \infty$. It is easy to check that

$$\langle N_i^{(k)}\rangle \to 0, \tag{42}$$

for $i > 1$ since $Z^{(k)} \to 0$, while t_l^* and $\langle N\rangle$ are fixed under repeated application of the renormalization transformation. It is clear that we simply end up with a linear chain length $\langle N\rangle$. The flow is shown in Fig. 1.

4. Discussion

The flow of coupling constants under the action of the renormalization group transformation used here is such that we move *away* from the critical line corresponding to branched polymers and directly to a linear chain of the same volume. The usual situation when applying the renormalization group transformations starting at a point in coupling constant space close to the critical surface is that one moves towards the critical surface in the first couple of steps since the irrelevant operators will dominate the blocking in the first few iterations. Eventually, after repeated applications of the renormalization group transformations one moves away from the critical surface in the direction dictated by the most relevant operator. In the BP-model all coupling constants t_n, $n > 1$ correspond to relevant couplings and repeated application immediately move us away from the critical surface. The only trace of being close to critical surface is that the approach to t^* will be slower as $\langle N\rangle$ increases. If we place ourself *at* the critical surface the iteration of the renormalization group will not remove us away from this surface and we will move towards the critical point $t_1^* = 1$. The approach to t^* will not be exponentially fast in the number of iterations k, as was the case for finite $\langle N\rangle$ in Eqs. (39) and (40). Rather, it is replaced by a power approach in k:

$$\varepsilon_k \approx \frac{2}{-R'(\alpha^*)k}, \quad t_0^{(k)} \approx -\frac{1}{2}R'(\alpha^*)\varepsilon_k, \tag{43}$$

which results in the following behavior

$$\frac{\langle N_0^{(k)}\rangle}{\langle N\rangle} \approx \frac{1}{k}, \quad \frac{\langle N_2^{(k)}\rangle}{\langle N\rangle} \approx \frac{1}{k},$$
$$\frac{\langle N_i^{(k)}\rangle}{\langle N\rangle} = O((1/k)^{i-1}). \tag{44}$$

Approaching the critical point $t_1^* = 1$ the branched polymer becomes more and more like a weakly branched polymer of the simplest kind, i.e. corresponding to $n = 2$.

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