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Optimal Dividends and ALM under Unhedgeable Risk[*]

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Abstract

In this paper we develop a framework for optimal investment decisions for insurance companies under un-hedgeable risk. The perspective that we choose is from an insurance company that tries to maximise the stream of dividends paid to its shareholders. The policy instruments that the company has are the dividend policy and the investment policy. The insurance company can continue to pay dividends until bankruptcy, and hence the time of bankruptcy is also endogenously controlled by the dividend and investment policies. Using stochastic control theory, we derive simultaneously the optimal investment policy and the optimal dividend policy, taking the insurance risks to be given.

Keywords: Optimal dividends, ALM, Stochastic control theory, HJB equation

JEL-Classification: G22, G31, G35

MSC-Classification: 93E20, 62P05

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1. Introduction

Insurance companies are faced with risks of many types. These include financial risks such as risks inherent in the investment process, but also non-financial risks such as insurance claims. While financial risks are generally assumed to be hedgeable, which means that such risks can be replicated in the financial markets, insurance claims are generally considered to be unhedgeable as no replicating portfolio exists for most "insurance events".

In this paper we aim to develop a framework for optimal investment decisions for insurance companies under sources of un-hedgeable risks. The perspective that we choose is from an insurance company that tries to maximise the stream of dividends paid to its shareholders. The policy instruments that the company has to this end are the dividend policy and the investment policy. The insurance company can continue to pay dividends until bankruptcy, and hence the time of bankruptcy is also endogenously controlled by the dividend and investment policies.

The problem of optimizing dividends payout schemes has a long history in actuarial mathematics; see, for example, the early contributions by De Finetti (1957), Borch (1967, 1969), Bühlmann (1970) and Gerber (1972, 1979). More recently the study of the problem has received an important impulse by the application of controlled diffusion techniques; see for example Paulsen and Gjessing (1997) and the overview paper by Taksar (2000).

The starting point for this paper is the results obtained in the papers by Asmussen & Taksar (1997; AT in the following), Højgaard & Taksar (1999; HT99 in the following) and Højgaard & Taksar (2004; HT04 in the following). Especially the results of HT99 and HT04 are quite interesting. In HT99, they analyse the case where an insurance company finds both an optimal dividend policy and an optimal level of reinsurance. In HT04, they consider the case where also the investment risk can be controlled.

Unlike the HT99 and HT04 papers, which focus primarily on the mathematical derivation of the optimal policies, we want to investigate in our paper what the economic significance is of the results we find. In order to do this, we specify explicitly the asset and liability processes and we distinguish carefully between hedgeable risks (i.e. risks that are traded in financial markets) and non-hedgeable risks (i.e. insurance risks that cannot be traded in financial markets). Unlike HT04 who only consider a 1-dimensional case, we solve the general N-dimensional case for the space of investment opportunities. Given this setup, we can use our results to infer what price should be charged for the non-hedgeable risks such that the value of the insurance company is unchanged. We also derive what the probability distribution of the time of bankruptcy is, and we illustrate how this information can be used to calibrate the model such that the implied default probabilities are consistent with observed default probabilities for insurance companies.

The outline of this paper is as follows: In Section 2 we introduce our framework. In Section 3 we derive the optimal policies and in Section 4 we illustrate the derived solution by means of an example. Section 5 discusses the pricing of insurance and Section 6 studies the time of bankruptcy. Finally, Section 7 concludes.

2. Stylised insurance company

We fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $(\mathcal{F}_t)_{t \geq 0}$, which we assume to satisfy the usual assumptions (completed and right-continuous). . This filtration represents the flow of information on which decisions are based. All Brownian Motions that we consider below are defined on $(\Omega, \mathcal{F}, \mathbb{P})$ and adapted to this filtration.

Unlike the papers of AT and HT who postulate directly a stochastic process for the surplus (or "the reserves"), we will start by postulating the dynamics of liabilities and assets separately.

The process for the liabilities L_t is given by

(1) $\qquad dL_t = (\mu_B + \mu_M - m)dt + \sigma_I dW_I + \sigma_M dW_M$.

We assume that the liability process is driven by two sources of risk: the diffusion term $\sigma_I dW_I$ which represents the insurance risks, and the diffusion term $\sigma_M dW_M$ which represents financial market risk component of the liabilities. Many types of insurance liabilities like unit-linked or participating contracts have exposure to financial market risk. We will make the assumption that the Brownian Motions W_M and W_I are independent. The drift term consists of two parts: $\mu_B + \mu_M$ which represents the return of the financial market risks that the policyholders expect minus a margin m that the insurance company has built into its process to cover the insurance risks and management fees. We assume that there is competition in the insurance market and that m is exogenously given and not a control variable for the insurance company. Please note that the constants μ, m and σ are absolute quantities and not "percentages".

The assets A_t of the insurance company are given by the following stochastic differential equation:

(2) $\qquad dA_t = (\mu_B + \boldsymbol{\alpha'}\boldsymbol{\mu_A})dt + \boldsymbol{\alpha'}(\boldsymbol{\Sigma_A})^{\frac{1}{2}} \cdot d\mathbf{W_A}$.

We assume that the assets of the insurance company can only be invested in financial markets. However, the insurance company can choose from a universe of N investment categories. The $(N\text{x}1)$-vector $\boldsymbol{\mu_A}$ denotes the vector of expected excess investment returns over the risk-free rate, the $(N\text{x}N)$-matrix $\boldsymbol{\Sigma_A}$ denotes the covariance matrix of the investment returns (which implies that $\boldsymbol{\Sigma_A}$ has to be symmetric and positive semi-definite) and $\mathbf{W_A}$ is an N-dimensional Brownian Motion. The vector $\boldsymbol{\alpha}$ denotes the amounts that are invested in each of the N investment categories. Note that when $\boldsymbol{\alpha}=\mathbf{0}$ then the insurance company invests only in risk-free assets and earns the risk-free bond-return $\mu_B dt$ with certainty.

The surplus S_t of the insurance company is given by the difference in value between assets and liabilities. The insurance company remains solvent as long as $S_t > 0$. The surplus process is given by:

$$(3) \qquad dS_t = d(A_t - L_t) = (\boldsymbol{\alpha'}\boldsymbol{\mu_A} - \mu_M + m)dt + \begin{pmatrix} \boldsymbol{\alpha} \\ -1 \end{pmatrix}' \left(\begin{array}{c|c} \boldsymbol{\Sigma_A} & \boldsymbol{\sigma_{AM}} \\ \hline (\boldsymbol{\sigma_{AM}})' & \sigma_M^2 \end{array} \right)^{\frac{1}{2}} \cdot \begin{pmatrix} d\mathbf{W_A} \\ dW_M \end{pmatrix} - \sigma_I dW_I .$$

In equation (3) we have stacked the $N+1$ sources of financial market risk together in an $(N+1)$-vector, with an $(N+1)$x$(N+1)$ covariance matrix. The $(N$x$1)$-vector $\boldsymbol{\sigma_{AM}}$ denotes the covariance of each asset category with the insurance liability portfolio.

When the financial risk of the insurance liabilities is spanned by the N investment opportunities, then the vector $\boldsymbol{\sigma_{AM}}$ is collinear with the matrix $\boldsymbol{\Sigma_A}$ and as a consequence the $(N+1)$x$(N+1)$ covariance matrix is rank-deficient. In this case it is possible to choose a vector $\boldsymbol{\alpha}$ such that all financial risk drivers are eliminated. This is known as the *replicating portfolio*. In this case the surplus process reduces to $dS_t = mdt - \sigma_I dW_I$. This means that when the insurance company decides to invest in the replicating portfolio, the surplus process is driven by pure insurance risks only. The optimal dividend policy for this special case is investigated in the AT paper.

If the insurance company decides to deviate from the replicating portfolio then the surplus process may benefit from additional excess returns, but at the cost of increased risk. It is this risk/return trade-off which is the subject of so-called ALM (Asset-Liability Management) models.

To lighten the notation for the analysis the surplus process, we replace the $N+2$ Brownian Motions by a single diffusion term which has the same law:

$$(4) \qquad dS_t = d(A_t - L_t) = (\boldsymbol{\alpha'}\boldsymbol{\mu_A} - \mu_M + m)dt + \sqrt{\begin{pmatrix} \boldsymbol{\alpha} \\ -1 \end{pmatrix}' \left(\begin{array}{c|c} \boldsymbol{\Sigma_A} & \boldsymbol{\sigma_{AM}} \\ \hline (\boldsymbol{\sigma_{AM}})' & \sigma_M^2 \end{array} \right) \begin{pmatrix} \boldsymbol{\alpha} \\ -1 \end{pmatrix} + \sigma_I^2} \; dW .$$

[As a final point, we note that we will assume that (due to risk management or regulatory restrictions) there is an upper bound M on the investment position, that is, $\beta \leq M$.
AP1: we need a different constraint for this as α is now a vector...]

Remark 2.1
Although geometric (rather than arithmetic) specifications of the liability and asset processes (1) and (2) would perhaps be more appropriate, we consider arithmetic specifications for analytical tractability reasons. Also note that for a typical insurance companies the surplus S is a factor 10 or 20 smaller that the total asset portfolio A (or the liability portfolio L). Therefore an arithmetic specification of the resulting surplus process S_t seems a reasonable approximation.

3. Optimal policy

After HT we seek the optimal solution for the following dynamic programming problem:

$$\sup_{\boldsymbol{\alpha}} \quad \mathbf{E}\int_0^\tau e^{-ct}dD_t$$

(5)   s.t. $dS_t = d\left(A_t - L_t\right) = \left(\boldsymbol{\alpha'\mu_A} - \mu_M + m\right)dt + \sqrt{\begin{pmatrix}\boldsymbol{\alpha} \\ -1\end{pmatrix}'\left(\begin{array}{c|c}\boldsymbol{\Sigma_A} & \boldsymbol{\sigma_{AM}} \\ \hline (\boldsymbol{\sigma_{AM}})' & \sigma_M^2\end{array}\right)\begin{pmatrix}\boldsymbol{\alpha} \\ -1\end{pmatrix} + \sigma_I^2}\ dW$

$$S_0 = x - D_0$$

where D_t denotes the cumulative dividend payout process, and τ denotes the time of bankruptcy defined as $\tau = \inf\{t: S_t=0\}$, x denotes the initial surplus of the insurance company, and c denotes the (subjective) discount rate that shareholders use to discount future dividends. In our search for the optimal dividend and optimal investment policies we restrict ourselves to cumulative dividend payout processes that are adapted to $\left(\mathcal{F}_t\right)_{t \geq 0}$, that are non-decreasing and right-continuous and satisfy $D_{0-}=0$.

Remark 3.1
When deriving the optimal dividend and optimal investment policies it is assumed that the management of the insurance company acts in the shareholders' interests. We thus refrain from possible agency problems between shareholders and management.

Remark 3.2
In (5), the expected discounted dividend stream paid to the shareholders is maximised, implicitly assuming that shareholders admit a linear utility function. In particular the risk discount rate c of the shareholders does not react to change in riskiness of the balance sheet of the insurance company. The dynamic programming problem becomes much more complex in case one would consider non-linear utility functions; see Hubalek & Schachermayer (2004) or Thonhauser and Albrecher (2007) for extensions of the 1-dimensional case in this direction.

Remark 3.3
As is usual in ruin models, it is assumed that bankruptcy takes place when $S_t=0$ for the first time, even though in reality the insurance company may decide to raise external funds at (or prior to) such occasion. The decision whether or not to raise external funds would be based on a trade off between incurring high costs of external financing while realizing future profits on the one hand and not incurring high costs of external financing and not realizing future profits on the other hand. We refrain from making such trade off and assume that bankruptcy takes place with certainty as soon as $S_t=0$ for the first time.

Following HT, we define a value function $V(x)=\mathbf{E}\int e^{-ct}dD_t$, which is the expected value of the discounted dividends given the initial level of surplus x. Note that from this definition it follows that $V(0)=0$, because when $x=0$ the insurance company immediately goes bankrupt and no dividends will ever be paid to the shareholders.

Using similar arguments as HT we find that $V(x)$ satisfies the following HJB equation:

(6)

$$\max\left(\max_{\alpha}\left[\frac{1}{2}\left(\left(\begin{array}{c}\alpha\\-1\end{array}\right)'\left(\begin{array}{c|c}\Sigma_A & \sigma_{AM}\\ \hline (\sigma_{AM})' & \sigma_M^2\end{array}\right)\left(\begin{array}{c}\alpha\\-1\end{array}\right)+\sigma_I^2\right)V''(x)+\left(\alpha'\mu_A-\mu_M+m\right)V'(x)-cV(x)\right],1-V'(x)\right)=0$$

To construct a solution to (6), we assume there are three regions:
- $0<x<u_0$, where we follow a "dynamic" ALM policy with **[β<M]**, and no dividends are paid out;
- $u_0<x<u_1$, where we follow a "maximum risk" strategy with **[β=M]**, and no dividends are paid out;
- $u_1<x$, where we pay out immediately the excess surplus $x-u_1$ as dividends to the shareholders.

Such a dividend strategy is called a *barrier strategy*.

Remark 3.4
Using barrier strategy by insurance companies in the real world.

Let us start with the region $0<x<u_0$. In this region the function $V(x)$ must satisfy

(7)

$$\max_{\alpha}\left[\frac{1}{2}\left(\left(\begin{array}{c}\alpha\\-1\end{array}\right)'\left(\begin{array}{c|c}\Sigma_A & \sigma_{AM}\\ \hline (\sigma_{AM})' & \sigma_M^2\end{array}\right)\left(\begin{array}{c}\alpha\\-1\end{array}\right)+\sigma_I^2\right)V''(x)+\left(\alpha'\mu_A-\mu_M+m\right)V'(x)-cV(x)\right]=0.$$

The expression on the left-hand side is maximised for

(8)

$$\alpha*(x)=\left(-\frac{V'(x)}{V''(x)}\right)\underbrace{(\Sigma_A)^{-1}\mu_A}_{\text{Merton pf}}+\underbrace{(\Sigma_A)^{-1}\sigma_{AM}}_{\text{Hedge pf}}.$$

We can interpret optimal portfolio **α*** as follows: the optimal portfolio consists of two parts. The term $(\Sigma_A)^{-1}\sigma_{AM}$ is the hedge portfolio that replicates as much of the (financial) liability risks as possible. Note that this term does not depend on the level of the surplus x. The term $(\Sigma_A)^{-1}\mu_A$ is the mean-variance optimal "Merton portfolio". The exposure to the Merton portfolio depends only on the level of the surplus x through the quantity $-V'(x)/V''(x)$. Hence, we find that, like in the CAPM, we get a two-fund separation solution for the optimal portfolio, and therefore the optimal choice of N assets can be reduced to a 1-dimensional problem.

Remark 3.5
The result we have found has important consequences for the ALM-process for an insurance company. First, the insurance company can determine the optimal hedge portfolio $(\Sigma_A)^{-1}\sigma_{AM}$. This is a fixed portfolio that does not depend on the surplus position of insurance company, but is determined by the nature of the liability portfolio. Second, the insurance company can determine

the mean-variance optimal portfolio $(\mathbf{\Sigma_A})^{-1}\mathbf{\mu_A}$. This is the "speculative" portfolio that the insurance company uses to optimise its expected asset returns. The composition of this portfolio is given exogenously, only the amount invested in this portfolio depends on the surplus x. The variance σ_U^2 of the un-hedgeable risk consists of two components: $\left(\sigma_M^2 - \mathbf{\sigma_{AM}}'\mathbf{\Sigma_A^{-1}}\mathbf{\sigma_{AM}}\right) + \sigma_I^2$, the first term is the market risk of the portfolio of insurance liabilities that cannot be hedged by the optimal hedge portfolio, the second term is the variance σ_I^2 of the (non-traded) insurance risks. It is the variance σ_U^2 that determines the mean-variance trade-off in the ALM-process that the insurance company has to make.

If we define $\beta(x) = -V'(x)/V''(x)$ in expression (8) and substitute into (7) we obtain

$$\tfrac{1}{2}\left(\beta(x)^2\sigma_A^2 + \sigma_U^2\right)V''(x) + \left(\beta(x)\sigma_A^2 + \mu_U\right)V'(x) - cV(x) = 0$$

where

(7R)
$$\sigma_A^2 = \mathbf{\mu_A}'\mathbf{\Sigma_A^{-1}}\mathbf{\mu_A}$$
$$\sigma_U^2 = \left(\sigma_M^2 - \mathbf{\sigma_{AM}}'\mathbf{\Sigma_A^{-1}}\mathbf{\sigma_{AM}}\right) + \sigma_I^2$$
$$\mu_U = \left(\mathbf{\mu_A}'\mathbf{\Sigma_A^{-1}}\mathbf{\sigma_{AM}} - \mu_M\right) + m$$

We could substitute the definition for $\beta(x)$ back into equation (7R) and obtain a differential equation for $V(x)$. Unfortunately, the resulting non-linear differential equation is very difficult to solve directly. We therefore proceed along a different path and solve for $\beta(x)$.

Substituting $V''(x) = -V'(x)/\beta(x)$ into (7R) leads to

(9)
$$\frac{1}{2}\left(\sigma_A^2\beta(x) + 2\mu_U - \frac{\sigma_U^2}{\beta(x)}\right)V'(x) - cV(x) = 0.$$

Taking the derivative with respect to x of this equation leads to

(10)
$$\frac{1}{2}\left(\sigma_A^2 + \frac{\sigma_U^2}{\beta(x)^2}\right)\beta'(x)V'(x) + \frac{1}{2}\left(\sigma_A^2\beta(x) + 2\mu_U - \frac{\sigma_U^2}{\beta(x)}\right)V''(x) - cV'(x) = 0.$$

Substituting $V''(x) = -V'(x)/\beta(x)$ into (10) leads to:

(11)
$$\frac{1}{2}\left(\sigma_A^2 + \frac{\sigma_U^2}{\beta(x)^2}\right)\beta'(x)V'(x) - \frac{1}{2}\left(\sigma_A^2\beta(x) + 2\mu_U - \frac{\sigma_U^2}{\beta(x)}\right)\frac{V'(x)}{\beta(x)} - cV'(x) = 0.$$

As the value function V is an increasing function, we have that V' is strictly positive for all x. Hence, we are allowed to divide (11) by V' and we obtain a differential equation for $\beta(x)$:

(12)
$$\beta'(x) = \frac{\left(\sigma_A^2 + 2c\right)\beta^2(x) + 2\mu_U\beta(x) - \sigma_U^2}{\left(\sigma_A^2\beta^2(x) + \sigma_U^2\right)}.$$

This is a first order differential equation of the form

$$(13) \qquad \frac{d\beta}{dx} = \frac{A\beta^2 + B\beta - C}{\beta^2 + C} \quad \text{with} \quad A = \frac{\sigma_A^2 + 2c}{\sigma_A^2}, B = \frac{2\mu_U}{\sigma_A^2}, C = \frac{\sigma_U^2}{\sigma_A^2}.$$

We can express the solution for (13) in the form:

$$(14) \qquad \int \frac{\beta^2 + C}{A\beta^2 + B\beta - C} d\beta - C_\beta = \int dx.$$

The integration constant C_β can be found from setting $x{=}0$ in equation (9).

The integral on the right-hand side of (14) is trivial and is equal to x. The integral on the left-hand side of (14) is a rational function in β which can be integrated analytically. We find the following expression for $\beta(x)$:

$$(15) \qquad \frac{B^2 + 2A(1+A)C}{2A^2\sqrt{B^2 + 4AC}} \ln\left(\frac{2A\beta + B - \sqrt{B^2 + 4AC}}{2A\beta + B + \sqrt{B^2 + 4AC}} \right) - \frac{B}{2A^2} \ln\left(A\beta^2 + B\beta - C \right) + \frac{\beta}{A} - C_\beta = x.$$

If we set $x{=}0$ in equation (9), use $V(0){=}0$ and divide by $V'(0)$, we obtain

$$(16) \qquad \sigma_A^2 \beta(0) + 2\mu_U - \frac{\sigma_U^2}{\beta(0)} = 0.$$

If we multiply by $\beta(0)$ we obtain a quadratic equation. Selecting the positive root gives the following expression for $\beta(0)$:

$$(17) \qquad \beta(0) = -\frac{\mu_U}{\sigma_A^2} + \sqrt{\left(\frac{\mu_U}{\sigma_A^2} \right)^2 + \left(\frac{\sigma_U}{\sigma_A} \right)^2}.$$

If we substitute this expression for $\beta(0)$ into (15) for $x{=}0$ we can solve for C_β.

Note, that (15) is the expression for the *inverse function* of $\beta(x)$. Let us denote this inverse function by $x(\beta)$. Although we do not obtain an explicit expression for $\beta(x)$, the implicit equation (15) is quite useful. First, the inverse function $x(\beta)$ is strictly increasing. Hence $\beta(x)$ itself is also strictly increasing in x. So for increasing levels of the surplus x the optimal investment policy for the insurance company is to hold an ever increasing amount of market risk until the maximum level M is reached. The surplus level u_0 is defined as the first point where β reaches the maximum level M. If we substitute $\beta{=}M$ into (15) we obtain directly an analytical expression for u_0.

Let us now seek a solution for $V(x)$. The definition $\beta(x) = -V'(x)/V''(x)$ gives a differential equation for $V(x)$. If we take the reciprocal on both sides and integrate we obtain

(17a)
$$C_0 - \int \frac{1}{\beta(x)} dx = \int \frac{V''(x)}{V'(x)} dx \, .$$

The integral on the right-hand side evaluates easily to $\ln V'(x)$. The left-hand integral is slightly more complicated since we do not know an explicit expression for $\beta(x)$. We can evaluate the integral if we perform a change of variable from dx to $d\beta$. Using the Change of Variables Theorem $dx = (dx/d\beta)d\beta = 1/(d\beta/dx)\, d\beta$, we can substitute the expression for $\beta'(x)$ given in equation (14) into the left-hand side of equation (17a):

(17b)
$$C_0 - \int \frac{1}{\beta}\left(\frac{\beta^2 + C}{A\beta^2 + B\beta - C}\right) d\beta = \ln V'(x) \, .$$

The left-hand side is a rational function in β that can be integrated explicitly. After taking the exponential we obtain:

(17c)
$$V'(x) = e^{C_0}\beta(x)\frac{\left(\dfrac{2A\beta(x)+B+\sqrt{B^2+4AC}}{2A\beta(x)+B-\sqrt{B^2+4AC}}\right)^{\frac{B(A-1)}{2A\sqrt{B^2+4AC}}}}{\left(A\beta(x)^2+B\beta(x)-C\right)^{\left(\frac{(A+1)}{2A}\right)}}$$

$$= e^{C_0}\beta(x)\frac{\left(\dfrac{(\sigma_A^2+2c)\beta(x)+\mu_U+\sqrt{\mu_U^2+(\sigma_A^2+2c)\sigma_U^2}}{(\sigma_A^2+2c)\beta(x)+\mu_U-\sqrt{\mu_U^2+(\sigma_A^2+2c)\sigma_U^2}}\right)^{\frac{c\mu_U}{(\sigma_A^2+2c)\sqrt{\mu_U^2+(\sigma_A^2+2c)\sigma_U^2}}}}{\left(\dfrac{(\sigma_A^2+2c)\beta(x)^2+2\mu_U\beta(x)-\sigma_U^2}{\sigma_A^2}\right)^{\left(\frac{\sigma_A^2+c}{\sigma_A^2+2c}\right)}} \, .$$

If we substitute this result for $V'(x)$ into equation (9) we obtain the following expression for $V(x)$:

(17d)

$$V(x) = C_0^*\left(\sigma_A^2\beta(x)^2+2\mu_U\beta(x)-\sigma_U^2\right)\frac{\left(\dfrac{(\sigma_A^2+2c)\beta(x)+\mu_U+\sqrt{\mu_U^2+(\sigma_A^2+2c)\sigma_U^2}}{(\sigma_A^2+2c)\beta(x)+\mu_U-\sqrt{\mu_U^2+(\sigma_A^2+2c)\sigma_U^2}}\right)^{\frac{c\mu_U}{(\sigma_A^2+2c)\sqrt{\mu_U^2+(\sigma_A^2+2c)\sigma_U^2}}}}{\left((\sigma_A^2+2c)\beta(x)^2+2\mu_U\beta(x)-\sigma_U^2\right)^{\left(\frac{\sigma_A^2+c}{\sigma_A^2+2c}\right)}}$$

where C_0^* denotes an arbitrary constant that will be solved later.

On the interval $u_0 < x < u_1$, the insurance company will follow the "maximum risk" strategy by holding $\beta = M$. The value function $V_1(x)$ is therefore a solution of the equation

(18) $\frac{1}{2}\sigma_{MM}^2 V''(x) + \mu_{MM} V'(x) - cV(x) = 0$ with $\mu_{MM} = (M\mu_M + m), \sigma_{MM}^2 = (M^2\sigma_M^2 + \sigma_I^2)$.

The solution to this second order differential equation on the interval $u_0 < x < u_1$ is given by

(19) $V_1(x) = C_1 e^{\gamma_{M+}(x-u_0)} + C_2 e^{\gamma_{M-}(x-u_0)}$ with $\gamma_{M\pm} = \frac{1}{\sigma_{MM}^2}\left(-\mu_{MM} \pm \sqrt{\mu_{MM}^2 + 2c\sigma_{MM}^2}\right)$.

At the point $x=u_0$, we know $\beta(x)=M=-(V'/V'')$. We also know that $V(u_0)=V_1(u_0)$. These two pieces of information are sufficient to determine C_1 and C_2:

(20) $\begin{cases} V(u_0) = V_1(u_0) \\ 0 = \mu_M V_1'(u_0) + M\sigma_M^2 V_1''(u_0) \end{cases} \Rightarrow \begin{cases} V(u_0) = C_1 + C_2 \\ 0 = C_1\left(\mu_M\gamma_{M+} + M\sigma_M^2\gamma_{M+}^2\right) + C_2\left(\mu_M\gamma_{M-} + M\sigma_M^2\gamma_{M-}^2\right) \end{cases}$

The function $V_1(x)$ is therefore given by

(21) $V_1(x) = V(u_0)\dfrac{\left(\mu_M\gamma_{M+} + M\sigma_M^2\gamma_{M+}^2\right)e^{\gamma_{M-}(x-u_0)} - \left(\mu_M\gamma_{M-} + M\sigma_M^2\gamma_{M-}^2\right)e^{\gamma_{M+}(x-u_0)}}{\mu_M(\gamma_{M+} - \gamma_{M-}) + M\sigma_M^2(\gamma_{M+}^2 - \gamma_{M-}^2)}$.

We can now solve for the upper limit u_1. From equation (6) follows that for $x>u_1$, the insurance immediately pays out the excess surplus $x-u_1$ as dividends. Hence, the function $V(x)$ follows the equation $V'(x)=1$. The solution is given by $V_2(x)=C_3+x$. The point u_1 is the point where the function $V_1(x)$ makes a "smooth" contact with the function V_2. This means that the first and second derivatives should match at the point u_1.

Since $V_2(x)$ is a straight line, its second derivative is 0. We can solve u_1 from the equation $V_1''(u_1)=0$. This leads to:

(22) $u_1 = u_0 + \dfrac{1}{(\gamma_{M+} - \gamma_{M-})}\ln\left(\dfrac{\left(\mu_M\gamma_{M+} + M\sigma_M^2\gamma_{M+}^2\right)\gamma_{M-}^2}{\left(\mu_M\gamma_{M-} + M\sigma_M^2\gamma_{M-}^2\right)\gamma_{M+}^2}\right)$.

Given this value for u_1, we can now solve the constant $V(u_0)$ from the condition $V_1'(u_1)=1$. This leads to

(23) $V(u_0) = \dfrac{\mu_M(\gamma_{M+} - \gamma_{M-}) + M\sigma_M^2(\gamma_{M+}^2 - \gamma_{M-}^2)}{\left(\mu_M\gamma_{M+} + M\sigma_M^2\gamma_{M+}^2\right)\gamma_{M-}e^{\gamma_{M-}(u_1-u_0)} - \left(\mu_M\gamma_{M-} + M\sigma_M^2\gamma_{M-}^2\right)\gamma_{M+}e^{\gamma_{M+}(u_1-u_0)}}$.

Finally, we solve for C_3 from $V_1(u_1)-u_1=C_3$.
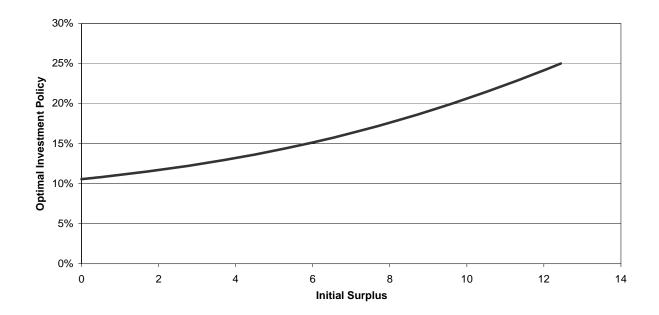
4. Example

Let us illustrate the derived solution with an example. The parameter specification is set as follows:

Table 4.1

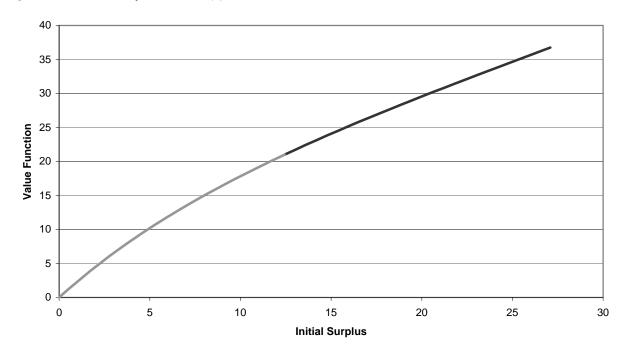| Parameter	Value
μ_M	5.25 (million)
σ_M	25.4 (million)
m	0.525 (million)
σ_I	4.60 (million)
M	25.0%
c	5.00%

Figure 4.1 displays the optimal investment policy as a function of the initial surplus.

Figure 4.1   *Optimal investment policy as a function of the initial surplus: β(x)*

Next, Figure 4.2 displays the expected value of the discounted dividends under the optimal investment and optimal dividend policy as a function of the initial surplus, i.e., the value function.

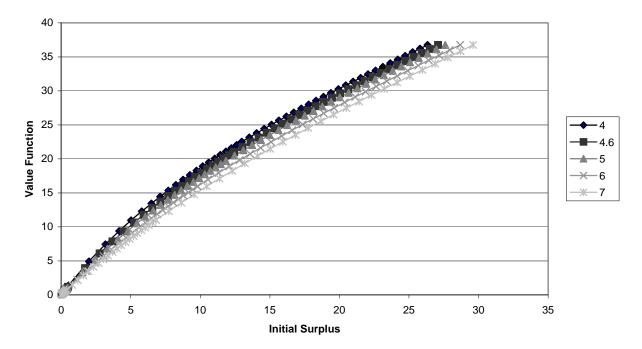***Figure 4.2***   *Value function: V(x)*

5. The pricing of insurance

The value function turns out to be decreasing in σ_I. The sensitivity of the value function with respect to σ_I can be interpreted as the *marginal price of insurance risk*. The marginal price of insurance risk is such that the shareholders are indifferent between bearing an additional unit of insurance risk (as measured by σ_I) while receiving an immediate dividend payout equal to the marginal price of insurance risk, and not bearing the additional unit of insurance risk.

Alternatively, to price insurance risk one may determine the increase of the margin *m* that offsets the decrease of the value function when an additional unit of insurance risk is borne. The increase of the margin *m* can be interpreted as the annual equivalent utility premium for one additional unit of insurance risk.

5.1 The example revisited

Let us consider again the example of Section 4. Figure 5.1 displays the sensitivity of the value function with respect to σ_I.

Figure 5.1 *Sensitivity of the value function with respect to σ_I*

6. The time of bankruptcy

In this section we study the distribution of the time of bankruptcy, τ. We denote by $\varphi(c,x)=\mathbf{E}\,e^{-c\tau}$, $c>0$, the *Laplace transform* of (the distribution function of) τ. It can be interpreted as the expected value of a payment 1 at the time of bankruptcy discounted by the dividend discount rate.

The function $\varphi(c, x)$ is a solution of the ode (7R) in the region $0<x<u_0$, and of the ode (18) in the region $u_0<x<u_1$.

Because the optimal dividend policy is a barrier strategy, the (modified) surplus process is a Brownian Motion with a *reflecting* barrier (at the level u_1 where the excess surplus is paid out as dividends to the shareholders) and an *absorbing* barrier (at the level 0 at which bankruptcy takes place). Hence, the function $\varphi(c, x)$ satisfies the following boundary conditions:

(24) $\qquad \varphi(c, 0) = 1$

(25) $\qquad d/dx\ \varphi(c, x{=}u_1) = 0$

6.1 Solution for φ in the region 0<x<u₀

In the region $0<x<u_0$, the function $V(x)$ given in (17d) is a particular solution to the ode (7R). The general solution for a linear second-order ode is given by the sum of two linearly independent functions. Given that we know one solution, the other solution $V_2(x)$ may be found using the so-called *reduction of order method* (see, for example, Weisstein (2004)).

Let us rewrite the ode (7R) as

$$V''(x) + P(x)V'(x) + Q(x)V(x) = 0$$

with

(26)
$$P(x) = \frac{\beta(x)\sigma_A^2 + \mu_U}{\frac{1}{2}\left(\beta(x)^2\sigma_A^2 + \sigma_U^2\right)}$$

$$Q(x) = \frac{-c}{\frac{1}{2}\left(\beta(x)^2\sigma_A^2 + \sigma_U^2\right)}.$$

The second solution $V_2(x)$ is now given by

(27)
$$V_2(x) = V(x)\int^x \frac{\exp\left\{-\int^y P(z)dz\right\}}{V(y)^2}\,dy.$$

We will proceed to build this solution in steps. The calculation is not straightforward as we only know the inverse function of $\beta(x)$ given in equation (15).

First, we determine the integral of $P(x)$ as follows:

$$\int P(x)dx = \int \frac{2\left(\beta(x)\sigma_A^2 + \mu_U\right)}{\left(\beta(x)^2\sigma_A^2 + \sigma_U^2\right)}dx$$

(28)
$$= \int \frac{2\left(\beta(x)\sigma_A^2 + \mu_U\right)}{\left(\beta(x)^2\sigma_A^2 + \sigma_U^2\right)}\frac{1}{\beta'(x)}d\beta(x)$$

$$= \int \frac{2\left(\beta(x)\sigma_A^2 + \mu_U\right)}{\left(\sigma_A^2 + 2c\right)\beta^2(x) + 2\mu_U\beta(x) - \sigma_U^2}d\beta(x)$$

where in the third line we have substituted the expression for $\beta'(x)$ given in (12).

The resulting integral can be evaluated explicitly and we obtain

$$\int P(x)dx = \frac{\sigma_A^2}{\left(\sigma_A^2 + 2c\right)}\ln\left(\left(\sigma_A^2 + 2c\right)\beta^2(x) + 2\mu_U\beta(x) - \sigma_U^2\right)+$$

(29)
$$\frac{2c\mu_U}{\left(\sigma_A^2 + 2c\right)\sqrt{\left(\sigma_A^2 + 2c\right)\sigma_U^2 + \mu_U^2}}\ln\left(\frac{\left(\sigma_A^2 + 2c\right)\beta(x) + \mu_U - \sqrt{\mu_U^2 + \left(\sigma_A^2 + 2c\right)\sigma_U^2}}{\left(\sigma_A^2 + 2c\right)\beta(x) + \mu_U + \sqrt{\mu_U^2 + \left(\sigma_A^2 + 2c\right)\sigma_U^2}}\right).$$

Substituting (17d) and (29) into equation (27) and simplifying yields:

$$V_2(x) = V(x)\int \frac{\left(\sigma_A^2 + 2c\right)\beta^2(x) + 2\mu_U\beta(x) - \sigma_U^2}{\left(\sigma_A^2\beta(x)^2 + 2\mu_U\beta(x) - \sigma_U^2\right)^2}dx$$

$$= V(x)\int \frac{\left(\sigma_A^2 + 2c\right)\beta^2(x) + 2\mu_U\beta(x) - \sigma_U^2}{\left(\sigma_A^2\beta(x)^2 + 2\mu_U\beta(x) - \sigma_U^2\right)^2}\frac{1}{\beta'(x)}d\beta(x)$$

(30)
$$= V(x)\int \frac{\left(\sigma_A^2\beta^2(x) + \sigma_U^2\right)}{\left(\sigma_A^2\beta(x)^2 + 2\mu_U\beta(x) - \sigma_U^2\right)^2}d\beta(x)$$

$$= V(x)\frac{-\beta(x)}{\sigma_A^2\beta(x)^2 + 2\mu_U\beta(x) - \sigma_U^2}.$$

When we substitute the expression for $V(x)$ given in (17d) we obtain finally:

(31)
$$V_2(x) = \beta(x)\frac{\left(\frac{(\sigma_A^2 + 2c)\beta(x) + \mu_U + \sqrt{\mu_U^2 + (\sigma_A^2 + 2c)\sigma_U^2}}{(\sigma_A^2 + 2c)\beta(x) + \mu_U - \sqrt{\mu_U^2 + (\sigma_A^2 + 2c)\sigma_U^2}}\right)^{\frac{c\mu_U}{(\sigma_A^2 + 2c)\sqrt{\mu_U^2 + (\sigma_A^2 + 2c)\sigma_U^2}}}}{\left((\sigma_A^2 + 2c)\beta(x)^2 + 2\mu_U\beta(x) - \sigma_U^2\right)^{\left(\frac{\sigma_A^2 + c}{\sigma_A^2 + 2c}\right)}}.$$

Please note that in the derivation for V_2 we have ignored any multiplicative constants, as they are irrelevant for obtaining a linearly independent solution. Also note that $V_2(x)$ we find here is equal (up to a constant) to $V'(x)$ given in (17c).

With this result, we can now derive an expression for $\varphi(c, x)$ as

$$\phi(c,x) = F_1 V(x) + F_2 V_2(x)$$

(32)

$$= \left(F_1\left(\sigma_A^2 \beta(x)^2 + 2\mu_U \beta(x) - \sigma_U^2\right) + F_2 \beta(x)\right)^{\left(\frac{(\sigma_A^2+2c)\beta(x)+\mu_U+\sqrt{\mu_U^2+(\sigma_A^2+2c)\sigma_U^2}}{(\sigma_A^2+2c)\beta(x)+\mu_U-\sqrt{\mu_U^2+(\sigma_A^2+2c)\sigma_U^2}}\right)^{\frac{c\mu_U}{(\sigma_A^2+2c)\sqrt{\mu_U^2+(\sigma_A^2+2c)\sigma_U^2}}}} \Big/ \left(\left(\sigma_A^2+2c\right)\beta(x)^2+2\mu_U\beta(x)-\sigma_U^2\right)^{\frac{\sigma_A^2+c}{\sigma_A^2+2c}},$$

where F_1 and F_2 are constant that have to be solved from the boundary conditions imposed on φ.

The lower boundary condition is given by (24) and is imposed for $x=0$. Note that at $x=0$ we have $V(x)=0$, and hence we can solve for F_2.

6.2 Solution for φ in the region u₀<x<u₁

In the region $u_0<x<u_1$ the function $\varphi(c,x)$ is given by the ode (18). This ode has a general solution given by (see also equation (19))

(33)
$$\varphi_1(c,x) = G_1 e^{\gamma_{M+}(x-u_0)} + G_2 e^{\gamma_{M-}(x-u_0)},$$

with $\gamma_{M\pm} = \gamma_{M\pm}(c)$ as defined in (19).

At the point $x=u_0$, we know $\beta(u_0)=M$. Furthermore, at $x=u_0$, the function φ must satisfy a "smooth pasting" condition. This implies that $\varphi(u_0)=\varphi_1(u_0)$ and that $\varphi'(u_0)=\varphi'_1(u_0)$. These two pieces of information are sufficient to determine G_1 and G_2:

(34)
$$\begin{cases} F_1 V(u_0) + F_2 V_2(u_0) = G_1 + G_2 \\ \left(F_1 2cM - F_2\right)\dfrac{V(u_0)}{\sigma_A^2 M^2 + 2\mu_U M - \sigma_U^2} = G_1 \gamma_{M+} + G_2 \gamma_{M-} \end{cases}$$

The solution for G_1 and G_2 is linear in the remaining free constant F_1.

Using the upper boundary condition $d/dx\ \varphi(c, x=u_1) = 0$, we can determine F_1. This is a linear equation in F_1 that is straightforward to solve, but the resulting expression is rather long and is omitted here for brevity.

6.3 Results for τ

The distribution function of τ can be obtained by inverting the Laplace transform $\varphi(c,x)$. Given the complicated expressions we find, this inversion has to be done numerically.

Furthermore, given our analytical solution for $\varphi(c,x)$ we can derive an analytical expression for the expected time to default for different levels of surplus x via $\mathbf{E}[\tau(x)] = d/dc\ \varphi(c{=}0,x)$, for both the intervals $0{<}x{<}u_0$ and $u_0{<}x{<}u_1$.
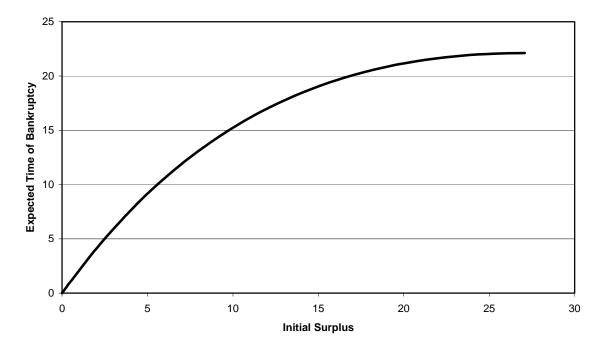
[Mathematica can do this calculation, but will lead to very messy expressions…]

The distribution function of τ can be useful in calibrating the model. In particular, it may be used to infer the dividend discount rate, c. Suppose that the insurance company aims at a probability of bankruptcy q over a 1-year time horizon, i.e., $\mathbb{P}[\tau \le 1] = q$. Then, using the distribution function of τ (which implicitly depends on c), one can solve for the value of c for which, under the corresponding optimal investment and optimal dividend policy, the probability of bankruptcy aimed at is achieved.

6.1 Example revisited once again

Let us consider again the example of Section 4. Figure 6.1 displays the expected time of bankruptcy under the optimal investment and optimal dividend policy as a function of the initial surplus.

Figure 6.1   *Expected time of bankruptcy as a function of the initial surplus*

7. Conclusion

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