Covariant w_∞ gravity and its reduction to W_N gravity

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By gauging the linear w_∞ algebra, imposing constraints on the curvatures and choosing suitable gauges, we obtain the covariant formulation of w_∞ gravity. This theory is coupled to matrix valued scalar fields, where the set of matrices is closed under symmetrized products. For the case of the vector representation of $u(N)$, the theory can be reduced to W_N gravity. We present closed expressions for the reduction to a covariant theory of W_3 gravity which we obtained previously.

Recently, a new gauge theory for the W_3 algebra was constructed in refs. [1,2]. The W_3 algebra is an example of a nonlinear algebra, in which products of generators appear on the right-hand side of the fundamental brackets. Gauge theories of this type go beyond the gauge theories for linear algebras, such as Yang–Mills and conventional gravity and supergravity theories, and could have interesting applications in particle physics. For example, in the context of supersymmetry, nonlinear algebras offer a mechanism which avoids supersymmetry breaking and still explains why supersymmetric theories do not have a supersymmetric spectrum [3].

The W_N algebras contain the Virasoro generators L_m and higher spin generators, and lead to theories of gravity which may be called "W gravities". Our work is intended to form a suitable starting point for the description of quantum W gravities [4] and, eventually, W strings [5].

After the construction of W_3 gravity in the chiral gauge [1] and the light-cone gauge [2], we derived by a systematic procedure a covariant formulation of W_3 gravity [6], which reproduced the previous formulations in refs. [1,2] and allows one to analyze the theory, in particular its anomalies, in the conformal gauge and in other interesting gauges. Another approach to classical W gravity was advocated in ref. [7], where it was proposed to view the W_N gravities as reductions of a w_∞ gravity. The latter corresponds to the w_∞ algebra which is, however, a linear algebra. The authors of ref. [7] constructed w_∞ gravity in the light-cone gauge, and indicated how the reduction to W_N gravity could be achieved.

In this letter we return to w_∞ gravity [7] and its reductions to W_N gravity. First we construct a covariant formulation of w_∞ gravity by gauging the w_∞ algebra, which does not have the complications that go with the gauging of nonlinear algebras. We couple the gauge fields corresponding to the generators of w_∞ to scalar fields that take values in an algebra of matrices, $\varphi=\varphi^I T_I$. The set of matrices T_I must be closed under the operation of taking symmetric products. Our final action for w_∞ is given by $I_\infty=\int \mathscr{L}_\infty$, where

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$$\mathscr{L}_\infty = -\tfrac{1}{2}e\,\mathrm{tr}(\partial_+\varphi\partial_-\varphi) - e\,\mathrm{tr}(F_+F_-)$$
$$+e\,\mathrm{tr}\Big(F_+\Big(\partial_-\varphi - \sum_{j\geqslant 3}\frac{1}{j}e_-{}^{(j-1)}(F_+)^{j-1}\Big)\Big) + e\,\mathrm{tr}\Big(F_-\Big(\partial_+\varphi - \sum_{j\geqslant 3}\frac{1}{j}\bar{e}_+{}^{(j-1)}(F_-)^{j-1}\Big)\Big)$$
$$-e\sum_{j\geqslant 3}e_+{}^{(j-1)}\,\mathrm{tr}\Big((F_+)^{j-1}\Big(F_- - \partial_-\varphi + \tfrac{1}{2}\sum_{k\geqslant 3}\bar{e}_-{}^{(k-1)}(F_-)^{k-1} + \sum_{k\geqslant 3}\frac{j-1}{j+k-2}e_-{}^{(k-1)}(F_+)^{k-1}\Big)\Big)$$
$$-e\sum_{j\geqslant 3}\bar{e}_-{}^{(j-1)}\,\mathrm{tr}\Big((F_-)^{j-1}\Big(F_+ - \partial_+\varphi + \tfrac{1}{2}\sum_{k\geqslant 3}e_+{}^{(k-1)}(F_+)^{k-1} + \sum_{k\geqslant 3}\frac{j-1}{j+k-2}\bar{e}_+{}^{(k-1)}(F_-)^{k-1}\Big)\Big). \tag{1}$$

The gauge fields are $e_\mu{}^{(j-1)}$ and $\bar{e}_\mu{}^{(j-1)}$ in the two chiral sectors with $j\geqslant 2$. The vielbein fields $e_\mu{}^+ \equiv e_\mu{}^{(1)}$ and $e_\mu{}^- \equiv \bar{e}_\mu{}^{(1)}$ (with $e = \det e_\mu{}^\pm$) are used to convert curved to flat indices (as in $e_+{}^{(j-1)}$), and F_+, F_- are 1.5 order fields [8] corresponding to $\partial_+\varphi$ and $\partial_-\varphi$, which are also matrix valued: $F_+ = F_+{}^I T_I$.

In the light-cone gauge, our action and transformation rules reduces to expressions which are similar but not identical to those of ref. [7]. The difference can be traced back to a different treatment of the gravitational gauge field $e_\mu{}^{(1)}$, $\bar{e}_\mu{}^{(1)}$.

For the reduction to W_N, we found it necessary to start from $\mathrm{u}(N)$ valued scalar fields. In this we differ from the approach in ref. [7] which uses $\mathrm{su}(N)$ valued fields. We find that the elimination of the scalar field φ^0 corresponding to the extra $\mathrm{U}(1)$ generator contributes in an essential way to the nonlinear extra terms in the W_N transformation rules. For the case of W_3 we find complete agreement with our results in ref. [6].

To obtain the action for w_∞ gravity, we proceed as in refs. [9,10,6], and gauge the classical w_∞ algebra. The latter is given by [11]

$$[t_m{}^j, t_n{}^k] = [(j-1)n - (k-1)m]\,t_{m+n}{}^{j+k-2}\,. \tag{2}$$

We consider henceforth two commuting copies of the subalgebra with j, $k\geqslant 2$ and $m\geqslant -j+1$, $n\geqslant -k+1$. Introducing ζ^+ or ζ^- dependent gauge fields $e_\mu{}^{(j-1)}$, $\bar{e}_\mu{}^{(j-1)}$ and local parameters $k^{(j-1)}$ and $\bar{k}^{(j-1)}$ as in $e_\mu{}^{(j-1)} = \sum_{m\geqslant -j+1} e_\mu{}^{(j-1),m}(x^+,x^-)(\zeta^+)^{m+j-1}$, we obtain for the gauge field transformation laws

$$\delta e_\mu{}^{(j-1)}(x^+,x^-,\zeta^+) = \partial_\mu k^{(j-1)} + \sum_{l=2}^{j}[(l-1)k^{(l-1)}\partial_{\zeta^+}e_\mu{}^{(j-l+1)} - (j-l+1)\partial_{\zeta^+}k^{(l-1)}e_\mu{}^{(j-l+1)}],\quad j\geqslant 2\,. \tag{3}$$

The result for $\delta\bar{e}_\mu{}^{(j-1)}(x^+,x^-,\zeta^-)$ is similar. (We shall from now on understand without mention that all steps are taken in both chiral sectors.) Note that $e_\mu{}^{(j-1)}$ varies only into $e_\mu{}^{(l-1)}$ with $l\leqslant j$ and that the vielbein $e_\mu{}^{(1)} \equiv e_\mu{}^+$ varies only under $k^{(1)}$ and $\partial_{\zeta^+}k^{(1)}$.

Next we evaluate the curvatures $R_{\mu\nu}{}^{(j-1)}$, $j\geqslant 2$, and impose the constraint that they all vanish. This leads to

$$e_-{}^\mu\partial_{\zeta^+}e_\mu{}^{(j-1)} \equiv A^{(j-1)}$$
$$= e_-{}^\mu\partial_+e_\mu{}^{(j-1)} - e_+{}^\mu\partial_-e_\mu{}^{(j-1)} + \sum_{l=2}^{j}(j-l+1)(e_+{}^\mu\partial_{\zeta^+}e_\mu{}^{(l-1)})e_-{}^{(j-l+1)} - \sum_{l=3}^{j}(l-1)e_+{}^{(l-1)}A^{(j-l+1)}\,. \tag{4}$$

Choosing in addition the gauge where all $e_+{}^\mu\partial_{\zeta^+}e_\mu{}^{(j-1)}$ vanish for $j\geqslant 2$ completely fixes the ζ^+ dependence of $e_\mu{}^{(j-1)}$ and determines $\partial_{\zeta^+}e_\mu{}^{(j-1)}$ in terms of the $e_\mu{}^{(k-1)}$ with $k<j$. For $j=2$ we have $A^{(1)} \equiv \mathscr{E}^+ = e_-{}^\mu\partial_+e_\mu{}^{(1)} - e_+{}^\mu\partial_-e_\mu{}^{(1)} = \omega_-$, and for $j=3$ we obtain $A^{(2)} \equiv \mathscr{B}^{++} = (e_-{}^\mu\partial_+e_\mu{}^{(2)} - e_+{}^\mu\partial_-e_\mu{}^{(2)}) - 2e_+{}^{(2)}\mathscr{E}^+$. These expressions agree with ref. [6] in the linear limit.

The residual gauge degrees of freedom after choosing the above gauge conditions, are parametrized by the values of $k^{(j-1)}$ and $\partial_{\zeta^+}k^{(j-1)}$ at $\zeta^+=0$, which we will denote by $k^{(j-1)}$ and $k_{\mathrm{W}}{}^{(j-1)}$. Note that $k_{\mathrm{W}}{}^{(1)}$ and $\bar{k}_{\mathrm{W}}{}^{(1)}$ denote the ordinary local Weyl and local Lorentz symmetries.

Matter is coupled as in ref. [6]: we introduce a set of matrix valued fields

$$\varphi = \sum \varphi^I(x^+, x^-, \zeta^+, \zeta^-) T_I , \tag{5}$$

which transform as $\delta\varphi = \sum\{k^{(l-1)}(\partial_{\zeta^+}\varphi)^{l-1} + \bar{k}^{(l-1)}(\partial_{\zeta^-}\varphi)^{l-1}\}$, with $l \geqslant 2$. We impose the constraint that their covariant derivatives $D_\mu\varphi$ vanish, where

$$D_\mu\varphi = \partial_\mu\varphi - \sum_{l=2}^{\infty} e_\mu{}^{(l-1)}(\partial_{\zeta^+}\varphi)^{l-1} - \sum_{l=2}^{\infty} \bar{e}_\mu{}^{(l-1)}(\partial_{\zeta^-}\varphi)^{l-1} . \tag{6}$$

This allows us to solve for $\partial_{\zeta^\pm}\varphi$. Denoting $\partial_{\zeta^\pm}\varphi$ by $F_\pm$, we obtain the constraints

$$F_+ = \partial_+\varphi - \sum_{l\geqslant 3} e_+{}^{(l-1)}(F_+)^{l-1} - \sum_{l\geqslant 3} \bar{e}_+{}^{(l-1)}(F_-)^{l-1} ,$$

$$F_- = \partial_-\varphi - \sum_{l\geqslant 3} \bar{e}_-{}^{(l-1)}(F_-)^{l-1} - \sum_{l\geqslant 3} e_-{}^{(l-1)}(F_+)^{l-1} . \tag{7}$$

By interpreting these expressions at $\zeta^\pm = 0$ as field equations we find a corresponding action, where we use an integrating factor (compare with ref. [6]). The integrating "factor" is such that the F_- field equation reads $\bar{F}_+ + \sum_{l\geqslant 3}(\bar{e}_-{}^{(l-1)}\sum_{k=0}^{l-2} F_-{}^{l-2-k}\bar{F}_+ F_-{}^k)$, where $\bar{F}_+ = F_+ - \partial_+\varphi + \ldots$ as in (7). This field equation, and its partner for F_+, satisfies the integrability condition and are used to construct a corresponding action. The result is the action given in (1). If we would be dealing with commuting matrices the field equations for $F_\pm$ would really factorize, but also in the general case it is ensured that they are equivalent to $\bar{F}_\pm = 0$.

A basic difference between our field equations (7) and those of ref. [7] is that the latter authors use the gravitational ($j=2$) covariantization $\delta_+{}^\mu\partial_\mu\varphi - h^{--}F_-$ rather than $\delta_+{}^\mu\partial_\mu\varphi - h^{--}\partial_-\varphi$ (for comparison we give both in the light-cone gauge, compare with ref. [2]). In the approach of ref. [7] one still obtains gravitational covariance after solving for F_+ and F_-, but the covariance is not manifest as is apparent from the fact that there is no vielbein determinant in front of the action; instead, the gravitational fields are treated on a par with the other gauge fields.

Our final theory is now formulated in terms of ζ independent fields. For the transformation rules for φ we obtain $\delta\varphi = \sum k^{(l-1)}(F_+)^{l-1} + \bar{k}^{(l-1)}(F_-)^{l-1}$. The gauge field rules were already obtained in (3), (4). For the proof of the invariance of the action we found it convenient to use rules which are manifestly covariant w.r.t. ordinary gravity ($j=2$). Choosing suitable linear combinations of the original rules yields

$$\delta e_\mu{}^{(j-1)} = \nabla_\mu k^{(j-1)} + \sum_{l=2}^{j} [(l-1)k^{(l-1)}\bar{e}_\mu{}^{(1)}A_{\rm cov}^{(j-l+1)} - (j-l+1)k_{\rm W}{}^{(l-1)}e_\mu{}^{(j-l+1)}] , \tag{8}$$

where $\nabla_\mu k^{(j-1)} = \partial_\mu k^{(j-1)} - (j-1)\omega_\mu k^{(j-1)}$, with ω_μ the spin connection. We have $A_{\rm cov}^{(1)} = 0$ and

$$A_{\rm cov}^{(j-1)} = \nabla_+ e_-{}^{(j-1)} - \nabla_- e_+{}^{(j-1)} - \sum_{l=3}^{j} (l-1)e_+{}^{(l-1)}A_{\rm cov}^{(j-l+1)} \tag{9}$$

for $j \geqslant 3$, where $\nabla_+ e_-{}^{(j-1)} = \partial_+ e_-{}^{(j-1)} - j\omega_+ e_-{}^{(j-1)}$. Using these transformation rules, and the 1.5 order rules in (7), one easily proves the invariance of the action in (1).

Having found the completely covariant action and transformation rules for w_∞ gravity, we will now discuss how this result can be used to derive similar results for W_N gravity, when N is some integer $\geqslant 3$. We discuss in some detail the case $N=3$, where we will make contact with the results obtained in ref. [6]. The theories with $N \geqslant 3$ can be recovered in a completely similar way.

For the reduction to W_3 gravity we start from scalar fields φ that take values in the algebra u(3) according to

$$\varphi = \varphi^0\sqrt{2}\,I + \sum_{i=1}^{8} \varphi^i T^i . \tag{10}$$

Our conventions for the hermitean matrices T^i, $i = 1, \ldots, 8$, are such that

$$T^{(i}T^{j)}=2\delta^{ij}I+d^{ijk}T^k\,, \tag{11}$$

where the d-symbols d^{ijk} are completely symmetric and satisfy

$$d^{m(ij}d^{kl)m}=\delta^{(ij}\delta^{kl)}\,. \tag{12}$$

As a consequence, we have the trace identities

$$\mathrm{tr}(T^{(i}T^{j)})=6\delta^{ij}\,,\quad \mathrm{tr}(T^{(i}T^jT^{k)})=6d^{ijk}\,,\quad \mathrm{tr}(T^{(i}T^{j)}T^{(k}T^{l)})=12\delta^{ij}\delta^{kl}+6d^{ijm}d^{klm}\,,\quad \text{etc.} \tag{13}$$

The set of matrix valued fields of the form (10) is closed under symmetrized products (notice that this would no longer be true if we would restrict to su(3) valued fields). Substitution into the action (1) and the transformation rules leads to a theory of w_∞ gravity in terms of 9 scalar fields φ^0, φ^i, $i=1,...,8$. The idea is now that this theory can be reduced to a theory of W_3 gravity with only the scalar fields φ^i, $i=1,...,8$. The correct transformation rules for the W_3 theory will follow from the requirement that the reduced action be gauge invariant. We will see that this requirement leads to certain nonlinear corrections to the transformation rules of the W_3 gauge fields $e_\mu{}^{(1)}$, $\bar{e}_\mu{}^{(1)}$, $e_\mu{}^{(2)}$ and $\bar{e}_\mu{}^{(2)}$. In ref. [6] the same terms were derived by a completely different procedure, namely by solving the curvature constraints for the nonlinear algebra.

Let us now consider the reduction to W_3 in some detail. Starting from the w_∞ theory we impose the constraints

$$e_\mu{}^{(j-1)}=\bar{e}_\mu{}^{(j-1)}=0\,,\quad j\geqslant 3\,,\qquad \varphi^0=0\,,\quad F_\pm{}^0=0\,, \tag{14}$$

which we will denote by $(*)$. At this point working with the 1.5 order formalism for the fields $F_\pm$ would lead to inconsistencies since the traces of the relations (7) do not vanish after restriction. We therefore use a completely first order formalism where the fields F_+ should give the following contribution to the variation of I_∞ under transformations with $k^{(j-1)}$, $k_W{}^{(j-1)}$ in order that the first order action be gauge invariant

$$\frac{\delta I_\infty}{\delta F_+}\,\delta(k^{(j-1)}\,,k_W{}^{(j-1)})F_+=\left(-F_-+\partial_-\varphi-\sum_{l\geqslant 3}\bar{e}_-{}^{(l-1)}(F_-)^{l-1}-\sum_{l\geqslant 3}e_-{}^{(l-1)}(F_+)^{l-1}\right)$$
$$\times\left(k^{(j-1)}\nabla_+(F_+{}^{(j-1)})+k_W{}^{(j-1)}\sum_{l\geqslant j}(l+1-j)e_+{}^{(l+1-j)}(F_+)^{l-1}\right), \tag{15}$$

with a similar contribution for the $\bar{k}^{(j-1)}$ and $\bar{k}_W{}^{(j-1)}$ variations through F_-.

We wish to perform the truncation $(*)$ in such a way that we do not lose the invariance under W_3 gauge transformations with parameters $k^{(2)}$, $\bar{k}^{(2)}$, $k_W{}^{(2)}$ and $\bar{k}_W{}^{(2)}$. Focussing on $k^{(2)}$ we first write the identity

$$\begin{aligned}0&=(\delta(k^{(2)})I_\infty|_*\\ &=e\,\mathrm{tr}([\nabla_+(\partial_-\varphi-F_+-\bar{e}_-{}^{(2)}F_-{}^2)-\nabla_-(F_++e_+{}^{(2)}F_+{}^2)]k^{(2)}F_+{}^2)\quad(\text{from }\varphi)\\ &\quad+e\,\mathrm{tr}((-F_-+\partial_-\varphi-\bar{e}_-{}^{(2)}F_-{}^2-e_-{}^{(2)}F_+{}^2)k^{(2)}\nabla_+(F_+{}^2))\quad(\text{from }F_+)\\ &\quad+e(\nabla_+k^{(2)})\,\mathrm{tr}(F_+{}^2(-F_-+\partial_-\varphi-\bar{e}_-{}^{(2)}F_-{}^2-\tfrac{1}{2}e_-{}^{(2)}F_+{}^2))\quad(\text{from }e_+{}^{(2)})\\ &\quad+e(\nabla_-k^{(2)})\,\mathrm{tr}(-\tfrac{1}{3}F_+{}^3-\tfrac{1}{2}e_+{}^{(2)}F_+{}^4)\quad(\text{from }e_-{}^{(2)})\\ &\quad-\tfrac{1}{2}e\,\mathrm{tr}(F_+{}^4)k^{(2)}A^{(2)}_{\mathrm{cov}}\quad(\text{from }e_-{}^{(j-1)}\,,j\geqslant 4)\,,\end{aligned} \tag{16}$$

where the contribution from the higher gauge fields $e_-{}^{(j-1)}$, $j\geqslant 4$, was derived by evaluating the infinite sum

$$\sum_{j\geqslant 4}\frac{\delta I_\infty}{\delta e_-{}^{(j-1)}}\bigg|_*(\delta(k^{(2)})e_-{}^{(j-1)})|_*=\sum_{j\geqslant 4}\left(-\frac{e}{j}\,\mathrm{tr}(F_+{}^j)-e_+{}^{(2)}\frac{2e}{j+1}\,\mathrm{tr}(F_+{}^{j+1})\right)[2k^{(2)}(-2e_+{}^{(2)})^{j-4}A^{(2)}_{\mathrm{cov}}]\,, \tag{17}$$

and we used that

$$A_{\rm cov}^{(j-1)}|_* = (-2e_+{}^{(2)})^{j-3} A_{\rm cov}^{(2)} . \tag{18}$$

Before we proceed, we would like to identify the correct nested covariant derivatives $(\hat{\partial}_\pm \varphi)^i$, which we obtain by simply extracting the ith component of the relations (7)$|_*$ and identifying $(\hat{\partial}_\pm \varphi)^i$ with $F_\pm{}^i$. We find

$$F_+{}^i = \partial_+ \varphi^i - e_+{}^{(2)} d^{ijk} F_+{}^j F_+{}^k - \bar{e}_+{}^{(2)} d^{ijk} F_-{}^j F_-{}^k , \tag{19}$$

with a similar relation for $F_-{}^i$. Since we expect the $F_\pm{}^i$ field equations to be covariant on $F_\pm{}^i$-shell, these *must* be the $F_\pm{}^i$ field equations modulo some integrating factor (the latter does not affect the covariance of the $F_\pm{}^i$ field equations as long as we are on $F_\pm{}^i$ shell).

We now remark that the relations (19) are *not* compatible with the truncated action $I_\infty|_*$, in the sense that the variations $\delta(I_\infty|_*)/\delta F_\pm{}^i$ do not vanish when the relations (19) are satisfied. We therefore propose the following action for the 8-scalar model of W_3 gravity:

$$I_3 = I_\infty|_* + 12 \int e\, e_+{}^{(2)} \bar{e}_-{}^{(2)} T_{++} T_{--} + 6 \int e(e_-{}^{(2)} e_+{}^{(2)} T_{++} T_{++} + \bar{e}_+{}^{(2)} \bar{e}_-{}^{(2)} T_{--} T_{--}) , \tag{20}$$

where we introduced the notation $T_{\pm\pm} = F_\pm{}^i F_\pm{}^i$. This action is chosen in such a way that the variations $\delta I_3/\delta F_\pm{}^i$ vanish if $F_\pm{}^i$ satisfy (19), and is obtained from the restriction of (1) by dropping the contributions that arise from the term $12\delta^{ij}\delta^{kl}$ in (13).

The variation of I_3 under the truncated w_∞ rules is easily obtained by using the result (16). We will evaluate this variation in a 1.5 formalism for the fields $F_\pm{}^i$, which simply means that we will use the relations (19). We find

$$\begin{aligned}
\delta(k^{(2)}) I_3 &= \delta(k^{(2)}) I_3 - (\delta(k^{(2)}) I_\infty)|_* \\
&= 12 e k^{(2)} T_{++} [\nabla_+ (\bar{e}_-{}^{(2)} T_{--}) + \nabla_- (e_+{}^{(2)} T_{++})] \quad (\text{from } \varphi^0) \\
&\quad + 12 e k^{(2)} (\nabla_+ T_{++})(\bar{e}_-{}^{(2)} T_{--} + e_-{}^{(2)} T_{++}) \quad (\text{from } F_+) \\
&\quad + 12 e (\nabla_+ k^{(2)}) T_{++} (\bar{e}_-{}^{(2)} T_{--} + \tfrac{1}{2} e_-{}^{(2)} T_{++}) \quad (\text{from } e_+{}^{(2)}) \\
&\quad + 6 e (\nabla_- k^{(2)}) T_{++} T_{++} e_+{}^{(2)} \quad (\text{from } e_-{}^{(2)}) \\
&\quad + 9 e k^{(2)} T_{++} T_{++} A_{\rm cov}^{(2)} \quad (\text{from } e_-{}^{(j-1)}, j \geqslant 4) \\
&= 3 e k^{(2)} T_{++} T_{++} A_{\rm cov}^{(2)} .
\end{aligned} \tag{21}$$

We now observe that this variation of I_3 under $k^{(2)}$ gauge transformations can be compensated by adding extra terms to the transformation laws of the W_3 gauge fields. Using the relation [6]

$$\frac{e}{2} T_{++} = \frac{1}{6} \frac{1}{1-2y} \left(e_+{}^\mu \frac{\delta I_3}{\delta e_-{}^\mu} + 3 e_+{}^{(2)} \frac{\delta I_3}{\delta e_-{}^{(2)}} + \bar{e}_+{}^{(2)} \frac{\delta I_3}{\delta \bar{e}_-{}^{(2)}} \right) , \tag{22}$$

where we defined $y = (e_+{}^{(2)})^2 T_{++}$, we derive the following extra terms in the $k^{(2)}$ transformations of the W_3 gauge fields:

$$\begin{aligned}
&\delta_{\rm extra} e_\mu{}^{(1)} = \frac{1}{1-2y} \bar{e}_\mu{}^{(1)} k^{(2)} T_{++} A_{\rm cov}^{(2)} , \quad \delta_{\rm extra} e_\mu{}^{(2)} = \frac{-2}{1-2y} \bar{e}_\mu{}^{(1)} k^{(2)} T_{++} e_+{}^{(2)} A_{\rm cov}^{(2)} , \\
&\delta_{\rm extra} \bar{e}_\mu{}^{(1)} = \delta_{\rm extra} \bar{e}_\mu{}^{(2)} = 0 .
\end{aligned} \tag{23}$$

With these terms added to the transformation rules the action I_3 is now gauge invariant.

In a similar way one reconstructs the complete transformation rules for gauge transformations $k^{(1)}$ (coordinate transformations), $k_{\rm W}{}^{(1)}$ (Weyl and Lorentz) and $k_{\rm W}{}^{(2)}$ (W_3 Weyl and Lorentz). These rules agree with the results of ref. [6], which were obtained in a different approach (for the $k_{\rm W}{}^{(2)}$ transformation the agreement is

up to a redefinition of the parameters: $\lambda^+ = [1/(1-2y)]k_W{}^{(2)}$, $k = [-1/(1-2y)]k_W{}^{(2)}e_+{}^{(2)}T_{++}$).

In the reduction from w_∞ to W_N gauge theories, we put φ^0, F^0 and the higher gauge fields equal to zero by hand. This suggests to look for an action whose field equations yield these constraints. A natural candidate is the gauge theory for $w_{1+\infty}$. The algebra of W_3 can be retrieved from the algebra of $w_{1+\infty}$ by considering operators $T^{(j)} = (1/2j)\ \mathrm{tr}(\partial_z\varphi)^j$ where φ is u(3) valued, and identifying $\partial_z\varphi^0$ with the u(1) current of the $w_{1+\infty}$ algebra. (Strictly speaking, one finds a central term in the $[U_m, U_n]$ commutator proportional to $m\delta_{m+n}$. However, for the process of gauging, we expect that only the truncation for $m \geqslant 0$ is needed, in which case the central term is not present.) By expanding the φ products into terms with φ^0 and terms independent of φ^0, one obtains an explicit rule how the generators of W(3) are constructed from the generators of $w_{1+\infty}$ (this construction is reminiscent of ideas discussed in ref. [12]). It could be interesting to extend these ideas to the corresponding gauge theories.

We finally mention that the constructed gauge multiplets for W_3 and w_∞ as presented here and in ref. [6] appear to be appropriate starting points for the analysis of topological quantum W gravity along the lines of ref. [13].

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