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A Mathematical Preliminaries

In this appendix we will collect various definitions and mathematical results for the purpose of self-containedness. Nevertheless, we do not seek to prove nor to explain them, since they can easily be found in various places in a compact and readable form. See for example [164, 165, 166, 167, 6] for some physics literature.

Definition (complex manifold) A complex manifold is a topological space M together with a holomorphic atlas. Equivalently, define the almost complex structure $\mathcal{J}$ to be a map between the tangent bundle $\mathcal{J} : TM \to TM$ satisfying $\mathcal{J}^2 = -\mathbb{1}$ and its Nijenhuis tensor $N : TM \times TM \to TM$

$$N[X,Y] = [X,Y] + \mathcal{J}[\mathcal{J}X,Y] + \mathcal{J}[X,\mathcal{J}Y] - [\mathcal{J}X,\mathcal{J}Y] \ .$$

Then

$$\text{N=0} \Leftrightarrow \ \mathcal{J} \text{ integrable} \Leftrightarrow M \text{ Complex Manifold}$$

In local coordinates, we can write $\mathcal{J} = -idz^i \otimes \frac{\partial}{\partial z^i} + id\bar{z}^{\bar{\imath}} \otimes \frac{\partial}{\partial \bar{z}^{\bar{\imath}}}$.

Definition (hermitian metric) A hermitian metric $g : TM \times TM \to \mathbb{R}$ of a complex manifold is a metric which satisfies

$$g(\mathcal{J}X,\mathcal{J}Y) = g(X,Y) \ .$$

In local coordinates it can be written in the form

$$g = g_{i\bar{\jmath}}\, dz^i \otimes dz^{\bar{\jmath}} + g_{\bar{\imath}j}\, d\bar{z}^{\bar{\imath}} \otimes dz^j$$

where the reality of the metric implies $g_{i\bar{\jmath}} = \overline{g_{j\bar{\imath}}}$.

Theorem A.0.6 *Every complex manifold admits a hermitian metric.*

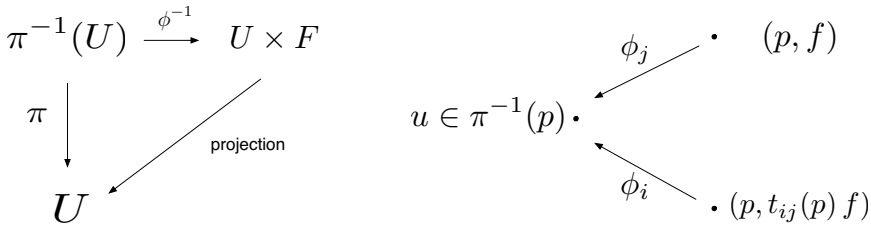

Figure A.1: The definition of (a) a fibre bundle (b) its structure group.

With the hermitian metric one can turn the complex structure $\mathcal{J}$ into a (1,1)-form, called the Kähler form

$$J = ig_{i\bar{\jmath}}\, dz^i \otimes d\bar{z}^{\bar{\jmath}} - ig_{\bar{\jmath}i}\, d\bar{z}^{\bar{\jmath}} \otimes dz^i = ig_{i\bar{\jmath}}\, dz^i \wedge d\bar{z}^{\bar{\jmath}}\,.$$

For a complex manifold with n complex dimension, the (n, n)-form

$$\underbrace{J \wedge \cdots \wedge J}_{n}$$

is nowhere vanishing can therefore serve as a volume element.

Definition (Kähler manifold) A Kähler manifold is a hermitian manifold with closed Kähler form $dJ = 0$. In local coordinates, this implies that the metric can be written as

$$g_{i\bar{\jmath}} = \partial_i \partial_{\bar{\jmath}} \mathcal{K}$$

for some Kähler potential $\mathcal{K}$. In the overlap of different coordinate charts, the Kähler potentials are related by $\mathcal{K} \to \mathcal{K} + f(z) + \bar{g}(\bar{z})$ for some (anti-)holomorphic function f ($\bar{g}$) and therefore have the same metric.

Definition (fibre bundle) A fibre bundle consists of data (E, B, π, F), often denoted by $E \xrightarrow{\pi} B$, where E (total space) , B (base space) and F (fibre) are differential manifolds, and π (projection) is a surjection $\pi : E \to B$ such that the "fibre at " $p \in B$ satisfies $\pi^{-1}(p) = F_p \simeq F$. A local trivialization ϕ_i in an open neighbourhood U_i in B is a map $\phi_i : U_i \times F \to \pi^{-1}(U_i)$. A transition function $t_{ij} : U_i \cap U_j \to G$, where G is a Lie group called the structure group, satisfies $\phi_j(p, f) = \phi_i(p, t_{ij}(p)f)$. A section v of a fibre bundle is a map $v : B \to E$ such that $\pi(v(p)) = p$ for all $p \in B$.

Definition (vector bundle) A vector bundle $E \xrightarrow{\pi} B$ is a fibre bundle whose fibre is a vector space.

In analogy to the concept of parallel transport on a Riemannian manifold, we can also define such a connection on a vector bundle. Formally, suppose $\Gamma(E)$ is the space of smooth sections of E, then a connection is a map $\mathcal{D} : \Gamma(E) \to \Gamma(E \otimes T^*B)$ such that the Leibnitz' rule

$$\mathcal{D}(f\zeta) = f\mathcal{D}\zeta + \zeta \otimes df$$

for any $\zeta \in \Gamma(E)$ and smooth function f on B. For a local coordinate e^a for E such that any $\zeta \in \Gamma(E)$ can be locally written as $\zeta_a e^a$, the connection one-form and the curvature two-form are given by the Cartan's structure equation

$$\begin{aligned} de^a + \omega^a{}_b \wedge e^b &= 0 \\ d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b &= R^a{}_b \end{aligned}$$

Definition (Chern class)

Given a complex vector bundle $E \xrightarrow{\pi} B$, let R be its curvature two-form, then the total *Chern class* is defined as

$$c(R) = \det\left(\mathbb{1} + \frac{i}{2\pi}Rt\right) = \sum_{i=0}^{n} t^i c_i(E) \ ,$$

where $c_i(R)$ is called the i-th Chern class. It would be useful to diagonalise the curvature two-form R into a diagonal $n \times n$ matrix of two-forms with eigen-two-forms $-2\pi i x_{i=1,\cdots,n}$. Then the Chern class can be written as

$$c(R) = \det\left(\mathbb{1} + \frac{i}{2\pi}Rt\right) = \prod_{i=1}^{n}(1 + x_i t)$$

For example, for a n-(complex) dimensional fibre we have

$$\begin{aligned} c_0(R) &= 1 \\ c_1(R) &= \frac{i}{2\pi}\mathrm{Tr}R = \sum_{i=1}^{n} x_i \\ &\vdots \\ c_n(R) &= (\frac{i}{2\pi})^n \det R = \prod_{i=1}^{n} x_i \ . \end{aligned}$$

Chern classes are topological invariants, meaning that all the curvatures in the same cohomology class have the same Chern classes, which capture many topological properties of a bundle. For example, when E is a line bundle the

first Chern class is the only non-trivial one. In this case the first Chern class completely fixes the topology of the line bundle.

Chern classes satisfies the following property for the Whitney sum of two bundles E and F, namely

$$c(E \oplus F) = c(E) \wedge c(F) \ . \tag{A.0.1}$$

Digression More Characteristic Classes and Some Index Theorems

For future use we list here more relevant characteristic classes and some index theorems related to them.

- The Chern character $\mathrm{ch}(R) = \mathrm{Tr}\exp(\frac{iR}{2\pi})$, is related to the Chern classes as

$$\mathrm{ch}(R) = \mathrm{Tr}\exp(\frac{iR}{2\pi}) = 2n + c_1(R) + \left(\frac{1}{2}c_1^2(R) - c_2(R)\right) + \cdots \ . \tag{A.0.2}$$

- For a real $2n$-dimensional vector bundle with structure group $O(2n)$, the field strength R' is anti-symmetric and can be diagonalise with $2n$ imaginary eigenvalues $(i2\pi x_1, -i2\pi x_1, \cdots, i2\pi x_n, -i2\pi x_n)$. Define the Pontrjagin class to be

$$p(R) = \det(\mathbb{1} + \frac{R'}{2\pi}t) = \det(\mathbb{1} - \frac{R'}{2\pi}t) = \sum_{i=0}^{n} t^i p_i(R) \ , \tag{A.0.3}$$

  By expressing both $p(R)$ and $c(R)$ in terms of the eigenvalues x_i, we obtain the following relations

$$\begin{aligned} p_1 &= c_1^2 - 2c_2 \\ p_2 &= c_2^2 - 2c_1c_3 + 2c_4 \qquad \text{etc.} \end{aligned}$$

  The special case in which the bundle E is the tangent bundle TM of a manifold M will be especially useful for us. We will often omit writing out the tangent bundle explicitly and just denote the characteristic classes by the manifold M, for example, it should be understood that $c_1(M) = c_1(TM)$ etc.

- The Dirac genus (A-roof genus) is defined as

$$\hat{\mathcal{A}}(M) = \prod \frac{x_i/2}{\sinh(x_i/2)} = 1 - \frac{1}{24}p_1 + \cdots$$
$$= 1 - \frac{1}{24}(c_1^2 - 2c_2) + \cdots \ . \qquad \text{(A.0.4)}$$

- The Hirzebruch $\hat{\mathcal{L}}-$polynomial is defined as

$$\hat{\mathcal{L}}(M) = \prod \frac{x_i}{\tanh(x_i)} = 1 + \frac{1}{3}p_1 + \cdots$$
$$= 1 + \frac{1}{3}(c_1^2 - 2c_2) + \cdots \ . \qquad \text{(A.0.5)}$$

- The Todd class is defined as

$$\mathrm{T}d(M) = \prod \frac{x_i}{1 - e^{-x_i}} = 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \cdots \ . \qquad \text{(A.0.6)}$$

- The Euler class is defined for an oriented real $(2n)$-dimensional fiber with SO(2n) structure group as

$$e(M) = x_1 \cdots x_n = (\, p_n(M)\,)^{1/2} = c_n(M) \ . \qquad \text{(A.0.7)}$$

- Gauss-Bonnet Theorem

  For an even dimensional manifold M

$$\chi(M) = \int_M e(T(M)) = \sum (-1)^i \, b^i \ , \qquad \text{(A.0.8)}$$

  where b^i denotes the i-th Betti number.

- Signature Index Theorem

  For a real $(4n)$-dimensional manifold M, the signature is given by the dimension of the self-dual and anti-self-dual (under the Hodge star operation) part of the middle cohomology as

$$\sigma(M) = \mathrm{dim} H_+^{2n}(M;\mathbb{R}) - \mathrm{dim} H_-^{2n}(M;\mathbb{R}) \ , \qquad \text{(A.0.9)}$$

  and the index theorem states

$$\sigma(M) = \int_M \hat{\mathcal{L}}(M) \ . \qquad \text{(A.0.10)}$$

- Hirzebruch-Riemann-Roch Theorem

  The betti numbers of bundle-valued forms satisfy the following relation

$$w(M;V) = \sum_{k=0}^{\dim M} (-1)^k \dim H^k(M;V) = \int_M ch(V)Td(M) \ . \quad \text{(A.0.11)}$$

Finally we are ready to introduce the kind of manifold which will play an important role in the discussion about the moduli space of Calabi-Yau three-folds, namely the special Kähler manifold. Historically a special Kähler manifold is first defined as the scalar manifold of the $\mathcal{N} = 2, d = 4$ supergravity theory. But since this kind of manifold appears also in other context, we will proceed by first define the manifold abstractly, and later show that the $\mathcal{N} = 2, d = 4$ supergravity scalar manifold is an example of them.

Definition (Special Kähler manifold) A (local) special Kähler manifold M is a n-(complex) dimensional manifold with a holomorphic Sp(2n+2,$\mathbb{R}$) vector bundle $\mathcal{E}$ over M, a line bundle $\mathcal{L}$ over M with (2π times) the first Chern class equal to the Kähler form, and a holomorphic section Ω in $\mathcal{L} \otimes \mathcal{E}$ such that the Kähler form is given by

$$J = i\partial\bar{\partial}\log[-i\langle\Omega,\bar{\Omega}\rangle] \quad \text{(A.0.12)}$$

and such that $\langle\partial_a\Omega,\Omega\rangle = 0$, where $\langle\ ,\ \rangle$ denotes the symplectic product [47, 45].

In order to define the Calabi-Yau manifolds themselves, we still have to introduce a last element, meaning the holonomy group of a manifold. With the Levi-Civita connection we have a concept of parallel transport on a n-(complex) dimensional Riemannian manifold. When parallel transporting an orthonormal frame around a closed loop, it doesn't come back to itself generically, but is rather related by a SO($2n$) transformation to the original frame. By combining different closed loops it's not hard to see that holonomy forms a group. There are special kinds of manifolds whose holonomy group does not cover the the whole SO($2n$) but only a subgroup of it. They will be called manifolds with special holonomy. As we will see later, a smaller holonomy group is crucial for having unbroken supersymmetry after compactification.

Definition (Calabi-Yau manifold) A Calabi-Yau manifold is a compact Kähler manifold with vanishing first Chern class.

Theorem A.0.7 *(Yau's Theorem) If X is a complex Kähler manifold with vanishing first Chern class and with Kähler form J, then their exists a unique Ricci-flat metric on X whose Kähler form is in the same cohomology class as J.*

From the above theorem and from the fact that Ricci-flatness is a sufficient condition for a vanishing first Chern class, we can conclude that an equivalent definition of a Calabi-Yau manifold is a compact Kähler manifold admitting a Ricci-flat metric. This is a very powerful theorem since it is in general very hard to find the explicit metric but relatively easy to compute the Chern class.

The Ricci-flatness also implies special holonomy properties. Consider the integrability condition for parallel transporting a spinor

$$[\nabla_k, \nabla_m]\eta = -\frac{1}{4}R_{kmpq}\gamma^{pq}\eta \ ,$$

where γ's are the gamma-matrices satisfying the Clifford algebra. From the index structure of a Kähler manifold we can already see that a $n-$(complex) dimensional Kähler manifold has holonomy group $U(n) \subset SO(2n)$. But this is not enough to ensure the existence of a constant spinor when $n = 3$. For that we need a holonomy group $SU(n) \subset SO(2n)$. The vanishing of the $U(1)$ part of holonomy is given exactly by Ricci-flatness. Furthermore, employing the covariantly constant spinors one can actually show the presence of another equivalent relation, namely the existence of a nowhere vanishing, holomorphic $(n, 0)$ form.

We can now therefore gather the above facts and give four equivalent definitions of a Calabi-Yau manifold.

Definition (Calabi-Yau manifold) A Calabi-Yau manifold is a compact, complex, n (complex)-dimensional Kähler manifold X which

$$\begin{aligned} &\text{has } c_1(TX) = 0 \\ &\Leftrightarrow \text{admits a Ricci-flat metric} \\ &\Leftrightarrow \text{has holonomy group } SU(n) \\ &\Leftrightarrow \text{admits a nowhere vanishing holomorphic } (n,0) \text{ form .} \end{aligned} \tag{A.0.13}$$

Furthermore this holomorphic (n,0) form

$$\Omega = \frac{1}{3!}\Omega_{i_1\cdots i_n}\, dz^{i_1} \wedge \cdots \wedge dz^{i_n} \tag{A.0.14}$$

is harmonic and covariantly constant when the Ricci-flat metric is chosen.

Before going into the details of Calabi-Yau manifolds of different dimensions, we would like to end this appendix by discussing the cohomology structure of Calabi-Yau manifolds.

Let's denote by $h^{r,s}$ the dimension of the vector space of harmonic (r,s)-form on X. From the Hodge star operation we see that these Hodge numbers have the symmetry $h^{r,s} = h^{n-r,n-s}$. Furthermore, from the complex conjugation and Kählerity $\Delta = 2\Delta_\partial = 2\Delta_{\bar\partial}$ we have $h^{r,s} = h^{s,r}$. What is special about Calabi-Yau manifolds compared to the usual Kähler manifolds are, first of all, $h^{n,0} = h^{0,n} = 1$ as mentioned before. Moreover, Using the Ricci-flatness it's not hard to show that $h^{r,0} = 0$ for $0 < r < n$, at least when the Euler characteristic is non-zero. In the following we sum up the above properties in the so-called Hodge diamond for Calabi-Yau two- and three-folds.

$$\begin{array}{ccccc} & & h^{0,0} & & \\ & h^{1,0} & & h^{0,1} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{2,1} & & h^{1,2} & \\ & & h^{2,2} & & \end{array} = \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array} \tag{A.0.15}$$

$$\begin{array}{ccccccc} & & & h^{0,0} & & & \\ & & h^{1,0} & & h^{0,1} & & \\ & h^{2,0} & & h^{1,1} & & h^{0,2} & \\ h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{0,3} \\ & h^{3,1} & & h^{2,2} & & h^{1,3} & \\ & & h^{3,2} & & h^{2,3} & & \\ & & & h^{3,3} & & & \end{array} = \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{1,2} & & h^{1,2} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

We immediately see that the Euler numbers

$$\chi(X) = \sum_{r,s} (-1)^{r+s} h^{r,s}$$

of these Calabi-Yau manifolds are

$$\chi(\mathrm{CY}_2) = 24 \ ; \ \chi(\mathrm{CY}_3) = 2(h^{1,1} - h^{1,2}) \ . \tag{A.0.16}$$

Notice that apart from the above discussion about the cohomology properties of Calabi-Yau n-folds in general, we have used an extra piece of information for the two-folds. Namely we have put in $h^{1,1} = 20$. This is because there is only one Calabi-Yau two-folds with $SU(2) = Sp(1)$ holonomy group (so

they are also hyper-Kähler) but not a subgroup of it, in the sense that all of them are diffeomorphic to each other. From now on we will refer to this unique two-fold as K3 manifold and reserve the name "Calabi-Yau" for the three-folds.

A construction of K3 manifolds of special interests is given by orbifolding a T^4. Let's for example consider an orbifold by $\mathbb{Z}_2$: $(z_1, z_2) \to (-z_1, -z_2)$ where $z_{1,2}$ are the coordinates for the square torus. There are clearly $2^4 = 16$ fixed points, each can be "blown up" by replacing the singularity with an Eguchi-Hanson metric. See (1.3.19) for an description of the Eguchi-Hanson space. This is a gravitational instanton and therefore the first Chern class indeed vanishes. From the 16 harmonic anti-self-dual $(1,1)$-form of the Eguchi-Hanson metric, combined with the 4 $(1,1)$- and one $(0,2)$-, one $(2,0)$-form from the original torus, we indeed obtain the above Hodge diamond. Furthermore, from the above analysis we also know the self-dual and anti-self-dual splitting of the two-forms

$$b_2^+(K3) = 3 \ ; \ b_2^-(K3) = 19 \ . \tag{A.0.17}$$

In the eigenbasis of the Hodge star operator on a four dimensional manifold S, which satisfies $\star^2 = 1$, the symmetric bilinear on the space of even-forms satisfies

$$(\alpha, \alpha) := \int_S \alpha \wedge \alpha = \pm \int_S \alpha \wedge \star\alpha \ . \tag{A.0.18}$$

Using the information about the dimensions of the eigenspace of the Hodge star operator (A.0.17), we conclude that the signature of the space of middle cohomology classes of the K3 surface S is (3,19), namely

$$H^2(S, \mathbb{Z}) \cong \Gamma^{3,19} \ . \tag{A.0.19}$$

Furthermore, we can incorporate the whole Hodge diamond (A.0.15) by introducing also the basis α_0 and α^0 for $H^0(S, \mathbb{Z})$ and $H^4(S, \mathbb{Z})$ respectively, which is dual to each other in the sense that $\int_S \alpha_0 \wedge \alpha^0 = 1$, and therefore enlarge the lattice with an extra piece

$$U = \Gamma^{1,1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ . \tag{A.0.20}$$

We therefore conclude that the space of integral cohomology classes of a K3 manifold is the following lattice

$$H^{2*}(S, \mathbb{Z}) \cong \Gamma^{3,19} \oplus \Gamma^{1,1} \cong \Gamma^{4,20} \ . \tag{A.0.21}$$

Finally let us remark that we have used the fixed point counting of the $\mathbb{Z}_2$ orbifold limit of the K3 manifold, and the fact that all K3 manifolds are diffeomorphic with each other and have therefore the same topological invariants,

to obtain the signature (A.0.17). But one can also obtain it from using the signature index theorem (A.0.10). See, for example, [6].

This situation is in stark contrast with that of Calabi-Yau three-folds, in which case we don't even know whether the possibilities of different Hodge numbers are finite. Or, in another (not incompatible) extreme, whether all these possible Calabi-Yau's with different Hodge numbers are actually all connected by non-perturbative conifold transitions and in this sense there is only one Calabi-Yau. For the purpose of our thesis we can just stick to thinking of the space of all CY's as... rather complicated.