PENSION REGULATION AND THE MARKET VALUE OF PENSION LIABILITIES – A CONTINGENT CLAIMS ANALYSIS USING PARISIAN OPTIONS

DIRK BROEDERS[‡] AND AN CHEN[*]

Abstract. Defined benefit pension plans provide their beneficiaries with a prespecified retirement income related to years of service and wage. However, as pension contracts are not entirely legally enforceable, such payments of defined benefit pension plans are not completely irreversible. The pension deal can be closed early or converted, i.e. the sponsor might be tempted to change the nature of the pension fund liabilities from defined benefit to defined contribution to avoid excessive premium increases when the pension plan funding ratio deteriorates and crosses the regulatory minimum. This paper is the first to analyze market–consistent valuation of option–adjusted pension liabilities in a contingent claim framework whereby a knock–out barrier framework is built into the model in order to capture early closure of a pension plan. We mainly investigate two cases which we call "immediate closure procedure" and "delayed closure procedure". In an immediate closure procedure, when the assets value hits the regulatory boundary, the pension plan is terminated immediately. Whereas in a delayed closure procedure, a grace period is given to the pension fund for reorganization and recovery and only an underfunding longer than a certain period leads to premature closure. The framework is then used to analyze fair pension deals.

Keywords: Pension funds; DB and DC pension plans, barrier options, Parisian barrier options.
JEL: G11, G23

1. Introduction

Since Sharpe's (1976) seminal contribution, defined benefit (DB) pension plans are often viewed as a combination of option contracts. The beneficiaries of such a pension plan are entitled to a prespecified amount related to years of service and salary. In some cases the beneficiaries also have a share in the pension fund's surplus. This surplus is to some extent accruable to the sponsor (often via contribution holidays) who is therefore considered to possess an option on the pension fund's assets. Unlike most other financial contracts,

Date. September 10, 2008.

[‡] De Nederlandsche Bank (DNB), Supervisory Policy Division, Strategy Department, PO Box 98, 1000 AB Amsterdam, The Netherlands, Tel: 0031-20-5245794, Fax: 0031-20-5241885. E–Mail: d.w.g.a.broeders@dnb.nl.

[*] Netspar and Department of Quantitative Economics, University of Amsterdam, Roetersstraat 11, 1018 WB Amsterdam, The Netherlands, Tel: 0031-20-5254125, Fax: 0031-20-5254349, E–Mail: a.chen@uva.nl.

pension plans have a peculiar legal status, i.e. they are in most cases not entirely legally enforceable. Opposed to a life insurance contract, a defined benefit pension promise is not completely irreversible. Most current pension contracts implicitly enable their sponsors to premature terminate or convert the deal along the way. This implies that the sponsor can avoid the payment of recovery premiums by changing the nature of the pension liabilities from a DB to a defined contribution (DC) pension plan when the pension fund is unable to settle its initially DB promises[1]. In reality, many sponsoring companies consider this as an ultimate escape route, via which they are able to get around DB pension obligations if the financial burden of maintaining them gets too high. For instance, in the early 2000s when funding ratios fell substantially after the stock market crash, some companies indeed changed the nature of their pension promise from DB to DC. Some closed the fund for new entrants and others transferred the liabilities to an insurance company. Furthermore, the extra burden related to improved longevity, lower interest rates and more volatile asset returns makes DB plans expensive to maintain. Some other important drivers of this conversion are changes in pension regulation and accounting, exposing mark-to-market values of pension liabilities and asset-liability-mismatch risks. More reasons (also with respect to the interests of employees) can be found in Aaronson and Coronado (2005) and Broadbent et al. (2006).

The conversion trend is most manifest in the US. According to the Flow of Funds Accounts of the United States, the division between assets held in private DB plans and private DC plans was 60% versus 40% in 1987, whereas in 2007 this ratio was exactly reversed; the turning point lay in 1995. Very recently large companies such as Ford, General Motors, IBM and Sears made a (partial) shift towards DC plans. The changes in the UK reveal a similar development. In 1979, final salary DB plans constituted 92% of all pension funds. However, in 2005 the Government Actuary's Department observed that 41% of all active members accrue their DB rights in pension plans closed to new entrants. This closure has been accompanied by the emergence of DC plans and average pay DB schemes. Starting from a relatively wealthy position with a funding ratio around 200% at the turn of the millennium, the Netherlands have been able to avoid a bulky shift towards DC plans. The perfect storm (negative stock returns combined with decreasing market interest rates) generated another change (see e.g. Bikker and Vlaar (2007)). Figures from De Nederlandsche Bank show that in 1998, 66% of all active members participated in a final pay DB plan. By 2007 this number was down to 3%. The vast majority of Dutch employees (85%) currently participate in a conditionally indexed average pay DB scheme. Under such schemes, the benefit depends not on the final but on the career average salary. Furthermore, both during the accrual and the benefit stage the pension rights are indexed

[1] A detailed description on the differences between DB and DC plans can be found e.g. in Bodie, Marcus and Merton (1988). Yang (2005) analyzes the factors that influence the choice between DB and DC plans for individuals.

to price or wage inflation conditional on a sufficient high cover ratio of the fund. As such, inflation and investment risks are shifted in part to active fund members. All these changes imply that DB pension plans do *not* provide their participants with a guaranteed amount and regular (unconditionally) options introduced by Sharpe, but the premature closure and conversion feature of the pension plans has made the various claims of the participants on the pension fund's assets more exotic. With this feature, the participants of the pension plans are in fact exposed to more risks. When the pension fund's assets value falls below the applicable regulatory boundary (roughly speaking in case of underfunding), the guaranteed payment may be provided only partially. This affects the value for the beneficiary.

The pension conversion feature has been pointed out or analyzed by some empirical studies in the existing literature. For instance, Petersen (1992) examines three hypotheses concerning the motivation underlying pension plan reversion and finds that all the hypotheses are empirically supported by the US data. Niehaus and Yu (2005) analyze the conversion of DB plans to cash balance plans in the US in the nineties. From a regulatory point of view cash balance plans are treated as DB plans, however beneficiaries conceive it as DC plans. More on the nature of pension contracts can be found e.g. in Treynor (1997), Bulow (1982) and Ippolito (1985).

However, to our knowledge, the premature closing or converting feature of pension plans has never been investigated analytically and theoretically. This paper aims to fill the gap. Our objective is to incorporate the closing feature in the valuation of DB pension liabilities, i.e. the contract payoff of the DB plan depends on the entire evolution of the pension fund's assets. When the funding ratio deteriorates, the DB plan might be closed or converted to a DC plan. In this paper, the emphasis is not put on how to model the DC plan, but on how the premature closing feature affects the market value of the DB plans. Therefore, we assume that the DB contract is terminated upon conversion. We set ourselves in a contingent claim framework and use knock–out barrier options to describe the closing feature. We distinguish between two procedures: "immediate closure procedure" and "delayed closure procedure". In an immediate closure procedure, when the assets value hits the regulatory boundary, the pension plan is terminated immediately. This immediate closure procedure does not reflect reality in all cases because pension funds are usually given time to reorganize and recover. The recovery period varies across jurisdictions. Germany and Luxembourg require in principle immediate recovery. Sweden and Denmark allow for a recovery period of 1 year, while in the Netherlands, Norway and Ireland it is typically 3 years. Spain even allows longer recovery up to 5 years. The UK on the other hand uses a flexible recovery period as it has explicitly stated the objective of preventing undue pressure on sponsors and protecting schemes from being wound up. In other words, there is a need to balance the ongoing viability of the employer against the long–term interests

of the members and continued DB pension provision. One of the triggers for additional supervisory scrutiny in the UK is the duration of the recovery plan. More information on supervisory rules across European countries is provided in a CEIOPS report from 2008. Therefore, in addition to the immediate closure procedure, the delayed closure procedure is analyzed to capture all possible regulatory situations. The main feature of this procedure is that the closure does not come into force immediately when default (or underfunding) occurs; instead, a grace period is given to them for recovery. Mathematically the immediate and delayed closure procedures can be realized by applying standard and Parisian down–and–out barrier options, respectively.

Barrier options belong to the family of exotic options and are first mentioned in the literature in Snyder (1969). The payoff of these products is not based on the final value of the underlying asset only, but linked to the additional conditions of the asset value evolution. Let us assume that we are interested in the modelling of a down–and–out barrier option. The option contract is knocked out if the underlying asset hits the barrier (from above) during the option life. The topic of Barrier options has been studied very widely in the literature, e.g. Rubinstein and Reiner (1991) and Rich (1994), to mention just a few. Recently, Grosen and Jørgensen (2002) incorporate a regulatory mechanism into the market valuation of equity and liabilities at life insurance companies by using a down–and–out barrier feature to describe the regulatory intervention rule.

Compared to standard barrier options, Parisian options do not have a long history in the literature on exotic options. They were introduced by Chesney et al. (1997) and subsequently developed by Moraux (2002), Anderluh and van der Weide (2004) and Bernard et al. (2005) etc. In a standard Parisian down–and–out option, the contract is knocked out if the underlying asset value remains consecutively below the barrier for longer than some predetermined time d before the maturity date. In the context of with–profit life insurance contracts, Chen and Suchanecki (2007) apply the Parsian barrier option framework to incorporate more realistic bankruptcy procedures (Chapter 11 bankruptcy procedure) in the market valuation of life insurance liabilities. Suppose a regulatory authority takes its bankruptcy filing actions according to a hypothetical default clock. The default clock starts ticking when the asset price process breaches the default barrier and the clock is reset to zero when the company recovers from the default. Thus, successive defaults are possible until one of these defaults lasts d units of time. Earlier defaults which may last a very long time but not longer than d do not have any consequences for eventual subsequent defaults.

The remainder of this paper is organized as follows. Section 2 describes the basic payoff structure of pension plans and the underlying contingent claim model setup. Additionally we introduce the theoretical background of barrier and Parisian barrier options. The next

section focuses on the valuation of the DB pension plans. Section 4 contains a variety of numerical analyses. Section 5 concludes the paper with a summary of the results.

2. Model

This section introduces the general framework for conditionally indexed DB pension plans and particularly various regulatory procedures at default, distinguishing between immediate and delayed closure of the pension plan. A standard down–and–out barrier option framework is used to describe the immediate closure procedure, and a Parisian down–and–out option framework explains the delayed closure procedure.

2.1. **Contract Specification.** We consider the pension plan of a single representative participant who has to work another T years. Let us assume the pension plan was issued at time $t_0 = 0$. At time 0, the pension fund issues a conditionally indexed defined benefit pension plan to a representative beneficiary who provides an upfront contribution P_0. The pension fund also receives an amount of initial contributions from the sponsor S_0 at time 0. Consequently, the initial asset value of the pension fund is given by the sum of the contributions from both the beneficiary and the sponsor, i.e. $A_0 = P_0 + S_0$. From now on, we shall denote $S_0 = \alpha A_0$ with $\alpha \in [0, 1]$. The pension fund invests the proceeds in a diversified portfolio of risky and non-risky assets.

At retirement T, the beneficiary receives a lump sum nominal pension of $\underline{L}$. In addition, the pension plan has the objective to increase pension rights by $i\%$ per annum, where $i\%$ might be related to, say, the average expected CPI or wage growth. Since the determination of this parameter should take into consideration many factors, in reality this procedure is fairly complicated. Here, for simplicity we assume i is deterministic and constant and a fully indexed pension is then $\bar{L} = \underline{L}e^{iT}$. However, it should be noted that the actual outcome of the pension plan is contingent on the funding ratio at maturity T, which is defined as the ratio of the pension fund's assets (A_T) to its liability. At maturity T, given that the assets are sufficiently high ($A_T > \bar{L}$), the beneficiary not only receives an indexed pension of $\bar{L}$, but is allowed to participate in the surplus of pension funds ($A_T - \bar{L}$) with a participation rate δ, where $\delta \in [0, 1]$ is the surplus distribution parameter. For instance, $\delta = 0.75$ means that the pension beneficiary receive 3/4 of the surplus. To entitle the beneficiary to share in the pension funds' surplus can be considered a reward for the fact that the beneficiary is exposed to converting risk. When the assets of the pension fund do not perform well, we distinguish between two scenarios: $A_T < \underline{L}$ and $\underline{L} \leq A_T < \bar{L}$. In the latter case, the assets value A_T is assigned to the beneficiary, whereas in the former case the guaranteed amount $\underline{L}$ is paid out to the beneficiary. Since a pension fund does not have external shareholders, there is instead the corporate pension plan sponsor, a pension guarantee fund or the government bearing the residual risk, when the assets are insufficient to cover the guaranteed benefits $\underline{L}$. To sum up, at the maturity date T the payoff to the

beneficiary is (assuming no early termination):

$$\psi_B(A_T) = \begin{cases} \underline{L}, & \text{if } A_T < \underline{L} \\ A_T, & \text{if } \underline{L} \leq A_T \leq \bar{L} \\ \bar{L} + \delta(A_T - \bar{L}), & \text{if } A_T > \bar{L} \end{cases}$$

as illustrated in Figure 1. It is observed that this payoff differs from that of a with–profit life insurance contract in Grosen and Jørgensen (2002) and Chen and Suchanecki (2007). More specifically, when the final asset's value is not sufficiently high ($A_T < \underline{L}$), in a with-profit life insurance contract, the contract holder will obtain A_T due to the limited liability of the equity holder, whereas in a pension plan, a floor (here $\underline{L}$, provided by the sponsor) is ensured to the beneficiary. When the assets perform moderately ($\underline{L} \leq A_T \leq \bar{L}$), a with-profit life insurance provides its contract holder with the guaranteed amount $\underline{L}$, whereas in a pension plan, the entire assets value is assigned to the beneficiary. Finally, if the assets perform well ($A_T > \bar{L}$), the surpluses–sharing feature of a pension plan is quite similar to that of a with–profit life insurance contract. Therefore, the main difference between the payoff of the pension plan and that of a with–profit life insurance contract is observed when the assets do not perform well, i.e. when $A_T < \underline{L}$, the pension plan provides a better guarantee by ensuring the amount $\underline{L}$. More compactly, we can split the above payoff into two parts:

$$\psi_B(A_T) = \min\{\max\{A_T, \underline{L}\}, \bar{L}\} + \delta \max\{A_T - \bar{L}, 0\}.$$

The first component on the right-hand side is capped by $\bar{L}$, i.e. it corresponds to the payoff of a traditional pension plan where the beneficiary sells off any payoffs above $\bar{L}$ and is not entitled to sharing in the surplus of the pension fund. The second component corresponds to the surplus participation which allows the beneficiary to share in the pension fund's surplus with a participation rate δ. Rephrasing the first component, we can rewrite this payoff to:

$$\psi_B(A_T) \;=\; \underline{L} + [A_T - \underline{L}]^+ - (1-\delta)[A_T - \bar{L}]^+, \tag{1}$$

where we have used $[x]^+ := \max\{x, 0\}$. This payoff consists of three parts: a promised amount $\underline{L}$, a long call option on the assets with strike equal to the promised payment $\underline{L}$, and a short call option with strike equal to $\bar{L}$ (multiplied by $1-\delta$). The latter represents the money returned to the pension plan sponsor by the beneficiaries to cover the shortfall risk.

Analogously, as a compensation, the residual of surplus, if any, is provided to the pension plan sponsor. The total payoff to the pension plan sponsor at maturity $\psi_S(A_T)$, is given by:

$$\psi_S(A_T) = \begin{cases} A_T - \underline{L}, & \text{if } A_T < \underline{L} \\ 0, & \text{if } \underline{L} \leq A_T \leq \bar{L} \\ (1-\delta)(A_T - \bar{L}), & \text{if } A_T > \bar{L} \end{cases}$$

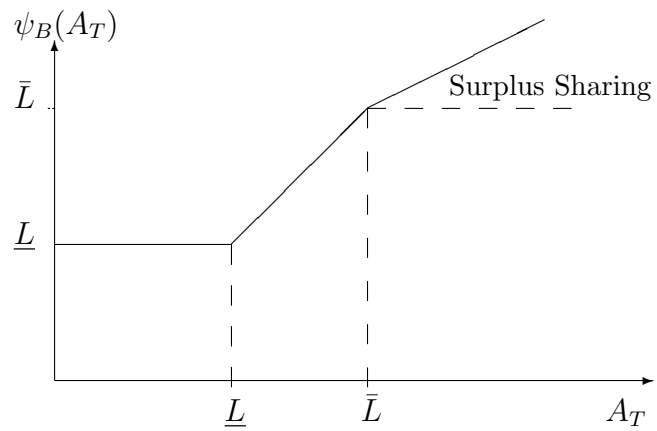

FIGURE 1. The payoff $\psi_B(A_T)$ to the beneficiary given no premature closure

or more compactly,

$$\psi_S(A_T) = (1-\delta)[A_T - \bar{L}]^+ - [\underline{L} - A_T]^+.$$

The payoff can be decomposed into two terms: a long call option which corresponds to the "bonus" received by the sponsor and a short put option reflecting the deficit which he covers in case of under-performance of the assets.

The above contract payoff can be regarded as a traditional DB pension plan which is relatively expensive because the plan provides the beneficiary with a long maturity guarantee $\underline{L}$. Due to the implicit early closing feature of the pension contracts, the sponsor is tempted to try to change the nature of the pension fund liabilities from DB to DC plans to avoid excessive premium increases, if for instance the pension plan funding ratio deteriorates. In other words, the above payoff at maturity to the beneficiary is not unconditional. In case of underfunding, it is likely that the beneficiary can not obtain the guaranteed payment $\underline{L}$. Therefore, the converting feature of the pension plan implies that a closure mechanism similar to a knock–out barrier option framework must be incorporated when analyzing the pension plan. Roughly speaking, when the pension fund's assets perform extremely poorly, the DB plan at some point is is terminated prematurely and a rebate payment is provided to the beneficiary. The rebate proceedings can e.g. be used to start a (collective) DC plan or to transfer the remaining liabilities to a pension guarantee fund or an insurance company. In the following subsection, we describe two default and premature closing procedures: the immediate and the delayed closing procedure.

2.2. **Default and premature closure formulation.** We distinguish between an immediate closure and a delayed closure procedure. In an immediate closure procedure, a premature default (underfunding) leads to immediate termination of the pension fund, i.e. default and closure are treated as equivalent events. Since the emphasis of this paper is to build in the early closure feature of the DB plans (to DC plans) in the market valuation

of the DB plans, we assume that the DB contract is terminated by regulatory intervention and a rebate is paid to the beneficiary. So, we leave the complexity of modeling possible conversion to a DC plan at that point for further research.[2] In a delayed closure procedure, default and pension plan termination are distinguishable events. A chance is given for reorganization and recovery during some "grace" period. If the pension fund is unable to recover during this period, the DB plan is converted to a DC plan and the contract is terminated prematurely.

An immediate closure procedure can be mathematically realized by using standard knock–out barrier options, similarly as an immediate default and liquidation procedure in the life insurance literature (c.f. Grosen and Jørgensen (2002)). A delayed closure procedure can be characterized by using the Parisian barrier option framework (similarly as Chen and Suchanecki (2007) in a life insurance context). In both articles, the regulatory intervention rule is introduced in the form of a boundary. Since the regulation objective is to provide the beneficiary with the guaranteed payment $\underline{L}$ at the maturity date, it is natural to assume an exponential barrier which increases over time as follows:

$$B_t = \lambda\, \underline{L}\, e^{-r(T-t)} = B_0 e^{rt}, \quad t \in [0, T], \tag{2}$$

where r is the prevailing market interest rate for maturity T, λ is the regulation parameter chosen by the legislator or regulator and $B_0 = \lambda \underline{L} e^{-rT}$. A λ of 1 for instance means that the funding ratio of the pension fund should always be in excess of 100%. Obviously, the specified contract contains standard down–and–out barrier options. Therefore, the requirement $A_0 > B_0 = \lambda \underline{L} e^{-rT}$ must be satisfied initially and leading to a reasonable range for the regulation parameter λ, i.e. $\lambda \in (0, A_0 e^{rT}/\underline{L}]$. In an immediate closure procedure, the pension fund defaults and the DB plan is converted to a DC plan immediately when the assets reach this boundary, namely, $A_\tau = B_\tau$ if $\tau \leq T$. Hence, the premature default and closure coincide and the premature closure time is given by

$$\tau = \inf\{t | A_t \leq B_t\}. \tag{3}$$

Upon premature closure, the contract is terminated and a rebate payment

$$\Theta_B(\tau) \;=\; \min\{\underline{L} e^{-r(T-\tau)}, B_\tau\} = \min\{1, \lambda\} \underline{L} e^{-r(T-\tau)} \tag{4}$$

is offered to the beneficiary immediately at the closure time τ. For $\lambda < 1$, the (discounted) guaranteed amount is not returned to the beneficiary, whereas in case of $\lambda > 1$, the (discounted) guaranteed amount is ensured and there will be a residual. The residual can be used to cover expenses in case the liabilities are transferred to an insurance company

[2]This conversion could also be modeled as an option, more specifically as an exchange option, i.e. the pension plan sponsor has the right to exchange cash flows from the DB pension plan to cash flows from the DC pension plan. In fact this must be a compound exchange option, because it is an exchange option on a combination of options which represent the terminal payoff. Furthermore, to include default risk and a possible delay in the conversion decision Parisian compound exchange options should be used.

or a guarantee fund. The sponsor is thus provided with the remaining assets as the rebate payment:

$$\Theta_S(\tau) \quad = \quad B_\tau - \min\{\underline{L}e^{-r(T-\tau)}, B_\tau\} = \max\{\lambda - 1, 0\}\underline{L}e^{-r(T-\tau)}. \tag{5}$$

The delayed closure procedure can be realized by adding a Parisian barrier option feature instead of the standard knock–out barrier option feature to the model (c.f. Chen and Suchanecki (2007)). In a standard Parisian barrier option framework, the closure of the pension fund is declared when the financial distress has lasted at least a period of length d. d can therefore be considered the maximum recovery period assigned to the pension fund to recover from the financial distress.

Before we come to the mathematical formulation of standard Parisian barrier options, it is convenient to specify the underlying assets process. Under the equivalent martingale measure Q, the price process of the pension fund's assets $\{A_t\}_{t\in[0,T]}$ is assumed to follow a geometric Brownian motion

$$dA_t = A_t(rdt + \sigma dW_t),$$

where σ denotes the deterministic volatility of the asset price process $\{A_t\}_{t\in[0,T]}$ and $\{W_t\}_{t\in[0,T]}$ the unique risk–neutral Q–martingale. Solving this differential equation, we obtain

$$A_t = A_0 \exp\left\{\left(r - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right\}.$$

There are several special cases of Parisian barrier options (a Parisian down–and–out call option is taken as an example)

- $A_t > B_t$ and $d \geq T - t$: In this case, it is impossible to have an excursion of A_t below B_t, between t and T, of length at least equal to d. Therefore, the value of a Parisian down–and–out call just corresponds to the Black–Scholes (Black and Scholes (1973)) price of a regular European call option.
- $d \geq T$: In this case, the Parisian option actually becomes a standard call option.
- $A_t > B_t$ and $d = 0$: When the time window d is set at 0, we are back in the immediate default and closure procedure.

Apart from these special cases, in the standard Parisian down–and–out option framework, the final payoffs $\psi_B(A_T)$, $\psi_S(A_T)$ are paid only if the following technical condition is satisfied:

$$T_B^- \quad = \quad \inf\{t > 0 | (t - g_{B,t}^A)\mathbf{1}_{\{A_t < B_t\}} > d\} > T \tag{6}$$

with

$$g_{B,t}^A = \sup\{s \leq t | A_s = B_s\},$$

where $g_{B,t}^A$ denotes the last time before t at which the value of the assets A hits the barrier B. T_B^- gives the first time at which an excursion below B lasts more than d units of time. In fact, T_B^- is the premature closure (or contract–termination) date if $T_B^- < T$. Figure 2

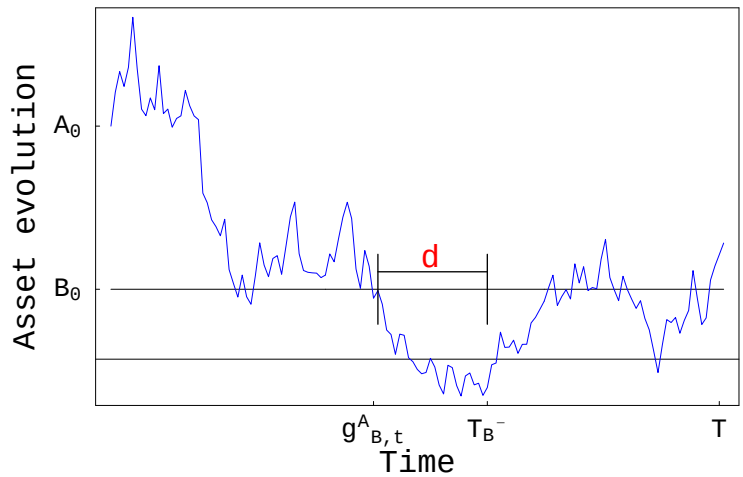

FIGURE 2. A path simulation of premature closure under the Parisian option framework.

simulates a path of the asset evolution which leads to premature closure of the DB plan under the Parisian option framework[3]. At $g^A_{B,t}$, the pension fund starts defaulting and the DB plan is converted at T_B^- because it does not recover from underfunding after d units of time.

It is noted that the condition in (6) is equivalent to

$$T_b^- := \inf\{t > 0 | (t - g_{b,t})\mathbf{1}_{\{Z_t < b\}} > d\} > T$$

where

$$g^Z_{b,t} \quad := \quad \sup\{s \leq t | Z_s = b\}; \qquad b = \frac{1}{\sigma} \ln\left(\frac{B_0}{A_0}\right); \quad B_0 = \lambda\, \underline{L}\, e^{-rT}$$

and $\{Z_t\}_{0 \leq t \leq T}$ is a martingale under a new probability measure P which is defined by the Radon–Nikodym density

$$\frac{dQ}{dP} = \exp\left\{mZ_T - \frac{m^2}{2}T\right\}, \qquad m = -\frac{\sigma}{2}, \tag{7}$$

i.e. $W_t = Z_t - m\,t$. Thereby, we transform the event "the excursion of the value of the assets below the exponential barrier $B_t = B_0 e^{rt}$" into the event "the excursion of the Brownian motion Z_t below a constant barrier $b = \frac{1}{\sigma} \ln \frac{B_0}{A_0}$". As specified, when $T_B^- = \inf\{t > 0 | (t - g^A_{B,t})\mathbf{1}_{\{A_t < B_t\}} > d\} < T$, the closure of the DB to a DC plan occurs and the contract is terminated prematurely. As already pointed out by Chen and Suchanecki (2007), we have to make a small change to the rebate term of the contract. The Parisian barrier option feature could lead to the result that at the closure time the asset price falls far below the barrier value, which makes it impossible for the pension fund to offer the rebate as in (4). Hence, a new rebate for the beneficiary is introduced to the model with

[3]For simplicity, we have used a constant barrier level in the figure.

the form

$$\Theta_B(T_B^-) = \min\{\underline{L}e^{-r(T-T_B^-)}, A_{T_B^-}\} \quad \text{with } A_{T_B^-} \leq B_{T_B^-} \tag{8}$$

where T_B^- is the closure time. The rebate term implicitly depends on the regulation parameter λ. Correspondingly, the new rebate for the pension fund sponsor can be expressed as follows:

$$\Theta_S(T_B^-) = A_{T_B^-} - \min\{\underline{L}e^{-r(T-T_B^-)}, A_{T_B^-}\} = \max\{A_{T_B^-} - \underline{L}e^{-r(T-T_B^-)}, 0\}, \tag{9}$$

i.e. the sponsor obtains the remaining assets value if there is any.

3. Valuation

We assume a continuous–time frictionless economy with a perfect financial market, no tax effects, no transaction costs and no other imperfections. Hence, we can rely on martingale techniques for the valuation of the contingent claims.

3.1. **Immediate closure procedure.** In a complete financial market, the price of a T–contingent claim with the payoff $\phi(A_T)$ corresponds to the expected discounted payoff under the risk–neutral probability measure Q, i.e.,

$$\mathbb{E}_Q\left[e^{-rT}\phi(A_T)\mathbf{1}_{\{\tau>T\}}\right].$$

The market–consistent value of the payoff to the beneficiary is hence determined by:

$$\begin{aligned} V_B(A_0,0) &= \mathbb{E}_Q[e^{-rT}\left(\underline{L} + [A_T - \underline{L}]^+ - (1-\delta)[A_T - \bar{L}]^+\right)\mathbf{1}_{\{\tau>T\}}] \\ &\quad + \mathbb{E}_Q[e^{-r\tau}\min\{1,\lambda\}\underline{L}e^{-r(T-\tau)}\mathbf{1}_{\{\tau\leq T\}}]. \end{aligned}$$

It is observed that the price of this contingent claim consists of four parts: a deterministic guaranteed or fixed part $\underline{L}$ which is paid at maturity when the value of the assets does not hit the barrier, a long down–and–out call option with strike $\underline{L}$, a shorted down–and–out call option with strike $\bar{L}$ (multiplied by $1-\delta$), and a rebate paid immediately upon premature closure. The expected fixed payment at maturity can be expressed as follows:

$$\mathbb{E}_Q\left[e^{-rT}\underline{L}\mathbf{1}_{\{\tau>T\}}\right] = \underline{L}e^{-rT}\left[N\left(d^-(A_0,B_0,T)\right) - \left(\frac{A_0}{B_0}\right)N\left(d^-(B_0,A_0,T)\right)\right].$$

The long down–and–out call option can be calculated further:

$$\begin{aligned} & e^{-rT}\mathbb{E}_Q\left[(A_T-\underline{L})^+\mathbf{1}_{\{\tau>T\}}\right] \\ &= A_0 N\left(d^+\left(A_0, \max\left\{\underline{L}e^{-rT}, B_0\right\}, T\right)\right) - \underline{L}e^{-rT}N\left(d^-\left(A_0, \max\left\{\underline{L}e^{-rT}, B_0\right\}, T\right)\right) \\ &\quad - \left(\frac{A_0}{B_0}\right)\left(\frac{B_0^2}{A_0}N\left(d^+\left(\frac{B_0^2}{A_0}, \max\left\{\underline{L}e^{-rT}, B_0\right\}, T\right)\right)\right. \\ &\quad \left. -\underline{L}e^{-rT}N\left(d^-\left(\frac{B_0^2}{A_0}, \max\left\{\underline{L}e^{-rT}, B_0\right\}, T\right)\right)\right) \end{aligned} \tag{10}$$

with $d^{\pm}(S,K,T) = \dfrac{\log\left(\frac{S}{K}\right) \pm \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$ and $N(x) = \displaystyle\int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{\frac{-u^2}{2}} du$. The shorted down–and–out call with strike $\bar{L}$ has the same form as (10) by replacing $\underline{L}$ by $\bar{L}$.

Finally, the expected rebate payment at the closure time has the form of

$$\begin{aligned} & \mathbb{E}_Q\left[e^{-r\tau}\min\{1,\lambda\}\underline{L}e^{-r(T-\tau)}\mathbf{1}_{\{\tau\le T\}}\right] \\ = \; & \min\{1,\lambda\}\underline{L}e^{-rT}\left[N\left(-d^-(A_0,B_0,T)\right) + \left(\frac{A_0}{B_0}\right)N\left(d^-(B_0,A_0,T)\right)\right]. \end{aligned}$$

The market–consistent value of the payoff to the sponsor is given by

$$\begin{aligned} V_S(A_0,0) \;=\; & \mathbb{E}_Q[e^{-rT}\left((1-\delta)[A_T-\bar{L}]^+ - (\underline{L}-A_T)^+\right)\mathbf{1}_{\{\tau>T\}}] \\ & + \mathbb{E}_Q[e^{-r\tau}\max\{\lambda-1,0\}\underline{L}e^{-r(T-\tau)}\mathbf{1}_{\{\tau\le T\}}]. \end{aligned}$$

The long down–and–out call option is determined analogous to (10) and the shorted down–and–out put option's value and the rebate payment are computed as follows:

$$\begin{aligned} & -\mathbb{E}_Q[e^{-rT}(\underline{L}-A_T)^+\mathbf{1}_{\{\tau>T\}}] \\ = \; & -\mathbf{1}_{\{\lambda<1\}}\Bigg\{\underline{L}e^{-rT}\Big[N\left(-d^-(A_0,\underline{L}e^{-rT},T)\right) - N\left(-d^-(A_0,B_0,T)\right)\Big] \\ & -A_0\Big[N\left(-d^+(A_0,\underline{L}e^{-rT},T)\right) - N\left(-d^+(A_0,B_0,T)\right)\Big] \\ & -\frac{A_0}{B_0}\Bigg[\underline{L}e^{-rT}\Big(N\left(-d^-(B_0^2,A_0^2\underline{L}e^{-rT},T)\right) - N\left(-d^-(B_0,A_0,T)\right)\Big) \\ & -\frac{B_0^2}{A_0}\Big(N\left(-d^+(B_0^2,A_0^2\underline{L}e^{-rT},T)\right) - N\left(-d^+(B_0,A_0,T)\right)\Big)\Bigg]\Bigg\} \\ & \mathbb{E}_Q[e^{-r\tau}\max\{\lambda-1,0\}\underline{L}e^{-r(T-\tau)}\mathbf{1}_{\{\tau\le T\}}] \\ = \; & \max\{\lambda-1,0\}\underline{L}e^{-rT}\left[N\left(-d^-(A_0,B_0,T)\right) + \left(\frac{A_0}{B_0}\right)N\left(d^-(B_0,A_0,T)\right)\right]. \end{aligned}$$

3.2. **Delayed closure procedure.** In the literature, various approaches are applied to valuing standard Parisian derivatives, such as Monte–Carlo algorithms (Andersen and Brotherton–Ratcliffe (1996)), binomial or trinomial trees (Avellaneda and Wu (1999); Costabile (2002)), partial differential equations (Haber et. al (2002)), finite–element methods (Stokes and Zhu (1999)) or the original inverse Laplace transform technique (initiated by Chesney et al. (1997)). However, most of these methods are very time–consuming if they are to obtain precise results. The inverse Laplace transform method is adopted here to price the standard Parisian claims. Further, as in Chen and Suchanecki (2007), the inverse procedure introduced by Bernard et al. (2005) is adopted to invert the Laplace transforms, which minimizes the computation time.

Under the new probability measure P (c.f. (7)), the value of the assets A_t can be expressed as

$$A_t = A_0 \exp\left\{\sigma\, Z_t\right\} \exp\{r\, t\}.$$

Under the risk–neutral probability measure Q, the price of a T–contingent claim with the payoff $\phi(A_T)$ corresponds to the expected discounted payoff

$$\mathbb{E}_Q\left[e^{-rT}\phi(A_T)\mathbf{1}_{\{T_B^- > T\}}\right].$$

This can be rephrased as follows:

$$e^{-(r+\frac{1}{2}m^2)T}\mathbb{E}_P\left[\mathbf{1}_{\{T_b^- > T\}}\phi(A_0 \exp\{\sigma Z_T\} \exp\{rT\}) \exp\{mZ_T\}\right].$$

The market-consistent value of the payoff to the beneficiary under the delayed closure procedure is hence determined by:

$$\begin{aligned}
& V_B(A_0, 0) \\
= \;& \mathbb{E}_Q[e^{-rT}\left(\underline{L} + [A_T - \underline{L}]^+ - (1-\delta)[A_T - \bar{L}]^+\right)\mathbf{1}_{\{T_B^- > T\}}] \\
& + \mathbb{E}_Q[e^{-rT_B^-}\min\{\underline{L}e^{-r(T-T_B^-)}, A_{T_B^-}\}\mathbf{1}_{\{T_B^- \le T\}}] \\
= \;& \mathbb{E}_Q\left[e^{-rT}\underline{L}\mathbf{1}_{\{T_b^- > T\}}\right] + e^{-\frac{1}{2}m^2T}\mathbb{E}_P\left[\left(A_0e^{\sigma Z_T} - \underline{L}e^{-rT}\right)^+ e^{mZ_T}\mathbf{1}_{\{T_b^- > T\}}\right] \\
& -(1-\delta)e^{-\frac{1}{2}m^2T}\mathbb{E}_P\left[\left(A_0e^{\sigma Z_T} - \bar{L}e^{-rT}\right)^+ e^{mZ_T}\mathbf{1}_{\{T_b^- > T\}}\right] \\
& + \mathbb{E}_P\left[e^{-\left(r+\frac{1}{2}m^2\right)T_b^-}\exp\{mZ_{T_b^-}\}\min\{\underline{L}e^{-r(T-T_b^-)}, A_{T_b^-}\}\mathbf{1}_{\{T_b^- \le T\}}\right] \\
:= \;& PDOC\left[A_0, B_0, \underline{L}e^{-rT}, r, r\right] - (1-\delta)\, PDOC[A_0, B_0, \bar{L}e^{-rT}, r, r] \\
& + \mathbb{E}_Q\left[e^{-rT}\underline{L}\mathbf{1}_{\{T_b^- > T\}}\right] + \mathbb{E}_P\left[e^{-\left(r+\frac{1}{2}m^2\right)T_b^-}\exp\{mZ_{T_b^-}\}\min\{\underline{L}e^{-r(T-T_b^-)}, A_{T_b^-}\}\mathbf{1}_{\{T_b^- \le T\}}\right].
\end{aligned}$$

It is observed that the price of this contingent claim consists of four parts: a deterministic guaranteed part $\underline{L}$ which is paid at maturity when the value of the assets has not remained below the barrier for a time longer than d, a long Parisian down–and–out call option with strike $\underline{L}e^{-rT}$, a shorted Parisian down–and–out call option with strike $\bar{L}e^{-rT}$ (multiplied by $1-\delta$), and a rebate paid immediately upon premature closure.

It is well known that the price of a Parisian down–and–out call option can be described as the difference of the price of a plain–vanilla call option and the price of a Parisian down–and–in call option with the same strike and maturity date, i.e.,

$$PDOC[A_0, B_0, \underline{L}e^{-rT}, r, r] = BSC[A_0, \underline{L}e^{-rT}, r] - PDIC[A_0, B_0, \underline{L}e^{-rT}, r, r].$$

Here the last component r in $PDIC$ and $PDOC$ is used to point out the fact that over time the barrier level increases by an exponential rate r. The price of the plain–vanilla call option is obtained by the Black–Scholes formula as follows:

$$\begin{aligned}
BSC\left[A_0, \underline{L}e^{-rT}, r\right] &= \mathbb{E}_Q\left[e^{-rT}\left(A_T - \underline{L}\right)^+\right] \\
&= A_0N\left(d^+(A_0, \underline{L}e^{-rT}, T)\right) - e^{-rT}\underline{L}N\left(d^-(A_0, \underline{L}e^{-rT}, T\right).
\end{aligned}$$

To calculate $PDIC$ we distinguish between $B_0 \leq e^{-rT}\underline{L}$ (i.e. $\lambda \leq 1$) and $B_0 > e^{-rT}\underline{L}$ (i.e. $\lambda > 1$) according to the relation between the barrier B_0 and the strike. A detailed derivation of $PDIC$ is conducted in Appendix 6.1.

The second term, the Parisian down–and–out call option with strike $\bar{L}$ can be computed analogously by distinguishing between $B_0 \leq \mathrm{e}^{-\mathrm{rT}}\bar{\mathrm{L}}$ and $B_0 > \mathrm{e}^{-\mathrm{rT}}\bar{\mathrm{L}}$. The cases $B_0 \leq \mathrm{e}^{-\mathrm{rT}}\bar{\mathrm{L}}$ and $B_0 > \mathrm{e}^{-\mathrm{rT}}\bar{\mathrm{L}}$ are equivalent to $\lambda \leq \mathrm{e}^{\mathrm{iT}}$ and $\lambda > \mathrm{e}^{\mathrm{iT}}$, respectively. The third term in the payoff function can be calculated as follows:

$$\begin{aligned}
\mathbb{E}_Q[e^{-rT}\underline{L}\mathbf{1}_{\{T_b^- > T\}}] &= e^{-rT}\underline{L} - \mathbb{E}_Q[e^{-rT}\underline{L}\mathbf{1}_{\{T_b^- \leq T\}}] \\
&= e^{-rT}\underline{L}\left[1 - e^{-\frac{m^2 T}{2}}\left(\int_{-\infty}^{b} h_2(T,y)e^{my}dy + \int_{b}^{\infty} h_1(T,y)e^{my}dy\right)\right].
\end{aligned}$$

In the calculation of the expected rebate, distinction of cases becomes necessary again. For the case of $\lambda < 1$, the beneficiary will get $A_{T_b^-}$ upon early closure. Therefore, the expected rebate can be calculated as follows:

$$\begin{aligned}
& \mathbb{E}_P\left[e^{-\left(r+\frac{1}{2}m^2\right)T_b^-}\exp\{mZ_{T_b^-}\}\min\{L_{T_b^-}e^{-r(T-T_b^-)}, A_{T_b^-}\}1_{\{T_b^- \leq T\}}\right] \\
= \; & A_0\mathbb{E}_P\left[e^{-\frac{1}{2}m^2 T_b^-}\exp\{(m+\sigma)Z_{T_b^-}\}1_{\{T_b^- \leq T\}}\right] \\
= \; & A_0\mathbb{E}_P\left[e^{-\frac{1}{2}m^2 T_b^-}1_{\{T_b^- \leq T\}}\right]\mathbb{E}_P\left[\exp\{(m+\sigma)Z_{T_b^-}\}\right].
\end{aligned}$$

The last equality follows from the fact that T_b^- and $Z_{T_b^-}$ are independent, which is shown in the appendix of Chesney et al. (1997). Furthermore, the corresponding laws for these two random variables are given in Chesney et al. (1997), too. As a consequence, we obtain

$$\mathbb{E}_P\left[\exp\{(m+\sigma)Z_{T_b^-}\}\right] = \int_{-\infty}^{b} e^{(m+\sigma)x}\,\frac{b-x}{d}\,\exp\left\{-\frac{(x-b)^2}{2d}\right\}dx$$

and

$$\mathbb{E}_P\left[e^{-\frac{1}{2}m^2 T_b^-}1_{\{T_b^- < T\}}\right] = \int_{d}^{T} e^{-\frac{1}{2}m^2 t}\,h_3(t)\,dt,$$

where $h_3(t)$ denotes the density of the stopping time T_b^-. This density can be calculated by inverting the following Laplace transform

$$\hat{h}_3(\eta) = \frac{\exp\{\sqrt{2\eta}b\}}{\psi(\sqrt{2\eta d})}.$$

For the case of $\lambda \geq 1$, we obtain

$$\begin{aligned}
& \mathbb{E}_P\left[e^{-\left(r+\frac{1}{2}m^2\right)T_b^-}\exp\{mZ_{T_b^-}\}\min\{\underline{L}e^{-r(T-T_b^-)}, A_{T_b^-}\}1_{\{T_b^-\leq T\}}\right] \\
= \; & A_0\mathbb{E}_P\left[e^{-\frac{1}{2}m^2T_b^-}\exp\{(m+\sigma)Z_{T_b^-}\}1_{\{T_b^-\leq T\}}1_{\{Z_{T_b^-}\leq k\}}\right] \\
& +\underline{L}e^{-rT}\mathbb{E}_P\left[e^{-\frac{1}{2}m^2T_b^-}\exp\{mZ_{T_b^-}\}1_{\{T_b^-\leq T\}}1_{\{k<Z_{T_b^-}<b\}}\right] \\
= \; & A_0\mathbb{E}_P\left[e^{-\frac{1}{2}m^2T_b^-}1_{\{T_b^-\leq T\}}\right]\mathbb{E}_P\left[\exp\{(m+\sigma)Z_{T_b^-}\}1_{\{Z_{T_b^-}<k\}}\right] \\
& +\underline{L}e^{-rT}\mathbb{E}_P\left[e^{-\frac{1}{2}m^2T_b^-}1_{\{T_b^-\leq T\}}\right]\mathbb{E}_P\left[\exp\{mZ_{T_b^-}\}1_{\{k<Z_{T_b^-}<b\}}\right].
\end{aligned}$$

with $k = \frac{1}{\sigma}\ln\frac{\underline{L}e^{-rT}}{A_0}$. All the expectations can be calculated in the same way as when $\lambda < 1$.

The present value of the payoff to the sponsor is given by

$$\begin{aligned}
V_S(A_0,0) \; = \; & \mathbb{E}_Q[e^{-rT}\left((1-\delta)[A_T-\bar{L}]^+ - (\underline{L}-A_T)^+\right)\mathbf{1}_{\{T_B^->T\}}] \\
& +\mathbb{E}_Q[e^{-rT_B^-}\max\{\lambda-1,0\}\underline{L}e^{-r(T-T_B^-)}\mathbf{1}_{\{T_B^-\leq T\}}] \\
:= \; & (1-\delta)\,PDOC[A_0,B_0,\bar{L}e^{-rT},r,r] - PDOP\left[A_0,B_0,\underline{L}e^{-rT},r,r\right] \\
& +\mathbb{E}_P\left[e^{-\left(r+\frac{1}{2}m^2\right)T_b^-}\exp\{mZ_{T_b^-}\}\max\{A_{T_b^-}-\underline{L}e^{-r(T-T_b^-)},0\}\mathbf{1}_{\{T_b^-\leq T\}}\right].
\end{aligned}$$

The long Parisian down–and–out call option has been determined. The shorted Parisian down–and–out put option's value can be derived by the following in–out–parity:

$$PDOP[A_0,B_0,\underline{L}e^{-rT},r,r] := BSP[A_0,\underline{L}e^{-rT},r] - PDIP[A_0,B_0,\underline{L}e^{-rT},r,r].$$

Here $BSP[A_0,\underline{L}e^{-rT},r]$ gives the price of the plain–vanilla put option and $PDIP[A_0,B_0,\underline{L}e^{-rT},r,r]$ the price of the Parisian down–and–in put option. $BSP[A_0,\underline{L}e^{-rT},r]$ is derived by the Black–Scholes formula:

$$\begin{aligned}
BSP\left[A_0,\underline{L}e^{-rT},r\right] \; &= \; \mathbb{E}_Q\left[e^{-rT}\left(\underline{L}-A_T\right)^+\right] \\
&= \; \underline{L}e^{-rT}N\left(-d^-(A_0,\underline{L}e^{-rT},T)\right) - A_0N\left(d^+(A_0,\underline{L}e^{-rT},T)\right).
\end{aligned}$$

Due to the different possible choices of the λ–value, different pricing formulas are obtained for the Parisian down–and–in put option (c.f. Appendix 6.2).
Concerning the rebate payment, the sponsor would possibly obtain a rebate payment in the case of $\lambda \geq 1$:

$$\begin{aligned}
& \mathbb{E}_P\left[e^{-\left(r+\frac{1}{2}m^2\right)T_b^-}\exp\{mZ_{T_b^-}\}\max\{A_{T_b^-}-\underline{L}e^{-r(T-T_b^-)},0\}1_{\{T_b^-\leq T\}}\right] \\
= \; & A_0\mathbb{E}_P\left[e^{-\frac{1}{2}m^2T_b^-}\exp\{(m+\sigma)Z_{T_b^-}\}1_{\{T_b^-\leq T\}}1_{\{k<Z_{T_b^-}<b\}}\right] \\
& -\underline{L}e^{-rT}\mathbb{E}_P\left[e^{-\frac{1}{2}m^2T_b^-}\exp\{mZ_{T_b^-}\}1_{\{T_b^-\leq T\}}1_{\{k<Z_{T_b^-}<b\}}\right].
\end{aligned}$$

Further calculations can be done analogously to the derivation of the expected rebate for the beneficiary.

4. Numerical analysis

In this section, we implement the valuation formulae obtained in Section 3 and the Appendix in Section 6. We carry out sensitivity analyses, i.e. calculate how the three regulatory/investment policy parameters (the investment policy's riskiness level σ, the regulation parameter λ and the recovery period d) affect the surplus participation rate δ based on a fair contract analysis. Put differently, how should the beneficiary and the sponsor divide the surplus for the pension deal to be fair given the investment policy of the pension fund and the regulatory environment[4]. The considered pension plan is a fair contract when:

$$\mathbb{E}_Q[e^{-rT}\psi_B(A_T)1_{\{\tau>T\}}] + \mathbb{E}_Q[e^{-r\tau}\Theta_B(A_\tau)1_{\{\tau\leq T\}}] \equiv P_0 = (1-\alpha)A_0 \tag{11}$$

where τ is the termination time of the pension fund under both the immediate and the delayed closure procedures. It says that a fair contract results when the initial market value of the pension liability equals the initial contributions made by the beneficiary. An alternative condition for a fair contract can be obtained from the viewpoint of the sponsor, i.e.

$$\mathbb{E}_Q[e^{-rT}\psi_S(A_T)1_{\{\tau>T\}}] + \mathbb{E}_Q[e^{-r\tau}\Theta_S(A_\tau)1_{\{\tau\leq T\}}] \equiv S_0 = \alpha\, A_0. \tag{12}$$

Equations (11) and (12) can both be used to conduct a fair contract analysis. For instance, if we are interested in determining the fair participation rate δ, these two equations lead to the same value for δ. For the following analysis, we fix the parameters:

$$A_0 = 100;\ \alpha = 0.1;\ \underline{L} = 120;\ \bar{L} = 188.20;\ T = 15;\ \sigma = 0.15;\ r = 0.04;\ \ d = 1.$$

A_0 has been chosen to be greater than $\underline{L}e^{-rT}$ (here the present value of the fixed payment discounted at the risk free rate equals 65.86), which reflects the fact that the initial contribution should be at least equal to the present value of the nominal pension liability. $\bar{L}$ is the fully indexed pension which is set equal to $\underline{L}e^{iT}$ with i being the parameter related to the expected CPI or wage growth etc. As a realistic i value must be smaller than the risk free rate r, we choose $i = 3\%$ which leads to an $\bar{L}$ of 188.20. Furthermore, the parameter λ is chosen to ensure that the initial asset value lies above the barrier level, i.e. $\lambda \in (0, A_0e^{rT}/\underline{L}]$ (for the chosen parameters λ shall be set in the interval $(0, 1.52]$). $T = 15$ is chosen because it is approximately the average duration of pension liabilities in reality. A volatility (σ) of 15% is also a reasonable number for a diversified portfolio.

Table 1 demonstrates how the various components of the contract values (for both the beneficiary and the sponsor) are influenced by the recovery period d and the regulation

[4]Fair contracts have been studied e.g. in Grosen and Jørgensen (2002), Chen and Suchanecki (2007) and Døskeland and Nordahl (2008).

parameter λ (d and λ are input parameters). In all rows the participation rate (δ) is chosen such that a fair contract results. For comparability, the simple case of DB plan where all the options expire at T (no barrier/Parisian barrier framework is involved) is also demonstrated in the first row. In this case, it is unnecessary to formulate the rebate payment. Compared to the standard barrier option framework, the resulting fair participation rate is rather low (0.27), due to the fact that the beneficiary is assured of the defined benefit given in Equation (1) in a simple DB plan and therefore does not have to be compensated that much for downside risk. Next, we turn to the barrier framework. For $d = 0$, there is no recovery time after default for the pension fund. Therefore, a standard Parisian option with $d = 0$ in fact corresponds to a standard down–and–out barrier option. Below, we observe the following three relations. First, a positive relation exists between the Parisian down–and–out call and the recovery period. The longer the allowed excursion, the larger the value of the option. In fact, the value of the call does not change much with the length of excursion when a certain level of d is reached, i.e. the value of the Parisian down–and–out call is a concave increasing function of d. Second, the value of the Parisian down–and–out call does not increase substantially in d when the barrier level is extremely low or the strike of the call is fairly high. In the extreme case, if the regulation parameter λ is set at zero, which results in a barrier level of zero, it then follows that the recovery period d has no effect on any of the components of the contract values (including the Parisian down–and–out call), because the asset price can never hit the barrier in this situation due to the log–normal assumption of the asset dynamics. For the given parameters which lead to a rather low barrier level and rather high strikes (both $\underline{L}$ and $\bar{L}$ are higher than A_0), the resulting Parisian down–and–out call values do not increase much in d. Third, the fixed payment arises only when the asset price process does not remain below the barrier for a time longer than d. Hence, as the size of d goes up, the probability that the fixed payment will become due increases. Consequently, the expected value of the fixed payment rises with d. Its magnitude is bounded from above by the payment $\underline{L}e^{-rT}$. In contrast, the rebate payment appears only when the pension fund is closed, i.e., when the asset price process stays below the barrier for a period longer than d. Therefore, the longer the recovery period, the smaller the expected rebate payment. The Parisian put option changes with the length of excursion in a similar way as Parisian call option. It increases with d but the extent to which it increases becomes smaller after a certain level of d is reached.

Although d has monotonic effects on the values of the call, the put, the fixed payment and the rebate payment (positive relation between the $PDOC$ and d, between the expected fixed payment and d, and negative relation between the shorted $PDOC$ and d, between the rebate payment and d), adding up their effects, a non-monotonic effect of d is observed on the fair participation rate δ. However, we point out that this argument depends on the parameter choice. If the effects of the call and the fixed payments dominate, a negative

	d	δ	C($\underline{L}$)	SC($\bar{L}$)	FP	RB	V_B	LC($\bar{L}$)	SP	RS	V_S
Simple case: all options expire at T											
Black&Scholes	–	0.27	45.39	-21.25	65.86	–	90	21.25	-11.25	–	10
$\lambda = 0.8 \Rightarrow B_0 = 52.69$											
Standard		0.52	39.13	-10.20	41.91	19.14	90	10.20	-0.20	0	10
Parisian	0	0.52	39.13	-10.20	41.91	19.14	90	10.20	-0.20	0	10
	0.25	0.51	39.88	-10.58	46.91	13.84	90	10.58	-0.58	0	10
	0.50	0.49	40.06	-10.83	48.88	11.96	90	10.83	-0.83	0	10
	1.00	0.47	40.25	-11.27	51.55	9.56	90	11.27	-1.27	0	10
	3.00	0.41	40.45	-12.77	57.47	4.76	90	12.77	-2.77	0	10
$\lambda = 0.9 \Rightarrow B_0 = 59.27$											
Standard		0.52	37.25	-10.02	35.03	27.75	90	10.02	-0.02	0	10
Parisian	0	0.52	37.25	-10.02	35.03	27.75	90	10.02	-0.02	0	10
	0.25	0.52	38.85	-10.17	40.74	20.46	90	10.17	-0.17	0	10
	0.50	0.52	39.29	-10.31	43.03	17.94	90	10.31	-0.31	0	10
	1.00	0.50	39.74	-10.60	46.19	14.79	90	10.60	-0.60	0	10
	3.00	0.45	40.31	-11.88	53.63	7.76	90	11.88	-1.88	0	10
$\lambda = 1 \Rightarrow B_0 = 65.86$											
Standard		0.52	34.14	-10.00	28.23	37.65	90	10.00	0	0	10
Parisian	0	0.52	34.14	-10.00	28.23	37.65	90	10.00	0	0	10
	0.25	0.52	36.99	-10.03	34.47	28.47	90	10.03	-0.03	0	10
	0.50	0.52	37.82	-10.08	37.00	24.99	90	10.08	-0.08	0	10
	1.00	0.52	38.71	-10.15	40.52	20.76	90	10.15	-0.15	0	10
	3.00	0.48	39.95	-11.19	49.24	11.97	90	11.19	-1.19	0	10
$\lambda = 1.1 \Rightarrow B_0 = 72.44$											
Standard		0.69	29.73	-5.57	21.71	44.14	90	5.57	0	4.43	10
Parisian	0	0.69	29.73	-5.57	21.71	44.14	90	5.57	0	4.43	10
	0.25	0.54	34.09	-9.22	28.27	37.26	90	9.22	0	0.78	10
	0.50	0.53	35.45	-9.63	31.00	33.35	90	9.63	-0.02	0.39	10
	1.00	0.52	36.97	-9.91	34.83	27.27	90	9.91	-0.08	0.17	10
	3.00	0.50	39.26	-10.63	44.28	17.94	90	10.63	-0.67	0.04	10
$\lambda = 1.2 \Rightarrow B_0 = 79.03$											
Standard		0.99	24.10	-0.15	15.66	50.20	90	0.15	0	9.85	10
Parisian	0	0.99	24.10	-0.15	15.66	50.20	90	0.15	0	9.85	10
	0.25	0.68	30.12	-5.91	22.28	43.52	90	5.91	0	4.09	10
	0.50	0.61	32.08	-7.40	24.81	41.00	90	7.40	0	2.60	10
	1.00	0.57	34.38	-8.68	29.23	35.40	90	8.68	-0.02	1.34	10
	3.00	0.53	38.07	-9.91	36.75	25.73	90	9.91	-0.30	0.39	10

TABLE 1. Decomposition of the fair contract value with parameters: $A_0 = 100$; $\underline{L} = 120$; $\bar{L} = 188.20$; $T = 15$; $\sigma = 0.15$; $r = 0.04$. $C(\underline{L})$ stands for the $PDOC(\underline{L})$, $SC(\bar{L})$ for the shorted $PDOC(\bar{L})$ (multiplied by $(1-\delta)$), FP the fixed payment, RB and RS the rebate payment of the beneficiary and the sponsor, SP the shorted put value, V_B and V_S the contract value of the beneficiary and the sponsor.

d	0.5	1	1.5	2	2.5	3
contract value ($\lambda = 0.9$)	93.06	93.16	93.96	94.80	95.20	95.31
contract value ($\lambda = 1.0$)	93.55	92.58	92.73	94.19	95.19	95.26
contract value ($\lambda = 1.1$)	94.16	93.36	91.13	92.57	95.10	95.96

TABLE 2. Contract value V_B as a function of d and different λ values ($\delta = 0.75$)

relation between the fair participation rate δ and the recovery period d results from fair contract analysis. The reversed effect is observed when the effects of the shorted call and the rebate payment dominate. Therefore, under certain circumstances, it is possible to observe a monotonic change of the contract value with respect to d. Specifically, for the considered parameters, a decreasing effect of d on the fair participation rate results. Concerning the regulation parameter λ, first of all, it is noted that different λ–values lead to different values of the barrier ($B_0 = \lambda \underline{L} e^{-rT}$). The higher the required funding ratio, the more likely that this barrier is hit (from above) and the values of the Parisian down–and–out call and put decrease, as does the value of fixed payment. In contrast, the expected value of the rebate increases with the barrier because the rebate payment is based on a countercondition as other components. The above non-monotonic effect of d on the contract value for the beneficiary V_B can be observed in Table 2 (for $\delta = 0.75$). Furthermore, d has a more apparent effect on the contract value if the regulation level (barrier level) is set higher.

Figures 3 and 4 depict the fair participation rate δ as a function of the investment policy σ for different recovery periods d. As the volatility σ goes up, the value of the Parisian down–and–out call increases, while the value of the Parisian down–and–out put increases with the volatility at first and then decreases (hump–shaped). The value of the fixed payment goes down and the rebate term behaves similarly to the Parisian down–and–out put, i.e. goes up at first and then goes down after a certain level of volatility is reached. In all, as pension funds pursue a riskier investment policy, the beneficiaries should be given a higher share in the surplus to make it a fair pension deal. Overall a positive relation between δ and σ is observed. As mentioned before, the recovery period d does not necessarily affect the fair participation rate monotonically. For $\lambda = 0.9$, the higher the length of the regulatory recovery period, the lower the fair participation rate δ. Whereas for $\lambda = 1.1$, δ does not decrease in d monotonically. In Figures 5 and 6, the relation between the fair participation rate δ and α is illustrated for different lengths of excursion. A negative relation between δ and α results, meaning the less money the sponsor contributes the higher the fair participation rate should be.

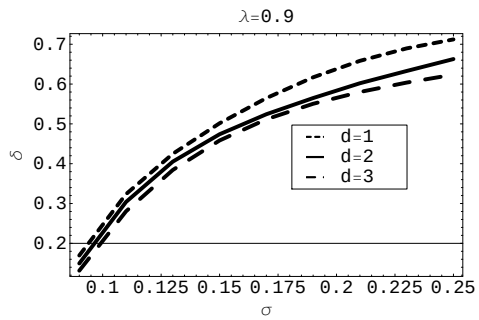

FIGURE 3. Fair combinations of δ and σ.

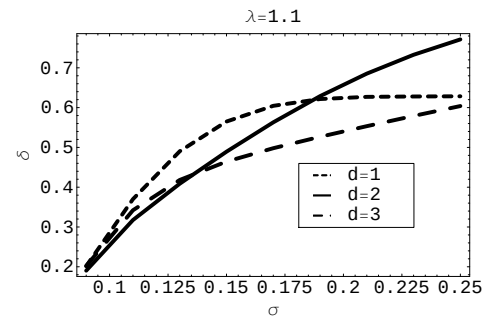

FIGURE 4. Fair combinations of δ and σ.

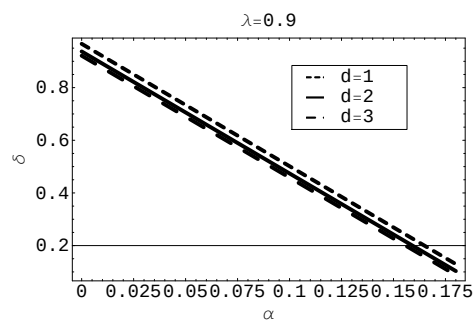

FIGURE 5. Fair combinations of δ and α.

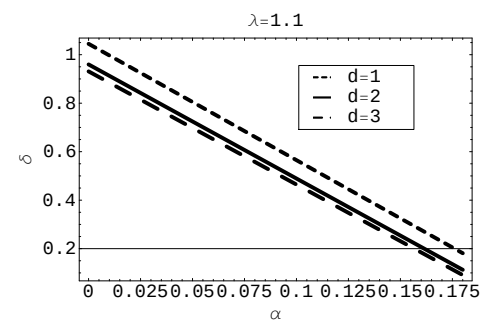

FIGURE 6. Fair combinations of δ and α.

5. CONCLUSION

The present paper considers the interaction of pension fund regulation and pension fund investment policy on the market-consistent valuation of defined benefit pension liabilities. Typically, premature closure of a DB plan is triggered by a low funding ratio, e.g. if this ratio of assets to liabilities hits the applicable regulatory minimum. We assume that early termination leads to an unwinding of the pension scheme and the assets are transferred to the beneficiaries. We distinguish between an immediate and a delayed closure procedure. In the former case, the moment the regulatory boundary is reached, the pension contract is immediately terminated. Whereas in the latter case, a grace period is given for reorganization and recovery. In several European jurisdictions this recovery period is 1, 3 or even 5 years by law. For both procedures, we derive closed–form formulae for the contracts which enable us to perform a fair contract analysis. A pension deal is defined economically fair if the initial contribution made by the participants to the pension fund equals the market–consistent value of the claim they get in return. Thereupon, the emphasis is placed particularly on how the interaction between the regulatory rules (required funding ratio and maximum recovery period) and the pension fund investment policy influences

the optimal amount by which the beneficiaries should participate in the pension fund's surplus. This is relevant for the contemporary discussion on "who owns the pension fund's surplus". Several ceteris paribus insights follow from this analysis. First, as the pension fund pursues a more risky investment strategy, the beneficiary should claim a higher stake in the pension fund's surplus for the deal to be fair. Otherwise, the higher return volatility transfers value from the beneficiary to the sponsor. Second, a longer regulatory recovery period can be accompanied by a somewhat lower beneficiary's claim on the surplus. A longer recovery period increases the probability that the fixed defined benefit payment at maturity will become due. Third, a lower required funding ratio can be accompanied with a lower claim on the surplus as it lowers the probability of a premature closure thereby lowering the value of any early rebate payment but increasing the value of the fixed payment at maturity.

Acknowledgements:

The authors would like to thank Jaap Bikker, Paul Cavelaars, Theo Nijman, Antoon Pelsser, Marc Pröpper and Walter Stortelder for helpful comments and suggestions and research assistance. A special gratitude is be given to Michael Suchanecki from whom the authors benefited much through numerous discussions with him. An Chen is grateful to Netspar and the Department of Quantitative Economics at the University of Amsterdam for their support.

6. Appendix

6.1. **Valuation of a Parisian down-and-in call.** To calculate $PDIC$ we shall distinguish between $B_0 \leq e^{-rT}\underline{L}$ (i.e. $\lambda \leq 1$) and $B_0 > e^{-rT}\underline{L}$ (i.e. $\lambda > 1$) according to the relation between the barrier B_0 and the strike. For $B_0 \leq e^{-rT}\underline{L}$, $PDIC[A_0, B_0, \underline{L}e^{-rT}, r, r]$ can be calculated as follows:

$$PDIC[A_0, B_0, \underline{L}e^{-rT}, r, r] \;=\; e^{-\frac{1}{2}m^2T}\int_k^\infty e^{my}\left(A_0e^{\sigma y} - \underline{L}e^{-rT}\right)h_1(T,y)dy$$

with $k = \frac{1}{\sigma}\ln\frac{e^{-rT}\underline{L}}{A_0}$. The density $h_1(T,y)$ is uniquely determined by inverting the corresponding Laplace transform which is given by

$$\begin{aligned} \hat{h}_1(\eta, y) &= \frac{e^{(2b-y)\sqrt{2\eta}}\psi(-\sqrt{2\eta d})}{\sqrt{2\eta}\psi(\sqrt{2\eta d})} \\ \text{with} \qquad \psi(z) &= \int_0^\infty x\exp\left(-\frac{x^2}{2} + zx\right)dx = 1 + z\sqrt{2\pi}e^{\frac{z^2}{2}}N(x), \end{aligned}$$

and η the parameter of Laplace transform. For the case of $B_0 > e^{-rT}\underline{L}$, we have

$$\begin{aligned} PDIC[A_0, B_0, \underline{L}e^{-rT}, r, r] &= e^{-\frac{1}{2}m^2T}\Bigg(\int_k^b e^{my}\left(A_0e^{\sigma y} - \underline{L}e^{-rT}\right)h_1(T,y)dy \\ &\quad + \int_b^\infty e^{my}\left(A_0e^{\sigma y} - \underline{L}e^{-rT}\right)h_1(T,y)dy\Bigg) \end{aligned}$$

As before, $h_1(T,y)$ and $h_2(T,y)$ are calculated by inverting the corresponding Laplace transform. $\hat{h}_1(T,y)$ has the same value as before and the Laplace transform of $h_2(T,y)$ is given by

$$\begin{aligned} \hat{h}_2(\eta, y) &= \frac{e^{y\sqrt{2\eta}}}{\sqrt{2\eta}\psi(\sqrt{2\eta d})} + \frac{\sqrt{2\eta d}e^{\eta d}}{\psi(\sqrt{2\eta d})}\Bigg(e^{y\sqrt{2\eta}}\left(N\left(-\sqrt{2\eta d} - \frac{y-b}{\sqrt{d}}\right) - N(-\sqrt{2\eta d})\right) \\ &\quad - e^{(2b-y)\sqrt{2\eta}}N\left(-\sqrt{2\eta d} + \frac{y-b}{\sqrt{d}}\right)\Bigg). \end{aligned}$$

6.2. **Valuation of a Parisian down-and-in call.** Due to the different possible choices of the λ–value, different pricing formulas are obtained for the Parisian down–and–in put option. A $\lambda < 1$, which leads to the fact that the strike (here $\underline{L}e^{-rT}$) is larger than the barrier (B_0), results in

$$\begin{aligned} PDIP[A_0, B_0, \underline{L}e^{-rT}, r, r] &= e^{-\frac{1}{2}m^2T}\Bigg(\int_{-\infty}^b e^{my}(\underline{L}e^{-rT} - A_0e^{\sigma y})h_2(T,y)dy \\ &\quad + \int_b^k e^{my}(\underline{L}e^{-rT} - A_0e^{\sigma y})h_1(T,y)dy\Bigg) \end{aligned}$$

with $k = \frac{1}{\sigma}\ln\left(\frac{\underline{L}e^{-rT}}{A_0}\right)$. As before, $h_1(T,y)$ and $h_2(T,y)$ are calculated by inverting the corresponding Laplace transform. $\hat{h}_1(T,y)$ and $\hat{h}_2(T,y)$ have the same values as before. Analogously, for the case of $\lambda \geq 1$, the Parisian down–and–in put option has the form of

$$PDIP[A_0, B_0, \underline{L}e^{-rT}, r, r] = e^{-\frac{1}{2}m^2T}\int_{-\infty}^k e^{my}(\underline{L}e^{-rT} - A_0e^{\sigma y})h_2(T,y)dy.$$

References

Aaronson, S. and Coronado, J. (2005): *Are firms or workers behind the shift way from DB pension plan?* Federal Reserve Board Finance and Economics Discussion Series Working paper no.2005-17.

Anderluh, J. and van der Weide, H. (2004): *Parisian options – the implied barrier concept.* Bubak, M., van Albada, G.D., Sloot, P.M.A., Dongarra, J.J. [eds.]: Computational Science – ICCS 2004, Lecture Notes in Computer Science, Springer, 851-858.

Andersen, L. and Brotherton-Ratcliffe, R. (1996): *Exact Exotics.* Risk 9(10), 85-89.

Avellaneda, M. and Wu, L. (1999): *Pricing Parisian-Style Options with a Lattice Method.* International Journal of Theoretical and Applied Finance, 2(1), 1-16.

Bernard, C., Le Courtois, O. and Quittard–Pinon, F. (2005): *A new procedure for pricing Parisian options*, The Journal of Derivatives, 12(4), Summer 2005, 45–53.

Black, F. and Scholes, M. (1973): *The Pricing of Options and Corporate Liabilities*, Journal of Political Economy, 81, 637–654.

Bikker, J.A. and Vlaar, P.J.G. (2007): *Conditional indexation in defined benefit pension plans in the Netherlands*, The Geneva Papers, 32, 494–515.

Bodie, Z., Marcus, A.J. and Merton, C. (1988): *Defined Benefit versus Defined Contribution Pension Plans: What are the Real Trade-offs?* Pensions in the U.S. Economy, edited by Zvi Bodie, John Shoven, and David Wise. Chicago, IL: University of Chicago Press. 139–162.

Bulow, J.I. (1982): *What are corporate pension liabilities?*, Quarterly Journal of Economics, 97(3), 435–452.

Broadbent, J., Palumbo, M. and Woodmam, E. (2006): *The shift from defined benefit to defined contribution pension plans - implications for asset allocation and risk management.* www.bis.org/publ/wgpapers/cgfs27broadbent3.pdf.

CEIOPS: *Survey on fully funded, technical provisions and security mechanisms in the European occupational pension sector*, CEIOPS-OPSSC-01/08 Final, 31 March 2008.

Chen, A. and Suchanecki, M. (2007): *Default risk, bankruptcy procedures and the market value of life insurance liabilities*, Insurance: Mathematics and Economics, 40(2), 231–255.

Chesney, M., Jeanblanc–Picqué, M. and Yor, M. (1997): *Brownian excursions and Parisian barrier options*, Advances in Applied Probability, 29, 165–184.

Costabile, M. (2002): *A combinatorial approach for pricing Parisian options*, Decisions in Economics and Finance 25(2), 111–125.

Døskeland, T.M. and Nordahl, H.A. (2008): *Optimal pension insurance design*, The Journal of Banking & Finance, 32, 382–392.

Grosen, A. and Jørgensen P.L. (2002): *Life Insurance Liabilities at Market Value: An Analysis of Insolvency Risk, Bonus Policy, and Regulatory Intervention Rules in a Barrier Option Framework*, The Journal of Risk and Insurance 69(1), 63–91.

Haber, R.J., Schönbucher, P.J. and Wilmott, P. (1999): *Pricing Parisian options*, The Journal of Derivatives 6(3), 71–79.

Ippolito, R.A. (1985): *The labor contract and true economic pension liabilities*, American Economic Review, 75(5), 1031–1043.

Karatzas, I. and Shreve, S.E. (1991): *Brownian Motion and Stochastic Calculus (2nd Edition)*, Springer.

Moraux, F. (2002): *On pricing cumulative Parisian options.* Finance 23, 127–132.

Niehaus, G. and T. Yu (2005): *Cash-balance plan conversions: evidence on excise taxes and implicit contracts*, The Journal of Risk and Insurance, Vol. 72(2), 321–352.

Petersen, M. (1992): *Pension reversions and worker-stockholder wealth transfers*, Quarterly Journal of Economics, 107(3), 1033–1056.

Rich, D. (1994): *The mathematical foundations of barrier options.* Advances in Futures and Options Research 7 (1994), 267-311.

Rubinstein, M. and Reiner, E. (1991): *Unscrambling the binary code*, Risk 4(10). 75–83.

Sharpe, W.F. (1976): *Corporate Pension Funding Policy*, Journal of Financial Economics, 3(3), 183–193.

Snyder, G.L. (1969): *Alternative forms of Options*, Financial Analysts Journal, 25(1), 93–99.

Stokes, N. and Zhu, Z. (1999): *A finite element platform for pricing path–dependent exotic options*, Proceedings of the Quantitative Methods in Finance Conference, Australia.

Treynor, J.L. (1997): *The principles of sponsor pension finance*, The Journal of Finance, 32(2), 627–638.

Yang, T. (2005): *Understanding the Defined Benefit versus Defined Contribution Choice*, PRC WP 2005-4, Pension Research Council Working Paper, Pension Research Council, The Wharton School, University of Pennsylvania (http://www.pensions-institute.org/workingpapers/WP2005-4.pdf).