A new gauge theory for W-type algebras ☆

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The rigid conformal and rigid W_3-type symmetries of the action for scalar fields in $d=2$ dimensions are promoted to local gauge invariances of both chiralities. The complete action and transformation rules are given; both are infinitely nonlinear.

In string theory, one may start with a real scalar field on a flat $d=2$ worldsheet with action

$$\mathscr{L}=\tfrac{1}{2}\partial_+\varphi\,\partial_-\varphi\,, \tag{1}$$

and add the constraints $T_{++}\equiv\partial_+\varphi\,\partial_+\varphi=0$ and $T_{--}=0$. They generate left- and right-handed rigid conformal symmetries, $\delta\varphi=k^\pm\,\partial_\pm\varphi$ with $\partial_-k^+=\partial_+k^-=0$. The quantum algebra follows then from the operator product expansion (OPE), and consists of two commuting copies of the Virasoro algebra. The BRST charge Q has the standard form and is nilpotent if the central charge of the matter sector has the value $c=26$. This can be achieved by taking 26 copies of (1), and choosing as constraints $T_{\pm\pm}=\sum_{i=1}^{26}T^i_{\pm\pm}=0$.

By introducing n real scalar fields $\varphi^i(i=1,\ ...,\ n)$ with action as in (1), and defining the constraints $W_{+++}\equiv\partial_+\varphi^i\,\partial_+\varphi^j\,\partial_+\varphi^k\,d^{ijk}=0$ and $W_{---}=0$, one obtains a theory with a current algebra that is reminiscent of the W_3-algebra [1]. The constraints produce left- and right-handed rigid symmetries of the action, $\delta\varphi^i=\lambda^{++}\,\partial_+\varphi^j\,\partial_+\varphi^k\,d^{ijk}$ and idem with λ^{--} where $\partial_-\lambda^{++}=\partial_+\lambda^{--}=0$. Classically, the rigid algebra of conformal and W-transformations on φ closes, but with φ-dependent structure constants, provided the totally symmetric d^{ijk} satisfy [2]

$$d^{k(ij}d^{l)mk}=\delta^{(ij}\delta^{l)m}\,. \tag{2}$$

Quantum mechanically, the requirement that $W_{\pm\pm\pm}$ be primary fields of spin 3 requires that the d symbol be traceless. The OPE of the properly normalized W-current,

$$\tilde{W}_{+++}=N\,\partial_+\varphi^i\,\partial_+\varphi^j\,\partial_+\varphi^k\,d^{ijk}\quad\text{with } N^{-1/2}=-9(n+2)\,, \tag{3}$$

yields, using (2) and $d^{ijk}=0$,

$$\begin{aligned}\tilde{W}(z)\tilde{W}(w)=&\frac{\frac{1}{3}n}{(z-w)^6}+\frac{2T(w)}{(z-w)^4}+\frac{\partial T(w)}{(z-w)^3}+\frac{1}{(z-w)^2}\left(\frac{4}{n+2}(TT)(w)-\frac{1}{2}\frac{n-2}{n+2}\partial^3\varphi^i\,\partial\varphi^i(w)\right)\\&+\frac{1}{z-w}\left(\frac{2}{n+2}\partial(TT)-\frac{1}{4}\frac{n-2}{n+2}\partial(\partial^3\varphi^i\,\partial\varphi^i)(w)-\tfrac{1}{12}\partial^3T(w)\right)+\dots\,,\end{aligned} \tag{4}$$

where $W=W_{+++}$, $T=T_{++}$, while ∂ and z stand for ∂_+ and x^+.

The W_3-algebra has the same form, except that the last two terms read

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$$\frac{1}{(z-w)^2}\left(\frac{32}{22+5n}(TT-\tfrac{3}{10}\partial^2 T)+\tfrac{3}{10}\partial^2 T\right)(w)+\frac{1}{z-w}\left(\frac{16}{22+5n}\partial(TT-\tfrac{3}{10}\partial^2 T)+\tfrac{1}{15}\partial^3 T\right)(w)\,. \tag{5}$$

There is clearly only agreement if $n=2$ [2,3]. The difference between our model and the W_3 result at position $(z-w)^{-2}$ is

$$\frac{-12}{22+5n}\frac{n-2}{n+2}\left((TT)+\tfrac{1}{8}(n+2)\partial^2 T+\frac{22+5n}{24}\partial^3\varphi^i\,\partial\varphi^i\right), \tag{6}$$

which is a primary field with spin 4, in agreement with ref. [4].

The quantum algebra for $n=2$ scalar fields consists then of two commuting copies of the W_3-algebra. The quantum BRST charge has been constructed in refs. [5,6]. It reads

$$Q=\oint \mathrm{d}z[c^+(T_{++}+\tfrac{1}{2}T^{\mathrm{gh}}_{++})+\gamma^{++}(W_{+++}+\tfrac{1}{2}W^{\mathrm{gh}}_{+++})+\tfrac{21}{4}\partial_+^2 c^+]\,,$$

$$T^{\mathrm{gh}}_{++}=-2b_{++}\,\partial_+ c^+-(\partial_+ b_{++})c^+-3\beta_{+++}\,\partial_+\gamma^{++}-2(\partial_+\beta_{+++})\gamma^{++}\,,$$

$$W^{\mathrm{gh}}_{+++}=\frac{25}{6\cdot 522}[20(\partial_+^3\gamma^{++})b_{++}+30(\partial_+^2\gamma^{++})\partial_+ b_{++}+18(\partial_+\gamma^{++})(\partial_+^2 b_{+++})+4\gamma^{++}\,\partial_+^3 b_{++}]$$

$$+c^+\,\partial_+\beta_{+++}+3(\partial_+ c^+)\beta_{+++}+\tfrac{16}{522}[2(\partial_+\gamma^{++})b_{++}T_{++}+\gamma^{++}\partial_+(b_{++}T_{++})]\,. \tag{7}$$

The requirement that it be nilpotent leads to two independent conditions which the central charge must satisfy, namely,

$$c=26+74\,,\quad c=1044\,\frac{c-50}{5c+22}\,. \tag{8}$$

The lowest eigenstate of $L_0+L_0^{\mathrm{gh}}$ has eigenvalue -4. Eq. (8) is satisfied for $c=100$, but taking fifty $n=2$ doublets would not achieve this, as $W_{+++}=\sum_{i=1}^{50}W^i_{+++}(n=2)$ would not satisfy the OPE in (4) or (5) due to the nonlinearity of the W_3-algebra.

In order to solve this dilemma, we have taken the following course of action. We start with the action in (1) for n scalar fields, with d^{ijk} only satisfying (2), and add a minimal coupling of $T_{\pm\pm}$, $W_{\pm\pm\pm}$ to new gauge fields $h^{\pm\pm}$ (the graviton) and $B^{\pm\pm\pm}$ (the W-graviton). We also let the parameters k^+, k^-, λ^{++}, λ^{--} become arbitrary, instead of being chiral. Giving these initial data, we then deduce the complete action and transformation rules using the time-honored Noether method. Our approach is similar to that of Hull [2], but he considers only one chiral sector. We hope that our treatment which includes the interactions between the two chiral sectors, will lead to a better understanding of the essential nonlinearity of the higher spin gauge theories and of the underlying geometric structure.

We start with the action for lowest nonminimal coupling

$$\mathscr{L}=\tfrac{1}{2}\partial_+\varphi^i\,\partial_-\varphi^i-\tfrac{1}{2}kh^{\pm\pm}T_{\pm\pm}-\tfrac{1}{3}kB^{\pm\pm\pm}W_{\pm\pm\pm}\,, \tag{9}$$

where k is the expansion parameter for the Noether method which will be set equal to one at the end. To order k^0 it is invariant under

$$\delta h^{++}=\frac{1}{k}\partial_- k^+ +(k^+\,\partial_+ +k^-\,\partial_-)h^{++}+h^{++}(\partial_- k^- -\partial_+ k^+)+(\lambda^{++}\,\partial_+ B^{+++}-B^{+++}\,\partial_+\lambda^{++})T_{++}\,,$$

$$\delta B^{+++}=(k^+\,\partial_+ +k^-\,\partial_-)B^{+++}+B^{+++}(\partial_- k^- -2\,\partial_+ k^+)+\frac{1}{k}\partial_-\lambda^{++}-h^{++}\,\partial_+\lambda^{++}+2(\partial_+ h^{++})\lambda^{++}\,,$$

$$\delta\varphi^i=(k^+\,\partial_+ +k^-\,\partial_-)\varphi^i+\lambda^{\pm\pm}\,\partial_\pm\varphi^j\,\partial_\pm\varphi^k d^{ijk}\,. \tag{10}$$

The rules for δB^{---} and δh^{--} follow by interchanging + and −. These rules give invariance to all orders if only one chiral sector is considered [2].

We then Noethered through third order in k. The results for the λ variation are given in table 1. It should be stressed that this method yields, order by order in k, in a systematic way the extra terms in transformation rules and action. Possible ambiguities at one level in k are fixed at the next level [7].

Inspection of these results reveals the following patterns:

(i) In the action and transformation rules one can find order $k+1$ terms from order k terms replacing ∂_- by $\partial_- - h^{++}\,\partial_+$ and ∂_+ by $\partial_+ - h^{--}\partial_-$. This suggests to introduce vielbeins $e^{\mu}_{\pm}$ where μ is the curved index, distinguished henceforth by a caret. In the gauge $e^{\hat{+}}_{+} = e^{\hat{-}}_{-} = 1$ we have

$$e^{\mu}_{\pm} = \begin{pmatrix} 1 & -h^{++} \\ -h^{--} & 1 \end{pmatrix}, \quad e^{\pm}_{\mu} = e \begin{pmatrix} 1 & h^{++} \\ h^{--} & 1 \end{pmatrix}, \tag{11}$$

where $e = \det(e) = (1 - h^{++}h^{--})^{-1}$. The compensating local Lorentz and local Weyl transformations, needed to preserve this gauge, lead to the following complete k transformation rules for $e^{+}_{\mu}\,\delta e^{\mu}_{-}$:

$$ke\,\delta e^{\hat{+}}_{-} = -\nabla_- \xi^+ \equiv -(e^{\mu}_{-}\,\partial_\mu + \omega_-{}^{+}{}_{+})\xi^+ , \tag{12}$$

with $\xi^{\pm} = k^{\mu} e^{\pm}_{\mu}$. The spin connection (with all indices flat) reads in this gauge

$$\omega_-{}^{+}{}_{+} = e^{-1}(-\partial_- e + h^{++}\,\partial_+ e) + \partial_+ h^{++} . \tag{13}$$

(ii) As a confirmation of the vielbein formulation, one may check that all B-independent terms in the action to third order in k are reproduced by $\frac{1}{2}e(e^{\mu}_{+}\,\partial_\mu\varphi)(e^{\nu}_{-}\,\partial_\nu\varphi)$ in the gauge chosen. The term with hB in the action can be viewed as vielbein covariantizations of the term with B, and it follows that $B^{\pm\pm\pm}$ are flat tensors. Hence, with the known Lorentz and Weyl compensations, we predict that $B^{\pm\pm\pm}$ must gravitationally transform as gravitational covariantization of (10). This rule agrees again with the Noether results.

(iii) A most important general observation is that in gauge theories, especially in supergravity, both the transformation rules and the field equations (but not the action!) can be written in terms of covariant derivatives.

Table 1
Action through order k^3 and transformation rules as obtained by the Noether method.

$$\begin{aligned}
\mathscr{L} = {} & \tfrac{1}{2}\partial_+\varphi^i\,\partial_-\varphi^i - \tfrac{1}{2}kh^{\pm\pm}\,\partial_\pm\varphi^i\,\partial_\pm\varphi^i - \tfrac{1}{3}kB^{\pm\pm\pm}d^{ijk}\,\partial_\pm\varphi^i\,\partial_\pm\varphi^j\,\partial_\pm\varphi^k + k^2h^{++}h^{--}\,\partial_+\varphi^i\,\partial_-\varphi^i \\
& + k^2B^{\pm\pm\pm}h^{\mp\mp}d^{ijk}\,\partial_\mp\varphi^i\,\partial_\pm\varphi^j\,\partial_\pm\varphi^k + k^2B^{+++}B^{---}d^{ipq}d^{irs}(\partial_+\varphi^p\,\partial_+\varphi^q)(\partial_-\varphi^r\,\partial_-\varphi^s) \\
& - \tfrac{1}{2}k^3(h^{++}h^{--})h^{\pm\pm}\,\partial_\pm\varphi^i\,\partial_\pm\varphi^i - \tfrac{1}{3}k^3h^{++}h^{--}B^{\pm\pm\pm}d^{ijk}\,\partial_\pm\varphi^i\,\partial_\pm\varphi^j\,\partial_\pm\varphi^k \\
& - k^3B^{\pm\pm\pm}h^{\mp\mp}h^{\mp\mp}d^{ijk}\,\partial_\mp\varphi^i\,\partial_\mp\varphi^j\,\partial_\pm\varphi^k - 2k^3B^{+++}B^{---}h^{\pm\pm}d^{ipq}d^{irs}(\partial_\pm\varphi^p\,\partial_\pm\varphi^q)(\partial_\pm\varphi^r\,\partial_\mp\varphi^s) \\
& - k^3B^{\mp\mp\mp}B^{\mp\mp\mp}B^{\pm\pm\pm}(d^{ijk}\,\partial_\pm\varphi^i)(d^{jpq}\,\partial_\mp\varphi^p\,\partial_\mp\varphi^q)(d^{krs}\,\partial_\mp\varphi^r\,\partial_\mp\varphi^s) + O(k^4) \\
\delta\varphi^i = {} & \lambda^{--}d^{ijk}\,\partial_-\varphi^j\,\partial_-\varphi^k - 2k\lambda^{--}h^{++}d^{ijk}\,\partial_-\varphi^j\,\partial_+\varphi^k - 2k\lambda^{--}B^{+++}(d^{ijk}\,\partial_-\varphi^j)(d^{kpq}\,\partial_+\varphi^p\,\partial_+\varphi^q) \\
& + k^2\lambda^{--}h^{++}h^{++}d^{ijk}\,\partial_+\varphi^j\,\partial_+\varphi^k + k^2\lambda^{--}B^{+++}B^{+++}d^{ijk}(d^{irs}\,\partial_+\varphi^r\,\partial_+\varphi^s)(d^{kpq}\,\partial_+\varphi^p\,\partial_+\varphi^q) \\
& + 2k^2\lambda^{--}B^{+++}h^{++}(d^{ijk}\,\partial_+\varphi^j)(d^{kpq}\,\partial_+\varphi^p\,\partial_+\varphi^q) + 4k^2\lambda^{--}B^{+++}B^{---}(d^{ijk}\,\partial_-\varphi^j)(d^{kpq}\,\partial_+\varphi^p)(d^{qrs}\,\partial_-\varphi^r\,\partial_-\varphi^s) \\
& + 4k^2\lambda^{--}B^{+++}h^{--}(d^{ijk}\,\partial_-\varphi^j)(d^{kpq}\,\partial_+\varphi^p\,\partial_-\varphi^q) + O(k^3) \\
\delta h^{--} = {} & (\lambda^{--}\,\partial_-B^{---} - B^{---}\,\partial_-\lambda^{--})\partial_-\varphi^i(\partial_-\varphi^i - 2kh^{++}\,\partial_+\varphi^i - 2kB^{+++}d^{ijk}\,\partial_+\varphi^j\,\partial_+\varphi^k) \\
& + k[-(\partial_+h^{++})\lambda^{--}B^{---} - h^{++}\lambda^{--}(\partial_+B^{---}) + B^{---}h^{++}(\partial_+\lambda^{--})](\partial_-\varphi^i\,\partial_-\varphi^i) + O(k^2) \\
\delta B^{---} = {} & \frac{1}{k}\partial_+\lambda^{--} - h^{--}(\partial_-\lambda^{--}) + 2(\partial_-h^{--})\lambda^{--} - 2k\,\partial_+(h^{++}h^{--})\lambda^{--} + O(k^2)
\end{aligned}$$

Closer inspection of the λ-transformation rules for φ^i reveals a new type of covariant derivative, which is perhaps best described as "nested covariant derivative": higher B-terms are obtained by successive substitutions of the form

$$\partial_\pm \varphi^i \to \partial_\pm \varphi^i - B^{\mp\mp\mp}\, \partial_\mp \varphi^j\, \partial_\mp \varphi^k\, d^{ijk}\,. \tag{14}$$

This nesting implies the following closed form solution for the nested covariant derivative:

$$\hat{\partial}_\pm \varphi^i = e^\mu_\pm\, \partial_\mu \varphi^i - B^{\mp\mp\mp}\, \hat{\partial}_- \varphi^j\, \hat{\partial}_- \varphi^k\, d^{ijk}\,. \tag{15}$$

Iteration reproduces all results for $\delta\varphi^i$ in the table. We suspect this to be a general feature of future gauge theories for nonlinear algebras.

(iv) Spurred by this success, we turn to the λ-law for $\delta h^{\pm\pm}$. One may check that all results obtained so far agree with

$$e\,\delta h^{++} = (\lambda^{++}\, \nabla_+ B^{+++} - B^{+++}\, \nabla_+ \lambda^{++})\hat{\partial}_+ \varphi^i\, \hat{\partial}_+ \varphi^i\,. \tag{16}$$

A most dramatic confirmation is provided by the B-field equation: *all* $B^{+++}(B^{---})$ dependent terms in the action can be recovered if the $B^{\pm\pm\pm}$ field equation reads $-\frac{1}{3} e\, \hat{\partial}_\pm \varphi^i\, \hat{\partial}_\pm \varphi^j\, \hat{\partial}_\pm \varphi^k\, d^{ijk}$.

(v) To extend these results to an exact result at all orders, we introduce a new field $F^i_\pm$ which on-shell reduces to the nested covariant derivative $\hat{\partial}_\pm \varphi^i$. Our intent is to reduce the infinite nonlinearity to polynomial form. Therefore, we interpret (15) as a set of coupled field equations for F^i_+ and F^i_-: $F^i_\pm = \partial_\pm \varphi^i - B^{\mp\mp\mp} F^j_\mp F^k_\mp\, d^{ijk}$. A corresponding first-order action suggested by this observation and by dimensional arguments, is then given by

$$\mathscr{L} = -\tfrac{1}{2} e\, \partial_+ \varphi^i\, \partial_- \varphi^i - e F^i_+ F^i_- + e(\partial_+ \varphi^i - \tfrac{1}{3} B^{---} F^j_- F^k_-\, d^{ijk}) F^i_- + e(\partial_- \varphi^i - \tfrac{1}{3} B^{+++} F^j_+ F^k_+\, d^{ijk}) F^i_+\,. \tag{17}$$

This is the complete action.

(vi) To find the complete transformation laws, we start from

$$\delta I = \frac{\delta I}{\delta \varphi^i}\, \delta\varphi^i + \frac{\delta I}{\delta h^{\pm\pm}}\, \delta h^{\pm\pm} + \frac{\delta I}{\delta B^{\pm\pm\pm}}\, \delta B^{\pm\pm\pm} + \frac{\delta I}{\delta F^i_\pm}\, \delta F^i_\pm\,. \tag{18}$$

We now use the obvious (but little understood) 1.5 order formalism [7]. According to this formalism, one works in second order formalism, but one needs not determine $\delta F^i_\pm$ (by the chain rule) since they are multiplied by $\delta I/\delta F^i_\pm$ which vanishes *identically*. The complete transformation rules are now easily found. They read for λ^{--} transformations

$$\begin{aligned} &\delta\varphi^i = \lambda^{--} F^j_- F^k_-\, d^{ijk}\,, \quad e\,\delta h^{--} = X^{--}\,, \quad e\,\delta h^{++} = Y^{++}\,, \\ &\delta B^{---} = \nabla_+ \lambda^{--} - B^{---} h^{++} X^{--} + 2B^{---} h^{--} Y^{++}\,, \quad \delta B^{+++} = +2B^{+++} h^{++} X^{--} - B^{+++} h^{--} Y^{++}\,, \end{aligned} \tag{19}$$

where

$$\begin{aligned} &X^{--} = (1-x)^{-1}(\lambda^{--}\, \nabla_- B^{---} - B^{---}\, \nabla_- \lambda^{--})(F^i_- F^i_-)\,, \quad Y^{++} = X^{--} B^{+++} B^{+++} F^i_+ F^i_+\,, \\ &x = (B^{+++})^2 (B^{---})^2 (F^i_+ F^i_+)(F^j_- F^j_-)\,. \end{aligned} \tag{20}$$

In these results one must replace $F^i_\pm$ by its on-shell value. The rules for λ^{++} follow by interchanging + and −.

(vii) In first order formalism the rules for $F^i_\pm$ follow by keeping track of where the F-field equations were used in the proof of the invariance of the second order action. One finds

$$\begin{aligned} &\delta F^i_- = \nabla_- (\lambda^{--} F^j_- F^k_-\, d^{ijk}) - e(h^{--} F^i_- - \tfrac{1}{2} F^i_+ + \tfrac{1}{2} \partial_+ \varphi^i + \tfrac{1}{2} B^{---} F^j_- F^k_-\, d^{ijk})\delta h^{++}\,, \\ &\delta F^i_+ = -e(h^{++} F^i_+ - \tfrac{1}{2} F^i_- + \tfrac{1}{2} \partial_- \varphi^i + \tfrac{1}{2} B^{+++} F^j_+ F^k_+\, d^{ijk})\delta h^{--}\,. \end{aligned} \tag{21}$$

(viii) A purely cubic action is obtained by replacing $d^{ijk}F^j_- F^k_-$ by Q^i_{--}, and adding the terms $eS^i_{\pm\pm} \times(-Q^i_{\mp\mp}+d^{ijk}F^j_{\mp}F^k_{\mp})$.

We end with some comments and a list of open problems.

(i) Of course, the outstanding problem is the quantization of this model. In particular, it should be studied whether the conformal and W-anomaly both cancel for a suitable choice of d^{ijk} and n. From the BRST analysis one knows that if the algebra is exactly W_3, one needs a $c=100$ matter system, but for our present model with $n\neq 2$ this analysis must be modified. The quantum consistency of our classical action is still completely open. A solution of this problem would pave the way to W-strings, of which some preliminary considerations were given in ref. [8].

(ii) The gauge algebra can be studied. In particular, we intend to investigate whether (modified) transformation rules for the set of fields in (18) can lead to a closed gauge algebra, or whether additional auxiliary fields are needed.

(iii) A covariant formulation for $h_{\mu\nu}$ and $W_{\mu\nu\rho}$ should exist. Just as the absence of h^{+-} can be traced to the existence of Weyl gauge invariance, the absence of B^{++-} and B^{+--} hints at the existence of a W–Weyl-symmetry. It should appear in the commutator of a W-symmetry and a Weyl symmetry.

(iv) The W-geometry should be analyzed. As a starting point, the W-anomaly in the covariant formulation is expected to give the W-extension of the usual conformal anomaly which is proportional to the scalar curvature of gravity.

(v) The transformation rules of $h^{\pm\pm}$ and $B^{\pm\pm\pm}$ under which the anomalous conservation laws of the Noether currents $T^{\pm\pm}$ and $W^{\pm\pm\pm}$ are invariant as found in refs. [9,10], differ from the purely classical laws of ref. [2] and us: the λ variation of $h^{\pm\pm}$ contains additional higher derivative terms. We observe that these same terms appear in renormalized form in the quantum BRST charge (7).

(vi) Supersymmetrization.

References

[1] A.B. Zamolodchikov, Teor. Mat. Fiz. 65 (1986) 1205.
[2] C.M. Hull, Phys. Lett. B 240 (1990) 110.
[3] V.A. Fateev and A.B. Zamolodchikov, Nucl. Phys. B 280 [FS18] (1987) 644.
[4] A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov, Nucl. Phys. B 241 (1984) 333.
[5] J. Thierry-Mieg, Phys. Lett. B 197 (1987) 368.
[6] K. Schoutens, A. Sevrin and P. van Nieuwenhuizen, Commun. Math. Phys. 124 (1989) 87.
[7] P. van Nieuwenhuizen, Phys. Rep. 68 (1981) 189.
[8] A. Bilal and J.-L. Gervais, Nucl. Phys. B 326 (1989) 222.
[9] Y. Matsuo, Phys. Lett. B 227 (1989) 209.
[10] M. Bershadsky and H. Ooguri, Commun. Math. Phys. 126 (1989) 49.