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8 Counting the Dying Dyons

8.1 Introduction

In the last chapter we have reviewed the microscopic counting of 1/2- and 1/4-BPS states, in particular the derivation of a dyon-counting formula and various mathematical properties of the dyon-counting partition function.

Recently, various puzzles have been raised about this formula [145, 146]. First of all, a subtlety in checking their S-duality invariance has been observed. Secondly, there is an ambiguity in choosing the integration contour arising from the complicated pole structure of the modular forms that enter the formulas. Finally, it has been noted that the BPS spectrum in the macroscopic supergravity theory is subjected to moduli dependence due to the presence of walls of marginal stability for some multi-centered bound states. See section 4.3.1 for a discussion of this phenomenon in the $\mathcal{N} = 2$ context. Finally, either by using a duality argument [145], or by studying a specific example in great details [146], there have been some hints that all the above issues might actually have something to do with each other.

The goal of the first part of the chapter is to address these issues and provide a resolution to some of these puzzles. In particular, our aim is to present a precise contour prescription that will lead to a counting formula that is manifestly S-duality invariant and suitable for all moduli. To arrive at this prescription for contours, an important role is played by the one-to-one correspondence between various poles in the integrand of the counting formula, and the different decay channels in which a dyon can be split into two 1/2-BPS particles. This correspondence between poles and bound states was envisaged in [139], and was recently reiterated in [145, 146]. It turns out that the only poles that can be crossed when the choice of contour is varied are precisely the ones that admit such a correspondence. Moreover, we find that the contributions of the poles exactly match the expected number of states corresponding to the two-centered configurations of BPS dyons (See also the work by Sen around the same time [147]). The key observation which allows us to identify the correct contour prescription is that the resulting integration contour should render the counting formula explicitly S-duality invariant, and should furthermore automatically take into account the (dis-)appearance of the two-centered bound states when a wall of marginal stability is crossed. This leads to a moduli-dependent degeneracy (or index-) formula that counts all the living dyons in every region of moduli space. In particular, we note that the walls of marginal stability have the property that for large black hole charges (as opposed to "small black holes" with vanishing leading macroscopic entropy), none of the two-centered bound states of 1/2-BPS particles can exist when the background moduli are fixed at their attractor values. Using this fact we also propose a second, moduli-independent contour prescription, which has the property of counting only the "immortal dyons" which exist everywhere in the moduli space.

The goal of the second part of this chapter is to explore the role of the Borcherds-Kac-Moody algebra in the BPS spectrum. Despite the progress mentioned above which has been made in understanding the dyonic spectrum of the theory, so far no concrete interpretation is given to the appearance of the Borcherds-Kac-Moody algebra. We will address the following two aspects of this issue.

First of all, we note that the counting formula, now equipped with a moduli-dependent contour, can be identified with the character formula of a Verma module of the algebra with an appropriate choice of highest weight depending on the moduli. Specifically, the dictionary is such that the attractor moduli corresponds to a Verma module of dominant highest weight. This sheds light on the question of how the BPS states form a representation of the algebra.

Secondly, using the realisation that the walls of moduli stability of the supergravity theory are given by the walls of Weyl chambers of the Borcherds-Kac-Moody algebra, we show that the discrete dependence of the BPS spectrum on the moduli is described by the Weyl group of the algebra. From this we conclude that, just from the low-energy supergravity theory we should be able to derive the existence of such a group as the group of a discretized version of the attractor flow. More concretely, given a beginning point in the moduli space, there is a unique sequence of Weyl reflections, each represents the crossing of a wall of marginal stability, which brings the moduli to their attractor values. This path of wall-crossing is exactly the one taken by the usual attractor flow. In particular this gives an order among possible decay channels, which has the structure of an order given by an RG-flow. We hope that this chapter elucidates the relationship between the wall crossing, the counting formula, and the Borcherds-Kac-Moody algebra, in the context of the $\mathcal{N} = 4$ dyonic spectrum.

8.2 Dying Dyons and Walls of Marginal Stability

8.2.1 Determining the Walls

Earlier in this thesis we have seen in section 4.3 the phenomenon of moduli dependence of solutions of $\mathcal{N} = 2$, $d = 4$ supergravity. Here we are interested in the question whether there is a similar phenomenon in the present $\mathcal{N} = 4$, $d = 4$ theory. Especially, we are interested in the bound states of two 1/2-BPS objects since these are the only bound states whose degeneracies might jump in different regions in the moduli space, which can be understood either by zero-mode counting or the fact that the corresponding walls of marginal stability are of higher co-dimension in the moduli space in all other cases [148]

As discussed in chapter 4.3, a decay of a two-centered supergravity solution is possible only when the background moduli at infinity is such that the mass satisfies the condition $M = M_1 + M_2$.

We want to determine when a dyonic bound state might decay in this theory. First we concentrate on the specific decay channel of a dyonic, 1/4-BPS state with charges (P, Q) splitting into two 1/2-BPS particles with charges $(P, 0)$ and $(0, Q)$. For this case, the condition for a wall of marginal stability is

$$M_{P,Q} = M_{P,0} + M_{0,Q} \ , \tag{8.2.1}$$

which can be rewritten as

$$|Z_{P,Q}(\lambda)| = |Z_{P,0}(\lambda)| + |Z_{0,Q}(\lambda)| \ .$$

Notice that we have temporarily suppressed the dependence on the Narain part of the moduli in our notation, which determines the left-moving part of the charges $P_L = \mu{\cdot}P$ (3.3.3), since these fields μ do not transform under the extended S-duality group $PGL(2, \mathbb{Z})$.

Using the fact that the total central charge obeys

$$Z_{P,Q}(\lambda) = Z_{P,0}(\lambda) + Z_{0,Q}(\lambda) \ ,$$

one finds that the condition of marginal stability can only be satisfied when the phases of the central charges are aligned, a phenomenon that is already familiar to us from chapter 4.3.

Using the explicit expression for the central charge (7.1.1), the above equation leads to the condition

$$\frac{\lambda_1}{\lambda_2} + \frac{P_L \cdot Q_L}{|P_L \wedge Q_L|} = 0 \ , \tag{8.2.2}$$

where

$$|P_L \wedge Q_L| = \sqrt{P_L^2 Q_L^2 - (P_L \cdot Q_L)^2}$$

is the quantity defined earlier in (7.1.2).

The next step will be to consider the other ways in which a dyon can split into two 1/2-BPS particles, and determine the corresponding walls of marginal stability. The decay channels are determined by the fact that that a 1/2-BPS state must satisfy the condition on their charges that the magnetic and electric charges have to be parallel to each other (7.1.4). It is now easy to check that every element of $\gamma \in PGL(2,\mathbb{Z})$ gives the following split of charges

$$\begin{pmatrix} P \\ Q \end{pmatrix} = \begin{pmatrix} P_1 \\ Q_1 \end{pmatrix} + \begin{pmatrix} P_2 \\ Q_2 \end{pmatrix} = P_\gamma \, \gamma^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + Q_\gamma \, \gamma^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} , \tag{8.2.3}$$

where

$$\begin{pmatrix} P_\gamma \\ Q_\gamma \end{pmatrix} = \gamma \begin{pmatrix} P \\ Q \end{pmatrix}$$

as before. Furthermore, from the quantisation condition of the charges one can also show that the converse is also true [145]. Namely, for every possible 1/2-BPS split one can always find a (not unique) $PGL(2,\mathbb{Z})$ element γ such that the charges can be written in the above form.

To determine the location of walls of marginal stability in the moduli space for the two-centered solutions with the above charges, we can just plug in the above split of charges into the marginal stability equation $M = M_1 + M_2$ and solve for the solution. But it will turn out to be a much more economic way to study the transformation of the central charge matrix (7.1.1) under the extended duality group $PGL(2,\mathbb{Z})$ (7.2.18). It is easy to check that this transformation has the effect of shifting the phase of the central charges by

$$Z_{P,Q}(\lambda) = \begin{cases} e^{i\alpha_\gamma} Z_{P_\gamma,Q_\gamma}(\lambda_\gamma) & \text{for } \det\gamma = 1 \\ e^{i\alpha_\gamma} \bar{Z}_{P_\gamma,Q_\gamma}(\lambda_\gamma) & \text{for } \det\gamma = -1 \end{cases} \tag{8.2.4}$$

with some charge-independent phase α_γ which depends on the group element γ and the axion-dilaton λ. Due to the fact that the phase shift is independent of the charges, all magnitudes of the relative phases will be duality invariant. In particular, we have

$$|Z_{P,Q}(\lambda)| = |Z_{P_\gamma,0}(\lambda_\gamma) + Z_{0,Q_\gamma}(\lambda_\gamma)| = |Z_{P_\gamma,0}(\lambda_\gamma)| + |Z_{0,Q_\gamma}(\lambda_\gamma)| \, .$$

This shows that the position of the walls of marginal stability corresponding to the charge splitting (8.2.3) are simply the $PGL(2,\mathbb{Z})$ image of the one for the bound state of the purely electric and purely magnetic 1/2-BPS states. Namely, the walls of marginal stability for all two-centered 1/2-BPS splits are

$$\frac{\lambda_{\gamma,1}}{\lambda_{\gamma,2}} + \frac{(P_L \cdot Q_L)_\gamma}{|P_L \wedge Q_L|_\gamma} = 0 \;, \tag{8.2.5}$$

where $P_{L,\gamma}, Q_{L,\gamma}, \lambda_\gamma$ are given by (7.2.18) as before.

The above formula might not look too complicated, but we should keep in mind that these are really infinitely many equations since $PGL(2,\mathbb{Z})$ is not a finite group. Inspired by the fact that all the quantities involved in the above equation have simple transformation rules under the extended S-duality group, we would like to look for a way to organise the above equations of walls of marginal stability in a form that is directly an equation on the un-transformed fields P_L, Q_L and λ.

First note that, from the fact that

$$\Lambda_{P,Q} = \begin{pmatrix} P \\ Q \end{pmatrix} \cdot \begin{pmatrix} P & Q \end{pmatrix}$$

we see that the three-dimensional charge vector $\Lambda_{P,Q}$ transforms as

$$\Lambda_{P,Q} \to \gamma(\Lambda_{P,Q}) = \gamma \Lambda_{P,Q} \gamma^T$$

under the S-duality transformation (7.2.18)

$$\begin{pmatrix} P \\ Q \end{pmatrix} \to \gamma \begin{pmatrix} P \\ Q \end{pmatrix} := \begin{pmatrix} P_\gamma \\ Q_\gamma \end{pmatrix} \quad , \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PGL(2,\mathbb{Z}) \;. \tag{8.2.6}$$

Since the S-duality group leaves the Narain moduli μ invariant, we conclude that the same transformation rule holds for

$$\begin{pmatrix} P_L \cdot P_L & P_L \cdot Q_L \\ P_L \cdot Q_L & Q_L \cdot Q_L \end{pmatrix} \to \gamma \begin{pmatrix} P_L \cdot P_L & P_L \cdot Q_L \\ P_L \cdot Q_L & Q_L \cdot Q_L \end{pmatrix} \gamma^T \;.$$

Especially, the norm of the this vector $\sim |P_L \wedge Q_L|$ is invariant under the transformation. For the axion-dilaton, from our experience with the S-duality of type IIB supergravity (1.3.36), we have seen that the following matrix also transforms under $PSL(2,\mathbb{Z})$ in the same way as the charge vector

$$\frac{1}{\lambda_2} \begin{pmatrix} |\lambda|^2 & \lambda_1 \\ \lambda_1 & 1 \end{pmatrix} \to \gamma \, \frac{1}{\lambda_2} \begin{pmatrix} |\lambda|^2 & \lambda_1 \\ \lambda_1 & 1 \end{pmatrix} \gamma^T \;,$$

and it is trivial to check that the same transformation rule extends to the extended duality group $PGL(2,\mathbb{Z})$.

In particular, if we choose the following combination of these two vectors

$$\mathcal{Z} = \frac{1}{\sqrt{|P_L \wedge Q_L|}} \begin{pmatrix} P_L \cdot P_L & P_L \cdot Q_L \\ P_L \cdot Q_L & Q_L \cdot Q_L \end{pmatrix} + \frac{\sqrt{|P_L \wedge Q_L|}}{\lambda_2} \begin{pmatrix} |\lambda|^2 & \lambda_1 \\ \lambda_1 & 1 \end{pmatrix} , \quad (8.2.7)$$

then it transforms as

$$\mathcal{Z} \to \gamma(X) = \gamma \mathcal{Z} \gamma^T$$

under an extended S-duality transformation.

As a side remark, let us note that the above somewhat awkward-looking normalisation has the advantage that now the mass of a dyon, which involves 134 moduli fields (7.1.3), is nothing but the norm of this single vector

$$M^2_{P,Q} = |Z_{P,Q}|^2 = -\frac{1}{2}\|\mathcal{Z}\|^2 .$$

But we are going to see in a moment that it is only the direction, not the length, of the vector $\mathcal{Z}$ which determines the existence or not of a certain two-centered solution. For convenience we will therefore define a "unit vector" X by

$$X = \frac{\mathcal{Z}}{M_{P,Q}} = \frac{\mathcal{Z}}{\sqrt{-\frac{1}{2}\|\mathcal{Z}\|^2}} . \quad (8.2.8)$$

Apparently this vector also transforms in the same way as $\mathcal{Z}$ under the $PGL(2,\mathbb{Z})$ transformation.

With this notation, the walls of marginal stability for the $(P,0)$, $(0,Q)$ split of charges (8.2.2) lie on the co-dimensional one space characterised by the following equation

$$\big(X, \alpha_1\big) = 0 ,$$

and similarly for other splits (8.2.5)

$$\big(\gamma(X), \alpha_1\big) = \big(X, \alpha\big) = 0 \quad , \quad \alpha = \gamma^{-1}(\alpha_1) .$$

Now we have achieved the goal of organising the equations of walls of marginal stability as equations directly constraining the untransformed quantities P_L, Q_L and λ.

To discuss in more details the possible relationship between the contour dependence of the integral and the physical walls of marginal stability, it is necessary to label these walls. As we mentioned before, the map (8.2.3) between $PGL(2,\mathbb{Z})$ elements and the split of charges into two 1/2-BPS charges is not one-to-one. For example, the element which simply exchanges what we call the "1st" and the "2nd" decay products will not give a different split of charges. On the other hand, from the expression for the walls of marginal stability (8.2.9), we see that the element γ which gives $\gamma(\alpha_1) = \pm\alpha_1$ will not give a new physical wall. It can indeed be checked that these are also the elements which give the same split of charges when using the map (8.2.3). Using the fact that $PGL(2,\mathbb{Z})$ is the group of symmetry for the root system of the Coxeter (Weyl) group W (7.2.24) and by inspecting the action of the dihedral group (7.2.19),(7.2.20), it is not hard to convince oneself that the two-centered solutions with two 1/2-BPS charges discussed in this section are actually given by the positive real roots of the Borcherds-Kac-Moody algebra. So we arrive at the conclusion that the relevant two-centered solutions are in one-to-one correspondence with the positive real roots of the Borcherds-Kac-Moody algebra, whose walls of marginal stability are given by

$$(X,\alpha) = 0 \quad , \quad \alpha \in \Delta_+^{re} \tag{8.2.9}$$

and whose decay products can be represented by the split of the charge vector as

$$\begin{aligned} \Lambda_{P_1,Q_1} &= P_\alpha^2\,\alpha^+ \quad , \quad \Lambda_{P_2,Q_2} = Q_\alpha^2\,\alpha^- \\ \Lambda_{P,Q} &= P_\alpha^2\,\alpha^+ + Q_\alpha^2\,\alpha^- - (P\cdot Q)_\alpha\,\alpha\,, \end{aligned} \tag{8.2.10}$$

where for a given α, the set of the two vectors $\alpha^\pm$ is given by the requirement that (i) they are both lightlike and future-pointing and perpendicular to the root α, (ii) they lie in the weight lattice, which is in this case the lattice generated by half (in length) of the simple roots $\alpha_i/2$, (iii) they have inner product $(\alpha^+,\alpha^-) = -1$. See figure 8.1. In particular, the combination P_α^2 and Q_α^2, which is related to the "oscillation level" of the heterotic string as (7.1.6) and which determines the degeneracy of the 1/2-BPS states, can be thought of as the "affine length" of the lightlike charge vector Λ_{P_1,Q_1} and Λ_{P_2,Q_2} of the decay products. Conversely, given the charges of the two centers, the walls of marginal stability is given by the requirement that the moduli vector X is the linear combination of the two charge vectors Λ_{P_1,Q_1} and Λ_{P_2,Q_2}.

More concretely, what the above formula (8.2.10) means is the following: given a spacelike vector, any future-pointing timelike vector can be split into a component parallel to it and a future-pointing timelike vector perpendicular to it. And the latter can be further split into two future-pointing lightlike vectors lying on the plane perpendicular to the given spacelike vector. See figure 8.1. When the timelike vector is taken to be the original charge vector $\Lambda_{P,Q}$, the two future-pointing lightlike vectors are then the charge vectors of the two decay products.

For instance, suppose $\alpha = \gamma^{-1}(\alpha_1)$ for some $\gamma \in PGL(2,\mathbb{Z})$, then $\{\alpha^+,\alpha^-\}$ is given by

$$\gamma^{-1}(\mathrm{diag}(0,1))\ , \quad \gamma^{-1}(\mathrm{diag}(1,0))\ .$$

In this case one can show that $P_\alpha^2 = P_\gamma^2$ and similar for the Q's, and the ambiguity of relating a $PGL(2,\mathbb{Z})$ element γ to a specific decay channel lies in the fact that there are different γ's that give the same set $\{\alpha^+,\alpha^-\}$.

8.2.2 Stability Conditions from Supergravity Solutions

The meaning of the presence of a wall of marginal stability is that a BPS bound state of two particles exists on one side of the wall and disappears when crossing into the other side. After deriving the location of the walls for these bound states, we would like to know on which side these states are stable and on which side unstable. As discussed in section 4.3.1, this can either be determined using the heuristic argument that the attractor moduli of the single-centered black hole solution with the given total charges should lie on the unstable side of the wall, or by analysing the integrability condition of the supergravity solution. Now we will perform the latter analysis as a check. For this purpose we need more information about the corresponding supergravity solutions.

Let us now consider the four-dimensional $\mathcal{N}=4$ supergravity theory describing the low energy limit of the heterotic string compactified on a six-torus. The metric part of a stationary solution reads

$$\begin{aligned} ds^2 &= -e^{-2U}(dt+\vec{\omega}\cdot d\vec{x})^2 + e^{2U}d\vec{x}^2 \\ e^{2U} &= |\mathcal{P}\wedge\mathcal{Q}| \equiv \sqrt{\mathcal{P}^2\mathcal{Q}^2-(\mathcal{P}\cdot\mathcal{Q})^2} \\ \vec{\nabla}\times\vec{\omega} &= \mathcal{P}\cdot\vec{\nabla}\mathcal{Q}-\mathcal{Q}\cdot\vec{\nabla}\mathcal{P}\,, \end{aligned} \tag{8.2.11}$$

where the indices are contracted using the standard $SO(6,22)$-invariant 28×28 matrix η_{AB}, for example $\mathcal{P}^2 \equiv \mathcal{P}^A\mathcal{P}^B\eta_{AB}$.

The 56 harmonic functions appearing in the above solution are

$$\begin{aligned} \mathcal{P}^A(\vec{x}) &= C^A + \sum_i \frac{P_i^A}{|\vec{x}-\vec{x}_i|} \\ \mathcal{Q}_A(\vec{x}) &= D_A + \sum_i \frac{Q_{A,i}}{|\vec{x}-\vec{x}_i|} \end{aligned} \quad ,$$

with the 56 constants given by the asymptotic value of 23 complex scalar fields (the axion-dilaton moduli λ and the 22 complex moduli projected from the aforementioned 6×22 moduli) as[1]

$$C^A = -\mathrm{Im}\left(e^{-i\alpha_{P,Q}}\frac{\partial Z_{P,Q}}{\partial Q_A}\right)$$
$$D_A = \mathrm{Im}\left(e^{-i\alpha_{P,Q}}\frac{\partial Z_{P,Q}}{\partial P^A}\right) \quad ,$$

where the P^A's and the Q_A's denote the total charges coming from all the centers. From this expression one immediately sees that these coefficients satisfy $Q_A C^A = P^A D_A$, since the central charge is linear in all charges.

For the specific two-center bound state with charges $(P,0)$, $(0,Q)$ considered earlier, the corresponding supergravity solution has harmonic functions given by

$$\mathcal{P}^A = C^A + \frac{P^A}{|\vec{x}-\vec{x}_P|}$$
$$\mathcal{Q}_A = D_A + \frac{Q_A}{|\vec{x}-\vec{x}_Q|} \quad .$$

In this case the coordinate distance between the two centers $|\vec{x}_P - \vec{x}_Q|$ is fixed by the integrability condition [54], obtained by taking the divergence of the both sides of (8.2.11), and reads

$$\frac{P\cdot Q}{|\vec{x}_P-\vec{x}_Q|} = -C^A Q_A \; .$$

After some algebra this becomes

$$|\vec{x}_P - \vec{x}_Q| = -\frac{1}{\sqrt{|P_L\wedge Q_L|}}\,\frac{(\Lambda_{P,Q},\alpha_1)}{(X,\alpha_1)} \; .$$

Since the distance between the two centers is always a positive number, one finds that, in order for the bound state to exist, the expression on the r.h.s. should better be positive as well. We therefore conclude that the bound state only exists when

$$(\Lambda_{P,Q},\alpha_1)(X,\alpha_1) < 0 \; , \tag{8.2.12}$$

and decays when one dials the background moduli to hit the wall where (X,α_1) vanishes. More precisely, one finds that the distance between the two centers goes to infinity, and the bound state no longer exists as a localisable state.

[1] By evaluating the $\mathcal{N}=4$ central charge operator $\hat{Z}$ (7.1.1) in the eigen basis of $\hat{Z}^\dagger\hat{Z}$, one can write the BPS equations in a way analogous to the $\mathcal{N}=2$ case as in section 4.1. Only 22 complex moduli made out of the 6×22 real moduli fields play a role in the solution. It is indeed known that the $\mathcal{N}=4$ moduli space locally decomposes as a product of 22 vector-, 44 hyper-, and 1 tensor-multiplet scalars in the $\mathcal{N}=2$ language (see, for example, [149]).

Using the $PGL(2,\mathbb{Z})$ transformation as before, it is now easy to write down for all the other two-centered solutions with 1/2-BPS centers the expression for the coordinate distance between the centers

$$|\,\vec{x}_{P_\alpha} - \vec{x}_{Q_\alpha}| = -\frac{1}{\sqrt{|P_L \wedge Q_L|}} \frac{(\Lambda_{P,Q},\alpha)}{(X,\alpha)}\ , \tag{8.2.13}$$

and hence the stability condition reads

$$(\Lambda_{P,Q},\alpha)(X,\alpha) < 0 \quad , \quad \alpha \in \Delta_+^{re}\ . \tag{8.2.14}$$

We have therefore achieved the goal of studying the stability condition for all two-centered bound states of 1/2-BPS objects in the present $\mathcal{N}=4$, $d=4$ supergravity theory.

8.3 Contour Dependence of the Counting Formula

In section 7.1.3 we have derived the microscopic counting formula (7.1.14), expressed in terms of the generating function $1/\Phi(\Omega)$. Formally, to extract the actual degeneracies $D(P,Q)$ from the generating function we can invert the formula into a contour integral (7.2.8).

But in this formula we have not specified how the contour of integration should be chosen. It would not be a problem if the integral were contour-independent, but as it turns out it is not the case here. As discussed in (7.2.31), the generating function $1/\Phi(\Omega)$ has poles lying on the rational quadratic divisors which are related to each other by $Sp(2,\mathbb{Z})$ modular transformation. Due to the presence of these poles, one has to be careful with choosing the contour $\mathcal{C}$: the counting formula will "jump" when the contour crosses one of these poles. Therefore, strictly speaking the formula (7.2.8) for $D(P,Q)$ is not just a function of the charges P and Q but also depends on the contour.

To determine what the appropriate contour should be, the symmetry of the theory, in particular the extended S-duality $PGL(2,\mathbb{Z})$ symmetry in this case, will be provide us with important hints.

First recall the transformation of property of of charge vector $\Lambda_{P,Q}$ (8.2.6) under the extended S-duality group

$$\begin{pmatrix} P\cdot P & P\cdot Q \\ P\cdot Q & Q\cdot Q\end{pmatrix} \to \gamma \begin{pmatrix} P\cdot P & P\cdot Q \\ P\cdot Q & Q\cdot Q\end{pmatrix}\gamma^T\ .$$

Recall that $PGL(2,\mathbb{Z})$ acts as a Lorentz plus spatial reflection transformation in the space M_2 of 2×2 symmetric matrices. In particular, the inner product of two vectors is invariant under such a transformation

$$(\gamma(\Omega),\gamma(\Lambda_{P,Q})) = (\Omega,\Lambda_{P,Q})\ .$$

Together with the fact that $\Phi(\Omega)$ is invariant under $\Omega \to \gamma(\Omega)$ (7.2.29), we conclude that the integrand of the contour integral (7.2.8) is invariant under the following S-duality transformation

$$(-1)^{(P\cdot Q)_\gamma} \frac{e^{\pi i(\gamma(\Omega),\gamma(\Lambda_{P,Q}))}}{\Phi(\gamma(\Omega))} = (-1)^{P\cdot Q} \frac{e^{\pi i(\Omega,\Lambda_{P,Q})}}{\Phi(\Omega)} ,$$

where we have also used the fact that $P^2, Q^2 = 0 \bmod 2$ and as a consequence $(P\cdot Q)_\gamma = P\cdot Q \bmod 2$. Therefore, if we ignore the ambiguity of the contour, the degeneracy formula (7.2.29) indeed satisfies the physical condition of being invariant under the duality group.

This fact is not yet sufficient, however, to prove the invariance of the degeneracies. Namely, due to the presence of the poles, the expression for $D(P,Q)$ fails to be S-duality invariant, unless the contour $\mathcal{C}$ is also transformed to a new contour $\mathcal{C}_\gamma$. Explicitly, the equality

$$\oint_{\mathcal{C}} d\Omega\, (-1)^{P\cdot Q} \frac{e^{\pi i(\Omega,\Lambda_{P,Q})}}{\Phi(\Omega)} = \oint_{\mathcal{C}_\gamma} d\Omega\, (-1)^{(P\cdot Q)_\gamma} \frac{e^{\pi i(\gamma(\Omega),\gamma(\Lambda_{P,Q}))}}{\Phi(\gamma(\Omega))} \tag{8.3.1}$$

only holds when the new contour $\mathcal{C}_\gamma$ in the $\gamma(\Omega)$-plane is the same as $\mathcal{C}$ in the Ω-plane. If this is not true, in general different contours cannot be deformed into one another without picking up any residue and the answer we get for the degeneracies will therefore not be S-duality invariant.

A natural guess for a remedy for the present situation is to let the contour depend on the charges, and possibly also the moduli fields, since these quantities do transform under S-duality. Indeed, there is an important reason to suspect that the dyon counting formula is moduli-dependent, since as we have seen in the last section, certain multi-centered BPS solutions only exist in some range of background moduli and decay when a wall of marginal stability is crossed.

As a first step towards understanding the moduli dependence of the integration contour, we will now study the dependence of the degeneracy $D(P,Q)$ on the choice of the contour in the integral formula (7.2.8), by analysing the contribution from the poles of the partition function $1/\Phi(\Omega)$ to the integral.

Let us have a closer look at the possible choice of the contour, namely a choice of three-cycle on which we perform the integral over in the three-complex dimensional space parametrised by

$$\Omega = \begin{pmatrix} \sigma & -\nu \\ -\nu & \rho \end{pmatrix} \in M_2(\mathbb{R}) + iV^+ . \tag{8.3.2}$$

Due to the fact that we are dealing with a modular form, the contour will have to be inside a fundamental domain of the $Sp(2,\mathbb{Z})$ modular group. A natural

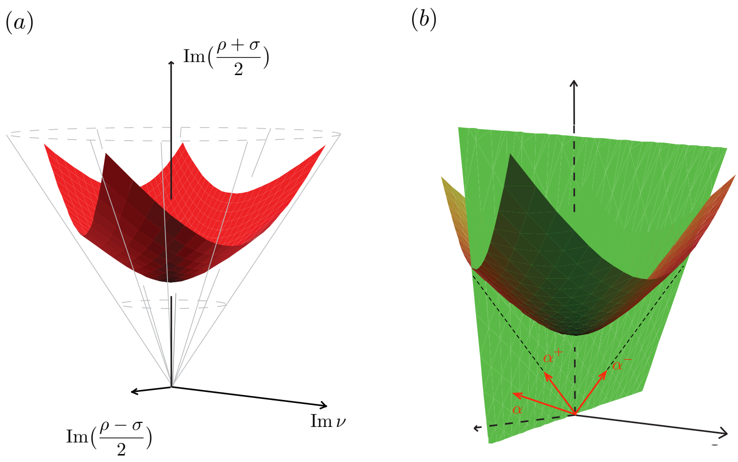

Figure 8.1: **(a)** The imaginary part of the Siegel upper-half plane for the modular form Φ is the future light-cone V^+ in the Minkowski space $\mathbb{R}^{2,1}$, and we consider the space of all contours to be a sheet of hyperboloid inside this light-cone, with all the points on the hyperboloid having the same large distance from the origin. **(b)** A plane $(X, \alpha) = 0$ given by a positive real root α always intersects the hyperboloid, or equivalently the upper-half plane or the Poincaré disk. And the root α defines two lightcone directions $\{\alpha^+, \alpha^-\}$ perpendicular to it, given by the intersection of the plane with the future light-cone.

choice of contour is to perform the integral over the real parts of ρ, σ and ν, while keeping the imaginary parts fixed. Specifically, the range of integration of the real variables is

$$0 \leq \mathrm{Re}\rho, \mathrm{Re}\sigma, \mathrm{Re}\nu < 1 \,. \tag{8.3.3}$$

The integration contour is thus a three-torus. The location of the contour is determined by a choice of the imaginary parts. To make sure that $1/\Phi$ has a well-defined expansion up to high order, we will choose these imaginary parts so that Ω lies well inside the Siegel upper-half plane, that is

$$-\frac{1}{2}\|\mathrm{Im}\Omega\|^2 = \mathrm{Im}\rho\,\mathrm{Im}\sigma - (\mathrm{Im}\nu)^2 \;= \varepsilon^{-2} \gg 1 \,. \tag{8.3.4}$$

To visualize the location of the poles relative to the contours, we note that the above condition defines a sheet of a hyperboloid high up inside the future light-cone. This is shown in Figure 8.1. As mentioned before, all the double poles of the generating function $1/\Phi$ are located at divisors given by the $Sp(2,\mathbb{Z})$ modular images of the divisor $(\Omega, \alpha_1) = -2\nu = 0$ in the Ω-space, where α_1 coincides with one of the simple real roots given in (7.2.9). As we have seen before (7.2.31), these poles take the form

$$r(\rho\sigma - \nu^2) + n\rho + m\sigma + \ell\nu + s = 0 \,.$$

The poles at divisors with $r = 1$ have exponentially dominant contribution to the degeneracy formula (7.2.8) compared to the rest in the case of large charges, as explained in the appendix of [139]. In [145] it was observed that the contour space (8.3.4) does not intersect any of the poles having $|r| \geq 1$. Indeed, a look at the real part of the above equation reveals that, since all the entries of $|\mathrm{Re}\Omega|$ run between 0 and 1, there is nothing to compensate the large contribution from $-\frac{1}{2}\|\mathrm{Im}\Omega\|^2 \gg 1$ contained in the real part of the term $-\frac{1}{2}\|\Omega\|^2 = \rho\sigma - \nu^2$. In other wods, these poles will always contribute to the degeneracy formula no matter which contour we choose, since they lie lower in the light-cone. Therefore, we never run into the danger of having a contour which crosses one of these poles. For our purpose of studying the contour dependence of the integral, it is hence sufficient to concentrate on the poles with $r = 0$.

Since we are only interested in the poles inside the real domain of integration (8.3.3), we can restrict our attention to the poles with $r = s = 0$. It is easy to see that an image of the pole $(\Omega, \alpha_1) = 0$ under $PGL(2, \mathbb{Z})$ transformation is another rational quadratic divisor, since the length condition (7.2.31)

$$\frac{1}{2}\|\alpha\|^2 = \frac{1}{2}\|\gamma^{-1}(\alpha_1)\|^2 = 1$$

follows directly from $\|\gamma^{-1}(\alpha_1)\|^2 = \|\alpha_1\|^2$.

On the other hand, it can be shown, by classifying both entries of $PGL(2, \mathbb{Z})$ elements and (k, ℓ, m) by their prime factorizations for example, that one can always find a (not unique) $PGL(2, \mathbb{Z})$ element γ for each pole (7.2.31) with $r = s = 0$ such that it is the image of the pole $(\Omega, \alpha_1) = 0$ under the group transformation.

In other words, these poles can be written in the form

$$\big(\gamma\Omega\gamma^T, \alpha_1\big) = \big(\gamma(\Omega), \alpha_1\big) = \big(\Omega, \gamma^{-1}(\alpha_1)\big) = 0 \quad \text{for some } \gamma \in PGL(2, \mathbb{Z}) \;,$$

and its imaginary part

$$\big(\mathrm{Im}\Omega, \alpha\big) = 0 \quad , \quad \alpha = \gamma^{-1}(\alpha_1) \quad \text{for some } \gamma \in PGL(2, \mathbb{Z}) \;.$$

defines a plane inside the space $\mathbb{R}^{2,1}$.

The fact that $\|\alpha\|^2 > 0$ implies that the normal vector to the plane is space-like, and hence these planes always intersect the contour space hyperboloid (8.3.4) along a hyperbola. Therefore, each plane divides the contours into two sub-classes $(\mathrm{Im}\Omega, \alpha) > 0$ and $(\mathrm{Im}\Omega, \alpha) < 0$. See Figure 8.1. Whether the corresponding poles contribute to the degeneracy formula for a given charge configuration will therefore depend on the contour we choose.

Let us now determine the condition under which these poles contribute to the integral, and, if they do contribute, what their contribution is. We first concentrate on the double pole at $(\Omega, \alpha_1) = 2\nu = 0$. Near the $\nu = 0$ divisor the generating function has the limit

$$\frac{1}{\Phi(\rho,\sigma,\nu)} = \frac{1}{4\pi^2}\frac{1}{\nu^2}\frac{1}{\eta^{24}(\rho)}\frac{1}{\eta^{24}(\sigma)}\,(1+\mathcal{O}(\nu^2))\;. \tag{8.3.5}$$

Notice that the last two factors in the limiting expression (8.3.5) are exactly the generating function for the 1/2-BPS degeneracies (7.1.7). By plugging the above expression into the degeneracy formula (7.2.8) and performing the integration over the real part of ρ and σ, one gets

$$\frac{(-1)^{P\cdot Q}}{4\pi^2} d(P)d(Q)\oint_{\mathcal{C}_\nu} d\nu\,\frac{e^{-2\pi i(P\cdot Q)\nu}}{\nu^2}\;,$$

where we have made use of (7.1.7). To evaluate the remaining integral over ν, we first consider a contour with

$$\big(\mathrm{Im}\Omega, \alpha_1\big) = 2\mathrm{Im}\nu > 0\;.$$

For this case the contour is shown in the Figure 8.2. When the charges under consideration satisfy

$$\big(\Lambda_{P,Q}, \alpha_1\big) = -2P\cdot Q > 0\;,$$

one can deform the contour to the upper infinity of the cylinder ($\mathrm{Im}\nu \to \infty$) where the integrand is zero without crossing any pole. One thus concludes that the integral yields zero. On the other hand, in the case $P\cdot Q > 0$, the contour can be moved to the lower infinity ($\mathrm{Im}\nu \to -\infty$) where the integrand is again zero, but now by doing so we pick up the contribution of the pole

$$-2\pi i\partial_\nu(e^{-2\pi i(PQ)\nu})|_{\nu=0} = -4\pi^2\,(P\cdot Q)\;,$$

where the extra minus sign comes from the fact that we are enclosing the pole in a clockwise direction. For the contours with $\mathrm{Im}\nu < 0$, a similar argument shows that the pole only contributes when $(P\cdot Q) < 0$, but now with the opposite sign as above due to the reverse orientation in which the pole is enclosed. One therefore concludes that the contribution of this specific pole to the degeneracy formula (7.2.8) is

$$(-1)^{(P\cdot Q)+1}\,|P\cdot Q|\,d(P)\,d(Q) \qquad \text{when}\;\; \big(\Lambda_{P,Q},\alpha_1\big)\big(\mathrm{Im}\Omega,\alpha_1\big) < 0 \tag{8.3.6}$$

and zero otherwise. The contributions of the other poles can be determined directly in a similar fashion. However, they are more easily obtained by making

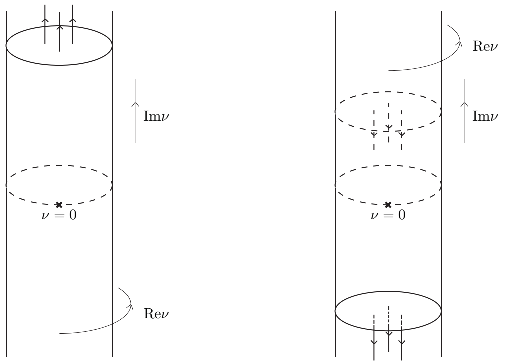

Figure 8.2: In this figure we show how the pole located at $\nu = 0$ contributes to the degeneracy formula for contours with $\text{Im}\nu > 0$. **(a)** For charges with $P \cdot Q < 0$, one can deform the contour to the upper infinity of the cylinder where the integrand goes to zero without hitting the pole. **(b)** For charges with $P \cdot Q > 0$, one can deform the contour to the lower infinity of the cylinder, and by doing so pick up the residue of the pole.

use of the fact that they are the $PGL(2,\mathbb{Z})$ images of the $\nu = 0$ pole. Together with the fact that the integrand is invariant under extended S-duality group (8.3.1), it follows that the double pole of $1/\Phi$ located at

$$(\Omega, \alpha) = 0 \quad , \quad \alpha = \gamma^{-1}(\alpha_1) \quad \text{for some } \gamma \in PGL(2,\mathbb{Z}) \tag{8.3.7}$$

gives the contribution

$$(-1)^{P_\gamma \cdot Q_\gamma + 1}\, |P_\gamma \cdot Q_\gamma|\, d(P_\gamma)\, d(Q_\gamma) \qquad \text{when} \quad \big(\Lambda_{P,Q}, \alpha\big)\big(\text{Im}\Omega, \alpha\big) < 0 \tag{8.3.8}$$

and zero otherwise. The equation (8.3.8) summarizes all the contour dependence in the degeneracy formula (7.2.8).

As we will see, the jumps in the counting formula when a contour crosses one of the poles are related to the decay of marginally bound 1/2-BPS particles. Specifically, we will argue that (8.3.6) precisely counts the number of states associated with the bound state of a purely electric 1/2-BPS object and a purely magnetic 1/2-BPS object, while (8.3.8) is associated with more general dyonic bound states that are obtained by electric-magnetic duality. This interpretation will be discussed in more details in section 8.4.

In order to make the above correspondence more precise, we will now discuss the labelling of the poles susceptible to contour-dependence studied above. As we mentioned earlier, the group element $\gamma \in PGL(2,\mathbb{Z})$ associated with a given such pole might not be unique. Indeed, the elements γ which gives $\gamma^{-1}(\alpha_1) = \pm\alpha_1$ will of course not give another equation. This redundancy is exactly the one we encountered in the last section when we were deriving the labelling of the decay channels (8.2.14). Hence we conclude that these poles are also labelled by the positive real roots of the Borcherds-Kac-Moody algebra. In other words, the full contour-dependence of the integral (7.2.8) can be summarised as follows: the hyperplanes

$$(\mathrm{Im}\Omega, \alpha) = 0 \quad , \quad \alpha \in \Delta_+^{re} \tag{8.3.9}$$

cut the space of contours into different regions, and the integral changes its value by the amount (8.3.8) when one of the "walls of contours" is crossed.

8.4 The Contour Prescription and its Interpretation

8.4.1 A Contour Prescription

Let us now return to the problem of identifying the contour that should be used in the counting formula, such that it counts the right number of states for a given value of the moduli, and therefore by definition yields a duality-invariant answer. The key observation which will allow us to find the correct prescription is that the contour dependence due to the crossing of the pole labelled by the positive real roots α should exactly match the physical decay process of the corresponding dyonic bound state. For example, at the wall of marginal stability of the bound state of an electric 1/2-BPS particle with charge $(P, 0)$ and a magnetic 1/2-BPS particle with charge $(0, Q)$, one expects the degeneracy $D(P, Q)$ to be adjusted by a certain amount corresponding to the degeneracy of this $(P, 0)$, $(0, Q)$ bound state. This degeneracy can be found in the following way [145, 146, 30]. Firstly, each of the two centers has its respective degeneracy $d(P)$, $d(Q)$, which is given by the 1/2-BPS partition function of the theory as (7.1.7). Secondly, there is an extra interaction factor due to the fact that the spacetime is no longer static. The conserved angular momentum, after carefully quantizing the system [97], turns out to be

$$2J + 1 = |P \cdot Q| \; . \tag{8.4.1}$$

One therefore concludes that the jump in the counting formula when one crosses the wall of marginal stability from the stable to the unstable side is given by

$$D(P, Q) \to D(P, Q) + (-1)^{P \cdot Q} \, |P \cdot Q| \, d(P) \, d(Q) \; . \tag{8.4.2}$$

This jump in the degeneracy is precisely the contribution from the pole at $(\Omega, \alpha_1) = 2\nu = 0$ that we found in (8.3.6)! Similar jumps occur when one crosses the walls of marginal stability for the other dyonic states labelled by positive real roots α. As explained in (8.2.10) and (8.3.8), both the jump in degeneracies and the contribution of the pole are given by

$$(-1)^{(P\cdot Q)_\alpha+1}\,|(P\cdot Q)_\alpha|\,d(P_\alpha)\,d(Q_\alpha)\;,$$

where $(P\cdot Q)_\alpha$, P_α^2 and Q_α^2 are given by α as (8.2.10)

$$\Lambda_{P,Q}=P_\alpha^2\,\alpha^+ + Q_\alpha^2\,\alpha^- - (P\cdot Q)_\alpha = \Lambda_{P_1,Q_1}+\Lambda_{P_2,Q_2}-(P\cdot Q)_\alpha\;,\;\;\alpha\in\Delta_+^{re}\;.$$

Since the amount of discontinuity matches between the contour side and the supergravity side, now the aim is to find the contour prescription such that the condition for contribution matches as well. Notice that it is a priori not clear whether this would be possible or not, since there are infinitely many potential two-centered solutions and infinitely many poles susceptible to contour dependence. However, from the condition for two-centered solution to exist (8.2.14) and for the poles to contribute (8.3.8), we see that this can be done simply by choosing the contour of the integral (7.2.8) to be the three-torus lying at

$$\mathrm{Im}\Omega=\varepsilon^{-1}X\;. \tag{8.4.3}$$

Here $\varepsilon \ll 1$ is taken to be small and positive to ensure that the series expansion of $1/\Phi$ converges rapidly. Moreover, as explained earlier, for sufficiently small ε the contour avoids all other poles except the ones given by the positive real roots as $(\Omega,\alpha)=0$.

To sum up, this prescription gives the location of the contour $\mathcal{C}$ in terms of the charges and moduli:

$$\begin{aligned} D(P,Q;\lambda,\mu) &= (-1)^{P\cdot Q}\oint_{\mathcal{C}(P,Q;\lambda,\mu)} d\Omega\,\frac{e^{\pi i(\Lambda_{P,Q},\Omega)}}{\Phi(\Omega)}\\ \mathcal{C}(P,Q;\lambda,\mu) &= \{\mathrm{Re}\Omega\in T^3, \mathrm{Im}\Omega=\varepsilon^{-1}X\}\;, \end{aligned} \tag{8.4.4}$$

where X is given in terms of total charges and moduli as (8.2.7), (8.2.8) and $\varepsilon \ll 1$ is some arbitrary small positive number.

Furthermore, from the fact that the contour transforms in the same way as the charge vector under the $PGL(2,\mathbb{Z})$ S-duality group

$$X\to\gamma(X)\quad,\quad \Lambda_{P,Q}\to\gamma(\Lambda_{P,Q})\;,$$

we can now finish the argument in (8.3.1) and show that the counting formula (8.4.4), now coming with a contour prescription, is indeed consistent with the S-duality symmetry of the theory.

A similar result also holds for the so-called CHL models [150, 151]. A dyon-counting formula has been proposed for these appropriate $\mathbb{Z}_N$ orbifolds of the above theory for $N = 2, 3, 5, 7$ [152, 153, 154, 155, 156, 157]. In these theories, the rank of the gauge group is reduced and the S-duality group is now the following subgroup of $SL(2,\mathbb{Z})$:

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \mid c = 0 \text{ mod } N \;, a, d = 1 \text{ mod } N \right\} .$$

Moreover, the family of the the contour-dependent poles of the proposed generating function $\frac{1}{\tilde{\Phi}_k(\tilde{\Omega})}$, which is now a modular form of a subgroup of $Sp(2,\mathbb{Z})$, and the ways in which a dyon can split into two 1/2-BPS particles, are both modified compared to the original theory. Nevertheless, we find that they can again both be given by the elements of the reduced S-duality group $\Gamma_1(N)$, and these poles again give the same jump of index as the decaying of these bound states. In particular, following the same arguments we make exactly the same proposal (8.4.3) for the integration contour for the dyon counting formula of this class of models.

8.4.2 The Attractor Contour for Large Charges

For large charges corresponding to a macroscopic black hole, it is natural to ask what happens to our prescription when one takes the moduli at infinity to be at the attractor value. Since the attractor values of the moduli are completely determined by the charges, this procedure leads to a degeneracy formula that is independent of the moduli. At the attractor point in moduli space the following equations hold for the Narain moduli

$$P_R|_{\text{attr.}} = 0, \qquad Q_R|_{\text{attr.}} = 0 \;, \tag{8.4.5}$$

and the axion and dilaton are given by

$$\lambda_1|_{\text{attr.}} = \frac{P \cdot Q}{Q^2}, \qquad \lambda_2|_{\text{attr.}} = \frac{|P \wedge Q|}{Q^2} \;. \tag{8.4.6}$$

In our favourite matrix notation, this reads

$$\frac{1}{|P_L \wedge Q_L|} \begin{pmatrix} P_L \cdot P_L & P_L \cdot Q_L \\ P_L \cdot Q_L & Q_L \cdot Q_L \end{pmatrix}\Big|_{\text{attr.}} = \frac{1}{\lambda_2} \begin{pmatrix} |\lambda|^2 & \lambda_1 \\ \lambda_1 & 1 \end{pmatrix}\Big|_{\text{attr.}} = \frac{1}{|P \wedge Q|} \begin{pmatrix} P \cdot P & P \cdot Q \\ P \cdot Q & Q \cdot Q \end{pmatrix} .$$

In this way, we find that at the attractor point our moduli-dependent contour reduces to the following moduli-independent expression

$$\text{Im}\Omega = \varepsilon^{-1} X|_{\text{attr.}} = \varepsilon^{-1} \frac{\Lambda_{P,Q}}{\sqrt{-\frac{1}{2}\|\Lambda_{P,Q}\|^2}} = \pi \varepsilon^{-1} \frac{\Lambda_{P,Q}}{S(P,Q)} \;. \tag{8.4.7}$$

Again the $PGL(2,\mathbb{Z})$ invariance is manifest, since both sides transform in the same way, and hence this prescription also leads to a S-duality invariant counting formula. But what are the states that are being counted by this formula?

In fact, we will now argue that these are precisely the 1/4-BPS states that are not given by the bound states of two 1/2-BPS particles, and therefore cannot decay. Namely, when one fixes the moduli to be at the attractor values, the stability condition (8.2.14) reduces to

$$(\Lambda_{P,Q},\alpha)^2 < 0 ,$$

which can clearly never be satisfied. In other words, none of the bound states of two 1/2-BPS particles can exist at the attractor moduli, which is a fact consistent with the general phenomenon that an attractor flow always flows from the stable to the unstable side, a fact that we are already familiar with from our discussion of the walls of marginal stability of the $\mathcal{N}=2$ theory in section 4.3.1.

In this sense, our moduli-independent contour prescription leads to a counting formula which counts only the "immortal" dyonic states that exist everywhere in the moduli space. Notice further that this class of contours is not defined for charges with negative discriminant, since they lie outside of the Siegel domain. This is consistent with the fact that they do not have an attractor point, and there is no single-centered supergravity solution carrying these charges.

Finally we would like to briefly comment on the role of the number ε in our proposed contours (8.4.3), (8.4.7). It can be seen as playing the role of a regulator for the convergence of the generating function. To see this, notice that when we take the contour according to our prescription (8.4.7), the contribution

$$\left| D(P,Q)\, e^{-i\pi(\Omega,\Lambda_{P,Q})} \right| = |D(P,Q)| e^{-2\varepsilon^{-1}\pi|P\wedge Q|} \sim e^S e^{-2\varepsilon^{-1}S} \tag{8.4.8}$$

of certain large charges to the partition function is highly suppressed when $\varepsilon \ll 1$, and we are therefore left with a rapidly converging generating function.

8.5 Wall-Crossing and Representations of the Algebra

Now we would like to pause and ask ourselves the following question: what does it mean that a different choice of contour gives a different answer for the BPS degeneracy? After all, the counting formula (7.1.14) we derived using the D1-D5 CFT does not seem to have any ambiguity. And since we know that the difference between different answers are exactly accounted for by the two-centered solutions in supergravity, what can we say about the states corresponding to these solutions?

First we begin with the first question. Although the formula (7.1.14) might look unambiguous, the ambiguity really lies in how we expand the right-hand side of the equation. For example [146], the two possible ways of expanding the following factor of the partition function

$$\begin{aligned}\frac{1}{(y^{1/2}-y^{-1/2})^2} &= \frac{1}{y(1-y^{-1})^2} = y^{-1}+2y^{-2}+\cdots \\ &= \frac{1}{y^{-1}(1-y^{1})^2} = y^{1}+2y^{2}+\cdots\ ,\end{aligned}$$

corresponding to two possible ranges for the parameter $y>1$ and $y<1$, will give different answers for the degeneracies. It is not hard to convince oneself that this ambiguity of choosing expansion parameters is exactly the same ambiguity as that of choosing integration contours when we invert the equation. To be more precise, rewrite the equation (7.1.14) in the following form as in section 7.2.3

$$\sum_{P,Q}(-1)^{P\cdot Q}D(P,Q)e^{-\pi i(\Lambda_{P,Q},\Omega)} = \left(\frac{1}{e^{-\pi i(\varrho,\Omega)}\prod_{\alpha\in\Delta^+}\left(1-e^{-\pi i(\alpha,\Omega)}\right)^{\text{mult}\alpha}}\right)^2 ,$$

apparently, we should expand the product factor in powers of $e^{-\pi i(\alpha,\Omega)}$ when $(\text{Im}\Omega,\alpha)<0$ and in powers of $e^{\pi i(\alpha,\Omega)}$ when $(\text{Im}\Omega,\alpha)>0$. Comparing with the integral formula with contour prescription (8.4.4), we see that the integral formula amounts to the statement that the degeneracies $D(P,Q)$ given a point in the moduli space is indeed counted by the above generating function, *provided* that the right-hand side be expanded in powers of $e^{-\pi i(\alpha,\Omega)}$ when $(\text{Im}\Omega,\alpha)<0$ and in powers of $e^{\pi i(\alpha,\Omega)}$ when $(\text{Im}\Omega,\alpha)>0$.

We can give the above prescription another interpretation which makes the role of the Borcherds-Kac-Moody algebra more manifest. Let's consider the Verma module $M(L)$ of this algebra with highest weight L and its super-character. Besides the denominator (7.2.33), the character formula also contains a numerator. Using the formal exponential introduced in section 7.2.3, the super-character reads

$$\begin{aligned}\text{sch}M(L) &= \sum_{\mu\leq L}\text{sdim}(M(L)_\mu)\,e(\mu) \qquad (8.5.1)\\ &= \frac{e(-\varrho+L)}{e(-\varrho)\prod_{\alpha\in\Delta^+}\left(1-e(-\alpha)\right)^{\text{mult}\alpha}} = \frac{e(L)}{\prod_{\alpha\in\Delta^+}\left(1-e(-\alpha)\right)^{\text{mult}\alpha}}\ ,\end{aligned}$$

where $M(L)_\mu$ denotes the weight-μ sub-module of the Verma module $M(L)$, and "$\mu \le L$" means that $L-\mu$ is a sum of simple roots. The "s" of "sdim" denotes the fact that we are dealing with the graded characters counting the graded degeneracies, taking the plus or minus sign depending on whether the root involved is even or odd. Indeed, recall that in section 7.2.3 we have defined the number multα to be the graded multiplicities of the root α.

Let's now compare this character formula with the integral

$$(-1)^{P\cdot Q} D(P,Q;\lambda,\mu) = \oint_{\mathcal{C}(P,Q;\lambda,\mu)} d\Omega \left(\frac{e^{i\frac{\pi}{2}(\Lambda_{P,Q},\Omega)}}{e^{-\pi i(\varrho,\Omega)} \prod_{\alpha\in\Delta^+} \left(1-e^{-\pi i(\alpha,\Omega)}\right)^{\text{mult}\alpha}} \right)^2$$

$$\mathcal{C}(P,Q;\lambda,\mu) = \{\text{Re}\Omega \in T^3, \text{Im}\Omega = \varepsilon^{-1}X\}\ , \tag{8.5.2}$$

we note that the integrand is exactly the square of sch$M(L_{P,Q})$, namely the square of the super-character of the Verma module with highest weight

$$L_{P,Q} = \varrho + \frac{1}{2}\Lambda_{P,Q}\ , \tag{8.5.3}$$

and the contour integral has the function of picking up the zero-weight sub-module. Hence the dyon degeneracy $(-1)^{P\cdot Q}D(P,Q)$ has the interpretation of counting the (graded) number of ways the weight $2L_{P,Q}$ can be written as a sum of two copies of positive roots.

What we just saw is that the dyon degeneracies have a nice interpretation in terms of positive roots of the algebra which seems to be free of ambiguities, so we might wonder where the contour/moduli ambiguities we have seen from the integral/supergravity viewpoint goes in this picture. The subtlety lies in the fact that, the character formula (8.5.1) in terms of the formal exponentials $e(\mu)$ satisfying $e(\mu)e(\mu') = e(\mu+\mu')$ contains the same information as the integral formula in terms of functions $e^{\pi i(\Omega,\mu)}$ of Ω only if we expand all the expressions in the latter formula in powers of $e^{\pi i(\Omega,\mu)}$. From our discussion above, this means that for this interpretation to be correct, one must make the (unique) choice of simple roots, and thereby the choice of positive roots, in the character formula such that the moduli vector X lies in the fundamental Weyl chamber of the root system

$$X \in \mathcal{W} := \big\{ x \in \Gamma_+ \otimes \mathbb{R},\ (x,\alpha_i) < 0, \alpha_i \in \{\text{simple roots}\} \big\}\ . \tag{8.5.4}$$

A manifest but crucial fact to keep in mind here is that the character formula (8.5.1) for Verma modules in *not* invariant under a change of simple roots.

What we have concluded from the above reasoning is that, for a super-selection sector with given total charges, a different choice of moduli corresponds to a different choice of positive roots in the algebra. This is nevertheless not very convenient. Instead we will use the equivalent description of letting the highest weight of the module be moduli-dependent while keeping the simple roots fixed.

No matter whether we choose to keep the highest weight fixed and vary the simple roots when we vary the moduli, or keep the simple roots fixed and vary the highest weight, from our contour condition (8.5.4) it's clear that only in the attractor region, characterised by

$$(X,\alpha)(\Lambda_{P,Q},\alpha) > 0 \quad , \quad \text{for all} \quad \alpha \in \Delta^{re} \; ,$$

does the counting formula corresponds to a Verma module of dominant highest weight. Recall that a dominant weight is a vector lying in the fundamental Weyl chamber and having integral inner product with all roots.

For a given set of total charges, a natural choice for the simple roots is therefore such that the charge vector lies in of the fundamental domain[2]

$$\Lambda_{P,Q} \in \mathcal{W} \; .$$

When the moduli do not lie in the attractor region, the corresponding Verma module will not have dominant highest weight. Indeed, when considering a point in the moduli space corresponding to another Weyl chamber

$$X \in w(\mathcal{W}) \quad , \quad w \in W \; ,$$

either from the contour integral (8.5.2) or from the character formula (8.5.1) we see that the dyon degeneracy is encoded in the Verma module of the following highest weight

$$L_{P,Q}|_w = \varrho + w^{-1}(L_{P,Q} - \varrho) \;\Leftrightarrow\; \Lambda_{P,Q}|_w = w^{-1}(\Lambda_{P,Q}) \; .$$

Notice that the ambiguity we discuss above does not involve the imaginary positive roots, defined as those positive roots that are timelike or lightlike, which give the majority of factors in the product formula, and are therefore responsible for the asymptotic growth of the degeneracies. This is because, because our X is in the future lightcone by construction, the convergence criterion

$$(X,\beta) < 0 \quad , \quad \beta \in \Delta^{im}_{+}$$

is guaranteed to be met, independent of the choice of X. This also justifies the fact that the Weyl group of a Borcherds-Kac-Moody algebra is the reflection group with respect to the real roots only.

[2]We ignore the special cases where the charge vector lies on the boundary of some fundamental domain, corresponding to the situation of having two-centered scaling solutions.

Therefore, we arrive at the conclusion that crossing a wall of marginal stability corresponds to a change of representation of the BPS algebra microscopically. More precisely, the change is such that the highest weight of the Verma module is changed by a Weyl reflection, and away from the attractor region, the highest weight of the representation will no longer be dominant.

8.6 Weyl Chambers and Discrete Attractor Flow Group

Classically, a moduli space is a continuous space in which the vev's of the moduli fields of the theory can take their values. A distinct path in this space for a given superselection sector, namely the total conserved charges, is the attractor flow of a single-centered black hole solution with a given starting point.

As we have established in section 8.2, the asymptotic values of these scalar fields play a role in the spectrum of BPS states through the presence/absence of certain 1/4-BPS bound states of two 1/2-BPS objects. The walls of marginal stability for these bound states divide the moduli space into different regions in which the BPS spectrum is predicted to be constant by our supergravity analysis. This is because the spectrum only jumps when a wall of marginal stability is crossed, and for the purpose of studying the BPS spectrum of the theory we can identify the region bounded by a set of walls to be a point.

From the above consideration, it is useful to consider a "discrete attractor flow", which brings one region in the moduli space to another. It is not difficult to see that such an operation forms a group. Recall that in this specific theory we study, the walls of marginal stability are in one-to-one correspondence with the positive roots of a Weyl group W, or the positive real roots of a Borcherds-Kac-Moody algebra introduced in section 7.2.3, as we have seen in section 8.2. This implies that we should be able to identify the group of discrete attractor flow to be the same Weyl group W. The aim of this section is to make this statement precise, and to address the implication of having such a group structure underlying the attractor flow.

8.6.1 Weyl Chamber and Moduli Space

To make the discussion more concrete let us visualise the situation. First recall that given a point in the 134-dimensional moduli space, whether a given two-centered solution exists only depends on the combination of the moduli field encapsulated in the following "unit vector" (8.2.8)

$$X = \frac{\mathcal{Z}}{\sqrt{-\frac{1}{2}\|\mathcal{Z}\|^2}}\ .$$

where

$$\mathcal{Z} = \frac{1}{\sqrt{|P_L \wedge Q_L|}} \begin{pmatrix} P_L \cdot P_L & P_L \cdot Q_L \\ P_L \cdot Q_L & Q_L \cdot Q_L \end{pmatrix} + \frac{\sqrt{|P_L \wedge Q_L|}}{\lambda_2} \begin{pmatrix} |\lambda|^2 & \lambda_1 \\ \lambda_1 & 1 \end{pmatrix} .$$

This can be understood in the following way. Recall that the largest central charge is related to the above vector as

$$|Z_{P,Q}|^2 = M_{P,Q}^2 = -\frac{1}{2}\|\mathcal{Z}\|^2 \tag{8.6.1}$$

and the fact that, in the $\mathcal{N} = 2$ language of chapter 4, the attractor flow is a gradient flow of the central charge $|Z_{P,Q}|$ (see Figure 4.1), it is not surprising that the relevant part of the moduli is encoded in its direction X.

Using the fact that the sheet of hyperbola of all future-pointing vector of fixed norm $\|X\|^2 = -2$ is equivalent to the upper-half plane $\mathcal{H}_1$ and the Poincaré disk (see Figure 1.2), we can therefore map the relevant part (the X-space) of the moduli space onto $\mathcal{H}_1$ or the Poincaré disk. Concretely, we will use the map (1.3.41)

$$\begin{aligned} X &= \frac{1}{\tau_2} \begin{pmatrix} |\tau|^2 & \tau_1 \\ \tau_1 & 1 \end{pmatrix} \\ z &= i\Big(\frac{\tau + e^{-\frac{i\pi}{3}}}{\tau + e^{\frac{i\pi}{3}}}\Big) . \end{aligned} \tag{8.6.2}$$

Now we are ready to draw the walls of marginal stability (8.2.9). First we will begin with α_1, which corresponds to the two-centered solution with charges $(P, 0)$, $(0, Q)$, for which the wall of marginal stability reads

$$(X, \alpha_1) = 0 .$$

This gives an arc of a circle on the Poincaré disk, which is a geodesic with respect to the hyperbolic metric, and a straight line (a degenerate circle) in the upper-half plane. See Figure 8.3.

As the next step, we will draw the walls given by other two roots α_2 and α_3 defined in (7.2.9). Notice that the three walls $(X, \alpha_i) = 0$, $i = 1, 2, 3$ bound a triangle on the disk, and furthermore it is easy to show that the interior of the disk satisfies $(X, \alpha_i) < 0$. For example, the center of the triangle is given by the normalised version $\varrho(\sqrt{-\frac{1}{2}\|\varrho\|^2})^{-1}$ of the Weyl vector ϱ satisfying $(\varrho, \alpha_i) = -1$.

As discussed in the previous section, given a charge vector one should choose the simple roots such that $\Lambda_{P,Q}$ lies inside the fundamental Weyl chamber.

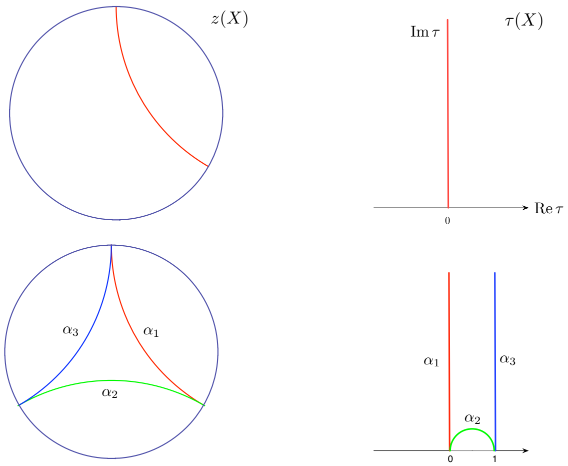

Figure 8.3: **(i)** The wall of marginal stability for the two-centered solution with charges (P,0) and (0,Q), projected onto the two-dimensional slice of moduli space equipped with a natural hyperbolic metric, and mapped to the Poincaré disk and the upper-half plane. **(ii)** The basic three walls $(X, \alpha_i) = 0$.

For concreteness of the discussion and without loss of generality, we will now assume that such a choice is given by the three simple roots we used in section 7.2.1, namely that the charge vector satisfies

$$(\Lambda_{P,Q}, \alpha_i) < 0 \quad , \quad i = 1, 2, 3$$
$$\alpha_1 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \quad , \quad \alpha_2 = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} \quad , \quad \alpha_3 = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} .$$

In other words, from now on we will consider charges such that the vector $\Lambda_{P,Q}$ lies inside the large triangle in Figure 8.3. By the virtue of the relation between the Weyl group and the extended S-duality group $W \subset PGL(2, \mathbb{Z})$, we can always use the duality group to map a set of charges (P, Q) to another set of charges for which the above is true.

By definition, all the other real roots, in particular all the other positive real roots, are related to these three simple roots by a Weyl reflection $\alpha = w(\alpha_i)$. We can thus draw the rest of the walls of marginal stability

$$(X, \alpha) = 0 \quad , \quad \alpha \in \Delta_+^{re}$$

by reflections of the triangle in Figure 8.3 with respect to the three sides. This gives a tessellation of the Poincaré disk as shown in Figure 8.4. Notice that the figure we draw can never be a faithful presentation of the real situation, since the group tessellates the disk with an infinite number of triangles.

By definition, the Weyl group divides the relevant part of the moduli space, namely the X-space, into different Weyl chambers bounded by the walls of orthogonalities with the positive roots. In other words, for any point in the moduli space, there exists a unique element of the Weyl group $w \in W$ such that the corresponding moduli vector X lies in the Weyl chamber

$$X \in w(\mathcal{W}) \Leftrightarrow (X, w(\alpha_i)) < 0 \ .$$

Because these walls, or mirrors, of the Weyl chambers are exactly the physical walls of marginal stability $(X, \alpha) = 0$ corresponding to the split into two centers (8.2.10), we conclude that the BPS spectrum does not jump when the moduli move inside a given triangle. In other words, the Weyl chambers are exactly the region in the moduli space where the BPS spectrum is constant, and there is a different dyon degeneracy associated to every different Weyl chamber $w(\mathcal{W})$.

8.6.2 A Hierarchy of Decay

Now we would like to discuss what the group structure of the moduli space discussed above implies for the dyon spectrum at a given moduli. In principle, given a point in the moduli space we know exactly which two-centered solutions exist by using the stability condition we worked out in section 8.2, namely that the solution given by the above split of charges exists if and only if

$$(X, \alpha)(\Lambda_{P,Q}, \alpha) < 0 \ . \tag{8.6.3}$$

But in fact we know less than it might seem. It is because there are infinitely many decay channels, or to say that there are infinitely many positive real roots in the Borcherds-Kac-Moody algebra, so given a moduli vector X, even equipped with the stability condition we will never be able to give a list of two-centered solutions within finite computation time. In this subsection we will see how the group structure changes this grim outlook.

First note that, in the central triangle, namely the fundamental Weyl chamber, there is no two-centered bound states. This can be seen as follows. Recall that we have chosen the simple roots such that

$$(\Lambda_{P,Q}, \alpha_i) < 0 \quad , \quad \alpha_i \in \{\text{simple real roots}\} \ ,$$

which implies

$$(\Lambda_{P,Q}, \alpha) < 0 \quad , \quad \text{for all } \alpha \in \Delta_+^{re}$$

simply from the definition of the positive real roots. Now the same thing holds for a point $X \in \mathcal{W}$ inside the fundamental Weyl chamber, namely

$$(X, \alpha) < 0 \quad , \quad \text{for all } \alpha \in \Delta_+^{re} \ ,$$

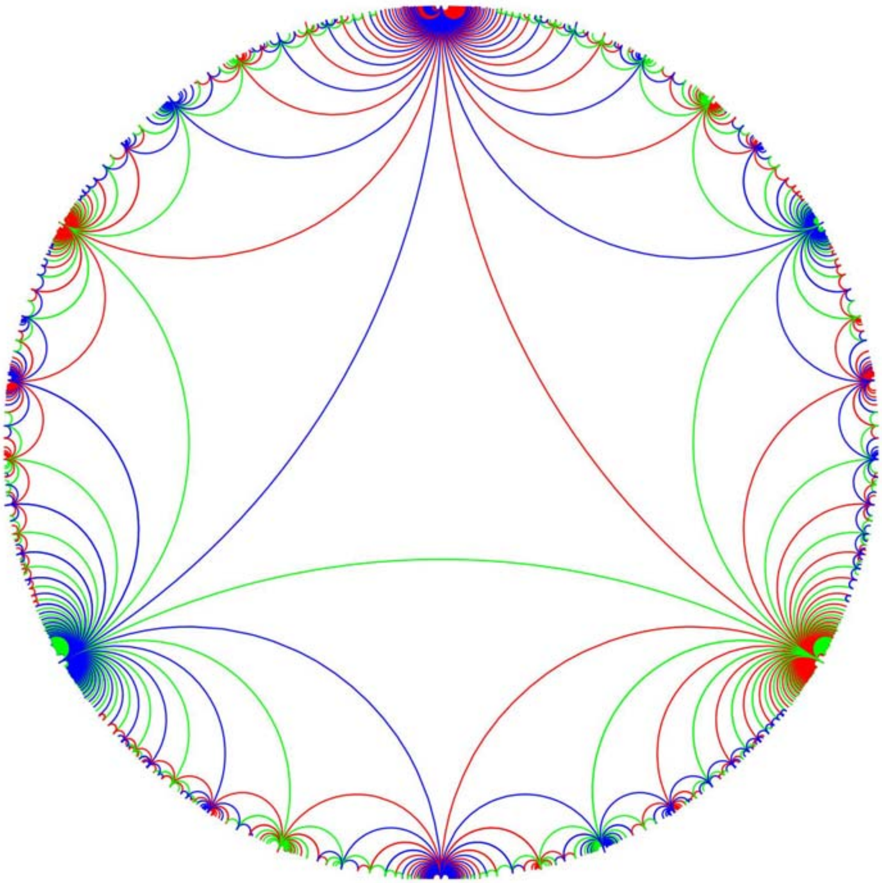

Figure 8.4: Tessellation of the Poincaré disk using the group W generated by the reflection with respect to the three "mirrors", namely the three sides of the regular triangle in the middle. Walls of the same color are mirror images of each other. Notice that each triangle has the same volume. The slight inhomogeneity of colours on the edge is an artifact of the computer algorithm we use.

hence we see that the stability condition (8.6.3) will never be satisfied for any split of charges.

We therefore conclude that the fundamental Weyl chamber represents the "attractor region" in the moduli space, namely the same region (chamber) where the attractor point lies and in which none of the two-centered solutions exists.

This is of course not true anymore once we move out of the fundamental Weyl chamber. Consider for example the neighbouring Weyl chamber $s_1(\mathcal{W})$, obtained by reflecting the fundamental chamber with respect to one of the simple roots α_1. Because this reflection takes $\alpha_1 \to -\alpha_1$ and permutes the rest of the positive real roots (8.9.3), we conclude that

$$(X, \alpha_1) > 0 \; , \; (X, \alpha) < 0 \quad \text{for all} \;\; X \in s_1(\mathcal{W}), \alpha \neq \alpha_1 \; , \alpha \in \Delta_+^{re} \; ,$$

which is also obvious from the picture. This means that there is now one two-centered solution corresponding to the split into charges $(P, 0)$, $(0, Q)$ (8.2.12) and no others.

We can now go on with this process to every Weyl chamber in the (reduced) moduli space: go to the next-neighbouring chamber, and the next-next-neighbouring, and so on, with the condition that the path doesn't walk "backwards", or more precisely that the length function (see 8.9.3) of the corresponding group element always increases.

In general, considering an arbitrary point on the disk

$$X \in w(\mathcal{W})$$

for some $w \in W$. For any group element, we can always decompose it as the following "shortest word" in terms of the three group generators

$$\begin{aligned} & w = s_{i_1} s_{i_2} \cdots s_{i_n} \\ & i_m \in \{1, 2, 3\} \;\; , \; i_m \neq i_{m\pm 1} \;\; \text{for all } m = 2, \cdots, n-1 \; . \end{aligned} \tag{8.6.4}$$

From the fact that a reflection with respect to the root $w(\alpha_i)$ is given by the group element $w s_i w^{-1}$:

$$w s_i w^{-1}\big(w(\alpha_i)\big) = w s_i(\alpha_i) = -w(\alpha_i) \; ,$$

we can describe the Weyl chamber $w(\mathcal{W})$ as given by the following successive reflection of the fundamental Weyl chamber:

$$w_0 = \mathbf{1} \xrightarrow[\alpha_{i_1}]{} w_1 \xrightarrow[w_1(\alpha_{i_2})]{} w_2 \cdots \xrightarrow[w_{n-1}(\alpha_{i_n})]{} \quad w_n = w \tag{8.6.5}$$

where "$\xrightarrow[\alpha]{}$" means "reflecting with respect to the wall (α, X)=0 given by the root α" and the intermediate group elements w_m are given by

$$w_m = s_{i_1} \cdots s_{i_m} \quad , \quad m \leq n \; .$$

In other words, for a given point in the moduli space and its Weyl chamber

$$X \in w(\mathcal{W}) \; ,$$

by following the above path from the attractor region $\mathcal{W}$, or the fundamental Weyl chamber, to $w(\mathcal{W})$, we can read out the two-centered solutions which exist at the point $X \in \mathcal{W}$. These are given by the charge splitting

$$\begin{aligned} \Lambda_{P_1,Q_1} &= P_\alpha^2 \, \alpha^+ \quad , \quad \Lambda_{P_2,Q_2} = Q_\alpha^2 \, \alpha^- \\ \Lambda_{P,Q} &= P_\alpha^2 \, \alpha^+ + Q_\alpha^2 \, \alpha^- - (P \cdot Q)_\alpha \, \alpha \; , \end{aligned}$$

with now

$$\alpha \in \{ w_{m-1}(\alpha_{i_m}) \; , \quad m = 1, \cdots, n \} \subset \Delta_+^{re} \; .$$

In other words, when we follow the journey from the attractor region $\mathcal{W}$ to the Weyl chamber $w(\mathcal{W})$ where the moduli is, namely when we follow the inverse attractor flow to the point under consideration, we will successively cross the walls of marginal stability corresponding to the roots α_{i_1}, $w_1(\alpha_{i_2})$, ... , and finally $w_{n-1}(\alpha_{i_n})$.

This gives a simple dictionary to read out the list of dying dyons for any given point in the moduli space, provided that we know the shortest decomposition of the group element w in terms of a string of generators ("letters").

To complete the algorithm, we also give a very simple algorithm to determine such a decomposition given an arbitrary lightlike, future-pointing vector X. Given a point X, we would like to determine the shortest-length string

$$w = s_{i_1} \cdots s_{i_n}$$

such that its corresponding chain of reflection induces the following successive mapping of the vector X into the fundamental Weyl chamber

$$\begin{aligned} &w_0 = \mathbf{1} \xrightarrow[\alpha_{i_1}]{} w_1 \xrightarrow[w_1(\alpha_{i_2})]{} \cdots \xrightarrow[w_{n-1}(\alpha_{i_n})]{} w_n = w \\ &X_0 \in \mathcal{W} \xrightarrow[\alpha_{i_1}]{} X_1 \xrightarrow[w_1(\alpha_{i_2})]{} \cdots \xrightarrow[w_{n-1}(\alpha_{i_n})]{} X_n = X \in w(\mathcal{W}) \; . \end{aligned}$$

One can show that the string is determined as follows: suppose

$$X_m = s_{i_1} \cdots s_{i_m} X_0 \quad , \quad m \leq n \; , \tag{8.6.6}$$

then $i_m \in \{1,2,3\}$ is given by the condition

$$(\alpha_{i_m}, X_m) > 0 \,, \tag{8.6.7}$$

which has at most one solution for i_m. We can go on with this process for X_{m-1} until the above equation has no solution anymore, corresponding to when the three expansion coefficients of $X_0 = \sum_{i=1}^{3} \alpha_i X_0^{(i)}$ satisfy the triangle inequality. This is when the decay ends and when the moduli flow to the attractor region given by the fundamental Weyl chamber.

Notice that there is a hierarchy of decay (or a "death row") in this process. Namely, considering another point in the moduli space which is in the Weyl chamber $w_m(\mathcal{W})$ with $m < n$, applying the same argument as above shows that the two-centered solutions existing in that Weyl chamber are given by the first m positive roots in the above list. Specifically, this argument shows that there is nowhere in the moduli space where the bound states given by the root $w_{n-1}(\alpha_{i_n})$ exists without all the other $n-1$ bound states given by $w_{m-1}(\alpha_{i_m}), m < n$ in front of it in the row.

This hierarchy among two-centered bound states clearly stems from the hierarchy among elements of the group W. Indeed, the ordering in (8.6.5) is an example of what is called the "weak Bruhat order" $w_m < w_{m+1}$ among elements of a Coxeter group. See (8.9) for the definition.

In our case this weak Bruhat ordering has an interpretation reminiscent of the RG-flow of the system. From the integrability condition (8.2.13)

$$\sqrt{|P_L \wedge Q_L|}\, |\vec{x}_{P_\alpha} - \vec{x}_{Q_\alpha}| = \left| \frac{(\Lambda_{P,Q}, \alpha)}{(X, \alpha)} \right| ,$$

now with $\alpha = \{w_{m-1}(\alpha_{i_m}), m \leq n\}$, we see that the ordering of the decay is exactly the ordering of the coordinate size of the bound state. In other words, roughly speaking, the ordering we discussed above can be summarised as the principle that the bound state which is the bigger in size are more prone to decay than the smaller ones, which is a fact parallel to the usual RG-flow phenomenon. See Fig 8.5 for a simple example of the flow beginning from a point in $s_1 s_3(\mathcal{W})$.

As was alluded to earlier, we can identify the path taken by such a string of reflections as the path taken by a discrete version of attractor flows. In other words, if we identify all points in a given Weyl chamber, justified by the fact that all points there have the same BPS spectrum, the usual continuous attractor flow reduces to the successive reflections discussed above. To argue this, note that the (single-centered) attractor flow also gives such a structure of hierarchy among multi-centered solutions, because an attractor flow can only cross a given wall at most once. Using the $\mathcal{N} = 2$ language of chapter

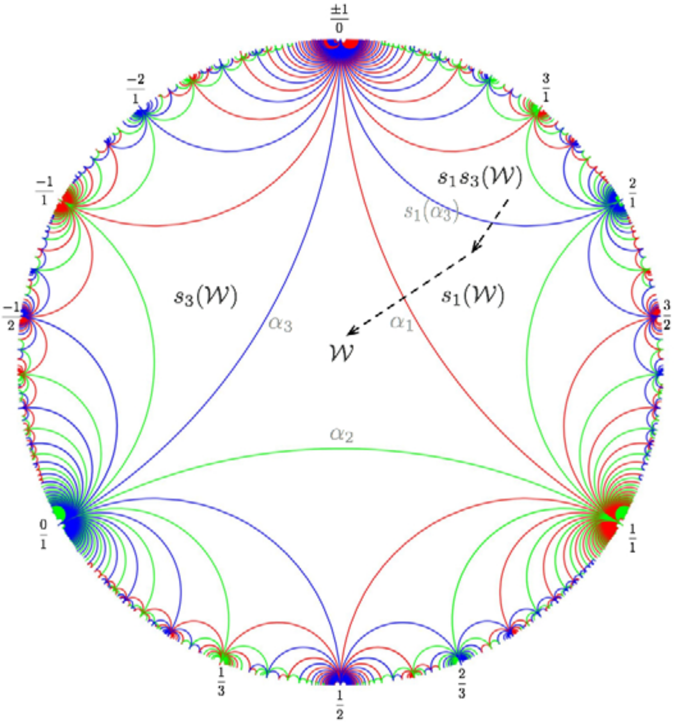

Figure 8.5: (i) An example of a discrete attractor flow from $X \in s_1 s_3(\mathcal{W})$ to the attractor region $\mathcal{W}$, passing through two walls of marginal stability $(s_1(\alpha_3), X) = 0$ and $(\alpha_1, X) = 0$. (ii) The boundary of the disk can be identified with the boundary of the light-cone, and in turn be identified with a compactified real line using the map (8.7.1). In this way there is a pair of rational numbers associated to each positive real roots, and a Weyl chamber is given by such a pair with its mediant. Furthermore, the discrete attractor flow can be thought of as a process of "coarse-graining" the rational numbers.

4, this is because for a single-centered solution, the quantity $\mathrm{Im}(\bar{Z}Z')/|Z|$ is linear in $1/r$, the inverse of the coordinate distance from the black hole, where Z' is the central charge of another arbitrary charge at the moduli where the flow is, and can therefore only pass zero at most once [83]. On the other hand, such a hierarchy among solutions can only be a property of the structure of the moduli space, since the stability condition is a local condition on the moduli and is in particular path-independent. Hence we have to conclude that the group and the flow must cross the walls in the same order. This justifies our claim that the group is simply the discretised group of attraction.

8.7 Arithmetic Attractor Flows

The title of this section is very similar the title "Arithmetic and Attractors" of the classic paper by Moore [149, 158]. As suggested in the title, in this section we will discuss the arithmetic aspects of our newly defined discrete attractor "flow" group.

First let us again review the equation for the walls of marginal stability (8.2.10), which states that for any positive real root α, the charge vector Λ_{P_1,Q_1}, Λ_{P_2,Q_2} of the associated two-centered solution are given by the component of the total charge vector $\Lambda_{P,Q}$ along the two lightcone direction $\alpha^{\pm}$ perpendicular to α. To be more precise, we defined the lightcone "unit vector" $\alpha^{\pm}$ to satisfy the following conditions : (i) $\|\alpha^{\pm}\|^2 = 0$ (lightlike), $\mathrm{Tr}(\alpha^{\pm}) > 0$ (future-pointing) (ii) As matrices $\alpha^{\pm}$ have integral entries on the diagonal (weight condition) (iii) $(\alpha^{+}, \alpha^{-}) = -1$ (normalisation condition).

From the above conditions, it is not difficult to see that the two-centered solutions can equivalently be given by a pair of rational numbers $\{b/a, d/c\}$ satisfying $ad - bc = 1$, such that the lightlike vectors are given by

$$\{\alpha^{+}, \alpha^{-}\} = \left\{ \begin{pmatrix} b^2 & ab \\ ab & a^2 \end{pmatrix}, \begin{pmatrix} d^2 & cd \\ cd & c^2 \end{pmatrix} \right\} \tag{8.7.1}$$

Notice that exchanging the two rational numbers amounts to exchanging (P_1, Q_1) and (P_2, Q_2), which obviously does not give a new solution. Without loss of generality, we now impose that $a \geq 0, c \geq 0$, while b, d can take any sign.

To be more precise, given such a pair of rational numbers, the corresponding positive roots is

$$\alpha = \begin{pmatrix} 2bd & ad + bc \\ ad + bc & 2ac \end{pmatrix} , \tag{8.7.2}$$

and the corresponding charge splitting is

$$\begin{pmatrix} P_1 \\ Q_1 \end{pmatrix} = (-cP + dQ) \begin{pmatrix} b \\ a \end{pmatrix} \quad , \quad \begin{pmatrix} P_2 \\ Q_2 \end{pmatrix} = (aP - bQ) \begin{pmatrix} d \\ c \end{pmatrix} . \tag{8.7.3}$$

In other words, the above formula gives an alternative labelling of the two-centered solutions of the theory by a pair of rational numbers

$$\left\{ \frac{b}{a}, \frac{d}{c} \right\} \quad , \quad ad - bc = 1 \quad , \quad a, c \geq 0 . \tag{8.7.4}$$

In particular, the three simple roots (7.2.9) correspond to the three sets of rational numbers $\{\frac{-1}{0}, \frac{0}{1}\}$, $\{\frac{0}{1}, \frac{1}{1}\}$, $\{\frac{1}{1}, \frac{1}{0}\}$ respectively.

Now we would like to know what the discrete attractor flow, defined in the last subsection, looks like in terms of the presentation in terms of rational numbers. From the figure 8.5 it is obvious that, for a positive root bounding the Weyl chamber $w(\mathcal{W})$, one of the following two roots $wOw^{-1}(\alpha)$ and $wO^2w^{-1}(\alpha)$ must be negative, where O is the order three generators of the dihedral group corresponding to a rotation of 120° (7.2.20), and this must be the root with respect to which the last reflection in the sequence (8.6.5) is. Therefore, from the computation which gives the following expression for $-wOw^{-1}(\alpha)$ and $-wO^2w^{-1}(\alpha)$

$$\begin{pmatrix} 2b(d-b) & bc+ad-2ab \\ bc+ad-2ab & 2c(c-a) \end{pmatrix} , \begin{pmatrix} 2d(b-d) & bc+ad-2cd \\ bc+ad-2cd & 2a(a-c) \end{pmatrix} ,$$

we conclude that given a two-centered solution corresponding to the pair rational numbers $\{b/a,\ d/c\}$ with $a, c \geq 0$ and $ad - bc = 1$, we can read out the next two-centered solution on the list of decadence (or the death row) as another pair of rational numbers

$$\left\{\frac{b}{a}, \frac{d-b}{c-a}\right\} \text{ if } c \geq a, bd \geq b^2 \quad , \quad \left\{\frac{d}{c}, \frac{b-d}{a-c}\right\} \text{ if } a \geq c, bd \geq d^2 \ .$$

This rule is actually much simpler than it might seem. Consider the boundary of the Poincaré disk as a compactified real line, namely with $\frac{\pm 1}{0}$ identified, then the map (8.7.3), (8.7.4) associates with each wall of Weyl chamber, or equivalently wall of marginal stability, a pair of rational numbers satisfying the above conditions. See Figure 8.5. Now what we saw above simply means: firstly, any triangle is bounded by a set of three rational numbers of the form

$$\left\{\frac{b}{a}, \frac{b+d}{a+c}, \frac{d}{c}\right\} \quad , \quad ad - bc = 1 \quad , \quad a, c \geq 0 \ . \tag{8.7.5}$$

The middle element $(b+d)/(a+c)$ is called the "mediant" of the other two, and can be easily shown to always lie between the two

$$\frac{b}{a} < \frac{b+d}{a+c} < \frac{d}{c} \ .$$

Secondly, the direction of the attractor flow is that it always flows to the numbers with smaller $|a| + |b|$, $|c| + |d|$, and can therefore be seen as a flow of coarse-graining the rational numbers.

As an illustration of the above statement, in Figure 8.6 we show the rational numbers corresponding to the first three levels of inverse attractor reflections from the attractor region, corresponding to the first three levels of the weak Bruhat tree.

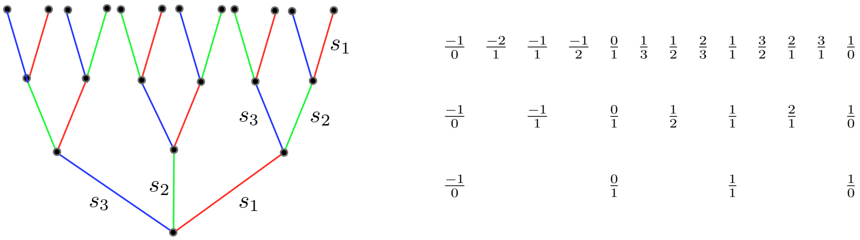

Figure 8.6: (l) The (first three levels of the) weak Bruhat ordering of the group W, which corresponds to the hierarchy of the wall-crossing of the theory. (r) The corresponding coarse-graining of the rational numbers, which is a part of the Stern-Brocot tree generalised to the whole real line.

Note that this is simply the generalisation of the Stern-Brocot tree to the negative part of the real line, and notice that *all* the rationals are contained in this tree. In particular, the Farey series is contained in the middle part of the tree.

8.8 Summary and Conclusion

In this chapter we study the moduli dependence of the BPS dyon degeneracies of a $\mathcal{N} = 4$, $d = 4$ string theory, and its relationship to a counting formula and a Borcherds-Kac-Moody algebra. In section 8.2 we study the stability condition for a solution in the low-energy supergravity theory which describes the bound state of two 1/2-BPS objects, to exist. We show that these are in one-to-one correspondence with the positive real roots of the Borcherds-Kac-Moody algebra. In section 8.3 we study the dependence of the dyon-counting formula on the possible choices of integration contours, and chart the difference between different BPS degeneracies predicted using different contours of integration. In the following section we show how a moduli-dependent contour prescription can relate the two ambiguities: that of choosing a contour and that of choosing the moduli of the theory, such that the dyon-counting formula correctly accounts for the moduli-dependence of the BPS spectrum.

After that we turn to the question of how a certain Borcherds-Kac-Moody algebra is related to the above phenomenon, since we have seen in the previous chapter that the counting formula is mathematically related to the denominator formula of this algebra. First we note in section 8.5 that the dyon-counting formula can be seen as the (square of the) character formula for the Verma module of the algebra, with an appropriate choice of highest weight which depends on the moduli of the theory. Secondly, we use the correspondence between the walls of marginal stability and walls of Weyl chambers of the algebra to identify the Weyl group of the algebra as the group of a discretised version of attractor flow, whose precise meaning is spelled out in section 8.6. Finally we comment on the arithmetics of this attractor flow group, in particular how a flow to the attractor region can be described as a process of coarse-graining the rational numbers.

This study further clarifies the issue of the moduli dependence of the spectrum in this theory and elucidates (part of) the role of the Borcherds-Kac-Moody algebra in structuring the BPS spectrum of the theory. Some interesting generalisation would be to extend the above analysis to the case where the "co-prime" condition on the charges (6.6.7) is not satisfied and when other possible two-centered solutions are possible. The other generalisation is to study the group and algebraic structure for the orbifolded $\mathcal{N} = 4$ theory, namely the CHL model[159]. Finally, a very interesting open question is to what extent these structures survive when supersymmetry is further broken down to $\mathcal{N} = 2$.

8.9 Appendix: Properties of Coxeter Groups

In this appendix we collect various definitions and facts about Coxeter groups. The proofs of them can be found in [160, 161, 162]. Our presentation is very similar to that of [163].

Definition (Coxeter System) A Coxeter system (W, S) consists of a Coxeter group W and a set of generators $S = \{s_i, i = 1, \cdots, n\}$, subjected to the relations

$$s_i^2 = 1 \qquad ; \qquad (s_i s_j)^{m_{ij}} = 1 \tag{8.9.1}$$

with

$$m_{ij} = m_{ji} \geq 2 \quad \text{for } i \neq j \; . \tag{8.9.2}$$

A Coxeter graph has n dots connected by single lines if $m_{ij} > 2$, with m_{ij} written on the lines if $m_{ij} > 3$.

Theorem 8.9.1 (Geometric Realization of Coxeter Groups) *Define a basis $\{\alpha_1, \cdots, \alpha_n\}$ of an n-dimensional vector space M. Define a metric by*

$$\begin{aligned} (\alpha_i, \alpha_i) &= (\alpha_j, \alpha_j) \qquad \forall \; i, j \\ (\alpha_i, \alpha_j) &= -(\alpha_i, \alpha_i) \cos(\frac{\pi}{m_{ij}}) \; , \end{aligned}$$

then the reflection

$$s_i : x \longmapsto x - 2\alpha_i \frac{(\alpha_i, x)}{(\alpha_i, \alpha_i)}$$

satisfies the definition of Coxeter group (8.9.1).

Notice that there are null directions of the metric $\alpha_i + \alpha_j$ if $m_{ij} = \infty$ and the metric is not decomposable iff the Coxeter graph is connected.

Definition (Length Function) Define the length function

$$\ell : W \to \mathbb{Z}_+ \tag{8.9.3}$$

such that an element has length ℓ if there is no way to write the element in terms of a product of less than ℓ generators. For $w \in G$, from

$$\begin{aligned} \ell(ws_i) &\leq \ell(w) + 1 \\ \ell(w) = \ell(ws_i^2) &\leq \ell(ws_i) + 1 \end{aligned}$$

we see that

$$\ell(w) - 1 \leq \ell(ws_i) \leq \ell(w) + 1 \ .$$

Furthermore, the length function defines a distance function on the group $d : W \times W \to \mathbb{Z}_+$ as

$$d(w, w') = \ell(w^{-1}w') = d(w', w) \ .$$

One can easily check that this is a metric, especially that the triangle inequality is satisfied.

Definition (Roots of the Coxeter Group) Define the set of roots

$$\Delta^{re} = \{w(\alpha_i)\, , w \in W,\ i = \{1, \cdots, n\}\}$$

especially α_i's are called the *simple roots*[3]. A root

$$\alpha = \sum_i a^{(i)} \alpha_i$$

is called a *positive root* if all $a^{(i)} > 0$ and a *negative root* if all $a^{(i)} < 0$. We will denote them as $\alpha > 0$ and $\alpha < 0$ respectively.

[3]The notation Δ^{re} is adapted to the fact that the real roots of the set of Borcherds-Kac-Moody algebra is the set of roots of the Weyl Coxeter group.

Theorem 8.9.2 *A root of a Coxeter group is either positive or negative, namely*

$$\Delta^{re} = \Delta^{re}_{+} \cup \Delta^{re}_{-}$$

where

$$\Delta^{re}_{+} = \{\alpha \in \Delta^{re} | \, \alpha > 0\} \quad , \quad \Delta^{re}_{-} = \{\alpha \in \Delta^{re} | \, \alpha < 0\} \, .$$

Furthermore

$$\Delta^{re}_{-} = -\Delta^{re}_{+} \, .$$

Theorem 8.9.3

$$\ell(ws_i) = \ell(w) + 1$$

iff $w(\alpha_i) > 0$.

Corollary 8.9.4 *A root is either positive or negative.*

Theorem 8.9.5 *For hyperbolic Coxeter group, the Tits cone, namely the image of a connected fundamental domain under the group action, is the future light-cone. Projected onto a constant length surface, this gives a tessellation of the hyperbolic space.*

Definition (Bruhat Order)
Given the Coxeter system (W, S) and its set of reflections $\mathcal{R} = \{ws_i w^{-1} | \, w \in W \; , \; s_i \in S\}$, and let $u, u' \in W$, then

1. $u \to u'$ means that $u^{-1}u' \in \mathcal{R}$ and $\ell(u) < \ell(u')$.
2. $u \leq_B u'$ means that there exists $u_k \in W$ such that

$$u \to u_1 \to \cdots \to u_m \to u' \tag{8.9.4}$$

Definition (weak Bruhat order) Given two elements $u, u' \in W$, repeating the above definition but now restrict further to $u^{-1}u' \in S$, we obtain the "weak Bruhat order" $u \leq u'$.

Two elements are said to be *comparable* if $u \leq u'$ or $u \geq u'$ and *incomparable* otherwise. See [161] for more details.