New Construction of Solvable Lattice Models Including an Ising Model in a Field

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In this Letter we report a new construction to obtain restricted solid-on-solid (RSOS) models out of loop models. The method is a generalization of ideas developed by Owczarek and Baxter, and by Pasquier. In particular we consider a solvable O(n) model and point out that some of the RSOS models thus obtained admit an off-critical extension. Among these models we find a spin-1 Ising model, which is solvable not only at the critical point, but also in a fieldlike deviation away from it. We calculate the critical exponent $\delta = 15$ directly from the relation between the free energy and the field. This is the first determination of this exponent without the use of scaling relations.

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The understanding of critical phenomena has greatly increased as a result of the exact solution of various model systems. In recent years many ways to construct solvable models have been given. In general these methods are based on algebraic techniques. Here we present a diagrammatic approach to obtain solvable models from known loop models. The resulting critical models include both new and known universality classes.

General loop model.—We consider the partition function defined as the sum over all graphs G consisting of closed nonintersecting polygons (or loops) on a square lattice. An edge can either be occupied by a loop segment or be empty. Each polygon has fugacity n and the nine possible vertices, shown in Fig. 1(a), have Boltzmann weights $\rho_1, \ldots, \rho_9$.

$$Z = \sum_G \rho_1^{n_1} \cdots \rho_9^{n_9} n^p, \tag{1}$$

where p is the total number of loops of G and n_i the number of vertices of type i. This model includes the polygon partition sum considered by [1], where only vertices of type 8 and 9 occur. However, it is in general distinct from the nonintersecting string models [2], where the strings or polygons each carry a variable, via which they interact.

General RSOS model.—We now define a restricted solid-on-solid (RSOS) model [3] on the square lattice and show how it is related to the loop model defined above.

For this purpose we define an arbitrary connected graph $\mathcal{G}$ consisting of L nodes, each node being labeled by an integer $a \in \{1, \ldots, L\}$, the "height" of the node. To such a graph we assign an adjacency or incidence matrix A, which has the following elements: $A_{a,b} = 1$ if the nodes a and b of $\mathcal{G}$ are adjacent ($a \sim b$), i.e., linked by a bond, and $A_{a,b} = 0$ otherwise. Let Λ be the largest eigenvalue of A and S the Perron-Frobenius vector. Then we define the Boltzmann weight of an elementary face of the RSOS model as follows:

$$W\begin{pmatrix} d\,c \\ a\,b \end{pmatrix} = \rho_1 \delta_{a,b,c,d} + \rho_2 \delta_{a,b,c} A_{a,d} + \rho_3 \delta_{a,c,d} A_{a,b} + \left(\frac{S_a}{S_b}\right)^{1/2} \rho_4 \delta_{b,c,d} A_{a,b} + \left(\frac{S_c}{S_a}\right)^{1/2} \rho_5 \delta_{a,b,d} A_{a,c} + \rho_6 \delta_{a,b} \delta_{c,d} A_{a,c} + \rho_7 \delta_{a,d} \delta_{b,c} A_{a,b} + \rho_8 \delta_{a,c} A_{a,b} A_{a,d} + \left(\frac{S_a S_c}{S_b S_d}\right)^{1/2} \rho_9 \delta_{b,d} A_{b,c} A_{a,b}, \tag{2}$$

where S_i is the ith entry of S and a, b, c, and d can take any of the L heights of the graph $\mathcal{G}$. The generalized Kronecker δ is defined as $\delta_{i_1,\ldots,i_m} \equiv \prod_{j=2}^m \delta_{i_1,i_j}$. This equation is a direct generalization of the weights introduced in [4, 5].

This model can be mapped onto the loop model if we identify $\Lambda = n$. We expand the partition function of the RSOS model, given by

$$Z = \sum_{\text{heights}} \prod_{\text{faces}} W\begin{pmatrix} d\,c \\ a\,b \end{pmatrix}, \tag{3}$$

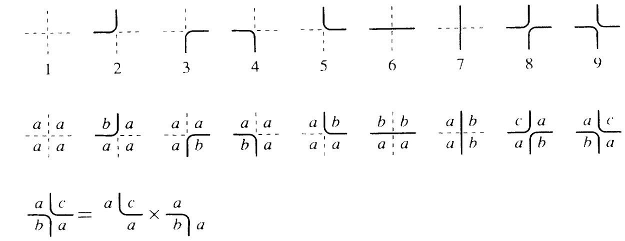

FIG. 1. (a) The nine vertices of the loop model. (b) The nine possible diagrams in the expansion of (3). (c) The factorization of diagram 9 of (b).

into a sum of 9^N terms, where N is the number of faces of the lattice. A given term in the expansion has one of the nine terms of (2) for each elementary face of the lattice. These nine possible terms can be represented by the diagrams shown in Fig. 1(b), in which the lines indicate domain walls separating regions of different height, adjacent on $\mathcal{G}$. The partition sum now consists of a summation on the configurations G of domain walls and on the height variables,

$$Z = \sum_G \rho_1^{n_1} \cdots \rho_9^{n_9} \sum_{\text{heights}} \prod_{a,b=1}^{L} \left(\frac{S_b}{S_a}\right)^{m_{ba}}, \tag{4}$$

where m_{ba} is the total power of S_a/S_b from vertices of types 4, 5, and 9. The summation on the heights is subject to the restriction that sites in the same domain take the same height, and that heights in neighboring domains are adjacent.

For each configuration G of domain walls, the dependence of the Boltzmann weight on the heights can be factorized into separate contributions associated with each domain wall. For that purpose vertex weights of type 9 of Fig. 1(b) are split into two factors as indicated in Fig. 1(c). We can now start to sum on the heights inside loops that do not surround other loops. Let a be the inner and b the outer height of such a loop. Then, since the contribution to $m_{ab} - m_{ba}$ for such a loop is always 1,

$$\sum_{a\sim b} \frac{S_a}{S_b} = \sum_{a=1}^{L} A_{b,a} \frac{S_a}{S_b} = \Lambda \equiv n. \tag{5}$$

Repeating this procedure from the inside out we indeed find, up to a proportionality constant depending on the specific choice of boundary conditions, that the summation over all heights yields a factor n^P, which proves the equivalence of (3) with (1).

Special solvable cases.—In general the loop model (1) is not solvable, but in the following we consider two cases for which this model satisfies the Yang-Baxter equation [6]. The generalization of Yang-Baxter equations to loop variables is treated in [7]. The solvable loop models we discuss are both critical for $n \leq 2$. The graphs $\mathcal{G}$ which have adjacency matrix A with largest eigenvalue $\Lambda \leq 2$ have been classified by the Dynkin diagrams of the classical and affine simply-laced Lie algebras, the A-D-E Lie algebras.

The Temperley-Lieb (TL) construction of [4, 5] builds on the loop model defined by

$$\rho_1 = \cdots = \rho_7 = 0, \quad \rho_8 = \frac{\sin(\lambda - u)}{\sin\lambda}, \qquad \rho_9 = \frac{\sin u}{\sin\lambda}, \quad n = 2\cos\lambda, \tag{6}$$

which is related to the six-vertex and self-dual Potts model [1]. The critical A and D RSOS models obtained from it can all be extended away from criticality while retaining their solvability. As these models have been discussed extensively we refer to the original literature for further details [3, 8, 9].

As the basis of the construction we now consider another solvable loop model [10], related to the Izergin-Korepin model [11]. It is defined by the following weights and fugacity:

$$\begin{aligned} &\rho_1 = \frac{\sin 2\lambda \cos 3\lambda + \sin u \cos(u+3\lambda)}{\sin 2\lambda \cos 3\lambda}, \quad \rho_2 = \rho_3 = \frac{\cos(u+3\lambda)}{\cos 3\lambda}, \quad \rho_4 = \rho_5 = \frac{\sin u}{\cos 3\lambda}, \\ &\rho_6 = \rho_7 = \frac{\sin u \cos(u+3\lambda)}{\sin 2\lambda \cos 3\lambda}, \quad \rho_8 = \frac{\sin(u+2\lambda)\cos(u+3\lambda)}{\sin 2\lambda \cos 3\lambda}, \quad \rho_9 = \frac{\sin u \cos(u+\lambda)}{\sin 2\lambda \cos 3\lambda}, \quad n = -2\cos 4\lambda. \end{aligned} \tag{7}$$

From a numerical investigation similar to that of Rietman [12], it seems that of the corresponding RSOS models, only those based on the Dynkin diagrams of the A_L series can be extended away from criticality, in contrast to the TL case. The orbifold construction of Fendley and Ginsparg [13], which relates the D_{L+2} and A_{2L+1} TL models, may readily be generalized to these new critical RSOS models. However, the requirement that the weights possess the same Z_2 symmetry as the Dynkin diagrams is not satisfied for the A_{2L+1} models away from criticality. Therefore the solvability of off-critical D_L models is not implied.

Off-critical A_L model.—The Dynkin diagram of A_L has adjacency matrix A with elements $A_{a,b} = \delta_{a,b-1} + \delta_{a,b+1}$, $a, b \in \{1, \ldots, L\}$. The weights of this model, with the usual definition of ϑ functions [14], are found to be

$$\begin{aligned} W\begin{pmatrix} a & a \\ a & a \end{pmatrix} &= \frac{\vartheta_1(u+6\lambda)\vartheta_2(u-3\lambda)}{\vartheta_1(6\lambda)\vartheta_2(3\lambda)} - \left(\frac{S(a+1)}{S(a)}\frac{\vartheta_4(2a\lambda-5\lambda)}{\vartheta_4(2a\lambda+\lambda)} + \frac{S(a-1)}{S(a)}\frac{\vartheta_4(2a\lambda+5\lambda)}{\vartheta_4(2a\lambda-\lambda)}\right)\frac{\vartheta_1(u)\vartheta_2(u+3\lambda)}{\vartheta_1(6\lambda)\vartheta_2(3\lambda)}, \\ W\begin{pmatrix} a\pm1 & a \\ a & a \end{pmatrix} &= W\begin{pmatrix} a & a \\ a & a\pm1 \end{pmatrix} = \frac{\vartheta_2(u+3\lambda)\vartheta_4(u\pm2a\lambda+\lambda)}{\vartheta_2(3\lambda)\vartheta_4(\pm2a\lambda+\lambda)}, \\ W\begin{pmatrix} a & a \\ a\pm1 & a \end{pmatrix} &= W\begin{pmatrix} a & a\pm1 \\ a & a \end{pmatrix} = \left(\frac{S(a\pm1)}{S(a)}\right)^{1/2}\frac{\vartheta_1(u)\vartheta_3(u\mp2a\lambda+2\lambda)}{\vartheta_2(3\lambda)\vartheta_4(\pm2a\lambda+\lambda)}, \\ W\begin{pmatrix} a & a\pm1 \\ a & a\pm1 \end{pmatrix} &= W\begin{pmatrix} a\pm1 & a\pm1 \\ a & a \end{pmatrix} = \left(\frac{\vartheta_4(\pm2a\lambda+3\lambda)\vartheta_4(\pm2a\lambda-\lambda)}{\vartheta_4^2(\pm2a\lambda+\lambda)}\right)^{1/2}\frac{\vartheta_1(u)\vartheta_2(u+3\lambda)}{\vartheta_1(2\lambda)\vartheta_2(3\lambda)}, \end{aligned} \tag{8}$$

$$W\begin{pmatrix} a\pm 1 & a \\ a & a\mp 1\end{pmatrix} = \frac{\vartheta_1(u+2\lambda)\vartheta_2(u+3\lambda)}{\vartheta_1(2\lambda)\vartheta_2(3\lambda)},$$
$$W\begin{pmatrix} a & a\mp 1 \\ a\pm 1 & a\end{pmatrix} = \left(\frac{S(a-1)S(a+1)}{S^2(a)}\right)^{1/2} \frac{\vartheta_1(u)\vartheta_2(u+\lambda)}{\vartheta_1(2\lambda)\vartheta_2(3\lambda)},$$
$$W\begin{pmatrix} a & a\pm 1 \\ a\pm 1 & a\end{pmatrix} = \frac{S(a\pm 1)}{S(a)}\frac{\vartheta_1(u)\vartheta_2(u\mp 4a\lambda+\lambda)}{\vartheta_2(3\lambda)\vartheta_1(\pm 4a\lambda+2\lambda)} - \frac{\vartheta_2(u+3\lambda)\vartheta_1(u\mp 4a\lambda-2\lambda)}{\vartheta_2(3\lambda)\vartheta_1(\pm 4a\lambda+2\lambda)},$$
$$S(a) = (-)^a\frac{\vartheta_1(4a\lambda)}{\vartheta_3(2a\lambda)}.$$

When the nome of the elliptic functions is taken to zero the critical weights (2) are recovered.

There are four different regimes in terms of the spectral and crossing parameters u and λ. The central charge is known from the equivalence with the O(n) model [15, 16],

$$\lambda = \frac{\pi}{4}\left(1\pm\frac{1}{L+1}\right), \quad 0<u<\frac{3\pi}{2}-3\lambda, \quad c = 1-\frac{3(\pi-4\lambda)^2}{2\pi\lambda},$$
$$\lambda = -\frac{\pi}{4}\left(1\pm\frac{1}{L+1}\right), \quad 0<u<-\frac{\pi}{2}-3\lambda, \quad c = \frac{3}{2}-\frac{3(\pi+4\lambda)^2}{\pi(\pi+2\lambda)}. \tag{9}$$

Other regimes can be mapped onto these four by the periodicity of the weights.

As remarked before, for L odd the weights do not satisfy the Z_2 symmetry of the Dynkin diagram

$$W\begin{pmatrix} d & c \\ a & b\end{pmatrix} \neq W\begin{pmatrix} L-d+1 & L-c+1 \\ L-a+1 & L-b+1\end{pmatrix}, \quad L \text{ odd}. \tag{10}$$

The simplest model that exhibits this broken symmetry is realized when $L = 3$. This is a three-state model, with states $\{+, 0, -\}$, of which a + and − spin cannot be neighbors. For $\lambda = \frac{3}{16}\pi$ and $0 < u < \frac{15}{16}\pi$, this model has central charge $c = \frac{1}{2}$ and belongs to the universality class of the ordinary Ising model. In fact, the nome of the elliptic functions in (8) serves as the magnetic field. When we calculate the free energy from the inversion relation [17] and extract the leading singularity, we find the magnetic exponent $\delta = 15$. Although this value is universally accepted, this is the first time that it has been calculated directly, rather than from scaling relations with other exponents.

Similar RSOS models related to the Izergin-Korepin model have been found by Kuniba [18], via a completely different, algebraic approach. His weights, however, do not break the symmetry of the adjacency diagrams. After this paper was submitted for publication, we learned that results equivalent to our Eqs. (2) and (7) have been obtained by Roche [19].

In this Letter we have described a new method to construct RSOS models out of loop models and have applied these ideas to the O(n) model. The critical models thus obtained can be classified by the *A-D-E* Lie algebras. Critical properties of these models can be extracted from known results for the O(n) model. Only the A_L model admits an off-critical elliptic extension, which for L odd turns out to be symmetry breaking. A more detailed investigation of this model is in progress.

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