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NOTE

A Comment on "A Note on 'Distance Transformations in Digital Images'"

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Recently, in a communication by Vossepoel [2] on optimal coefficients a, b (and c) for distance transformations [1], distance weights are assigned according to the following schemes:

b	a	b
a	0	a
b	a	b

$2b$	c	$2a$	c	$2b$
c	b	a	b	c
$2a$	a	0	a	$2a$
c	b	a	b	c
$2b$	c	$2a$	c	$2b$

The optimal parameter values for distance transformations are equal to the optimal parameter values for linear length estimates of type $l = a \cdot \Delta x + (b - a) \cdot \Delta y$. Therefore, to compute optimal values for (a, b) it is sufficient to evaluate the integrals for line length estimation [3, 4]:

$$\mathrm{Bias}(l) = \int\int_D (l - l_e(r, \varphi)) \cdot p(r, \varphi)\, dr\, d\varphi \tag{1}$$

and

$$\mathrm{MSE}(l) = \int\int_D (l - l_e(r, \varphi))^2 \cdot p(r, \varphi)\, dr\, d\varphi \tag{2}$$

where l_e is the Euclidean distance, $p(r, \varphi)$ is the probability of straight lines parameterized as indicated in Fig. 1, and D is the ensemble of all continuous straight lines in 2-dimensional space.

For the estimate $l = n \cdot (a + (b - a) \cdot \tan(\varphi))$, $0 < \varphi < \pi/4$, in [2] the author effectively evaluates the following set of integrals:

$$\mathrm{Bias}/n = \int_0^{\pi/4} (a + (b - a) \cdot \tan(\varphi) - \sec(\varphi))\, d\varphi \tag{3}$$

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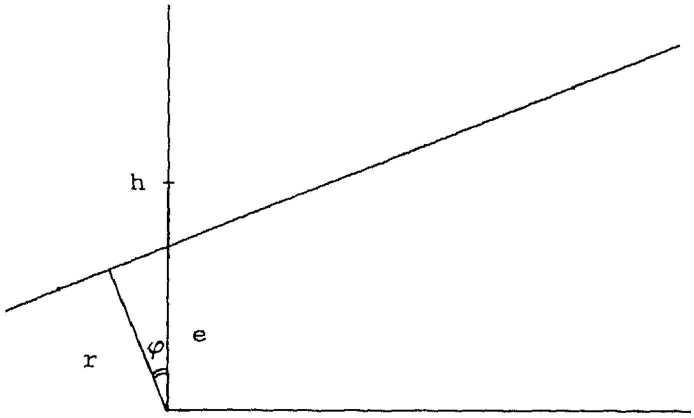

FIG. 1. The normal representation of a straight line.

and

$$\mathrm{MSE}/n^2 = \int_0^{\pi/4} (a + (b - a) \cdot \tan(\varphi) - \sec(\varphi))^2 \, d\varphi, \tag{4}$$

integrating over two columns of the grid separated by n pixels (see Eqs. (2)–(8) in [2]). To establish isotropy in the ensemble D of straight lines, a uniform distribution is assumed for φ. Minimizing (4) under the constraint that (3) equals zero, the author then finds the following optimal values for a and b:

$$a = \frac{(4 - \pi) \cdot \ln(1 + \sqrt{2}) - (\sqrt{2} - 1) \cdot \ln 4}{\pi \cdot (1 - \pi/4) - \ln^2 2} = 0.9413 \tag{5}$$

and

$$b = \frac{(4 - \pi - \ln 4) \cdot \ln(1 + \sqrt{2}) + (\sqrt{2} - 1) \cdot (\pi - \ln 4)}{\pi \cdot (1 - \pi/4) - \ln^2 2} = 1.3513.$$

Similarly for the (5×5) distance transformation, are found:

$$a = 0.9813, \qquad b = 1.4031, \qquad \text{and} \qquad c = 2.1953. \tag{6}$$

Both sets of values (5) and (6), however, are *not* optimal values for *isotropic* distance transformations. To appreciate this consider in Fig. 1 the entrance height e in the first column. Implicitly, when integrating over n columns in [2] it is assumed that $p(e, \varphi) = 4/\pi$ is uniform. The isotropic distribution of random lines, however, is equivalent with a uniform distribution of $p(r, \varphi)$. Hence $p(e, \varphi)$ is not uniform, but $p(e, \varphi) = p(r, \varphi) \cdot |J|$ is, where J is the Jacobian of the coordinate transition from (r, φ) to (e, φ). From Fig. 1 it is seen that $r = e \cdot \cos(\varphi)$, hence $|J| = |d(r, \varphi)/d(e, \varphi)| = \cos(\varphi)$ and $p(e, \varphi) = \sqrt{2} \cdot \cos(\varphi)$ over the first octant.

In [3, 4] the integrals (1) and (2) are calculated for a variety of length estimators (and distance transforms), using the proper $p(r, \varphi)$ to establish isotropy. For the two parameter length estimator the following optimal coefficients are found:

$$
\begin{aligned}
a &= \frac{\pi/2 \cdot \{\sqrt{2} \cdot \ln(\sqrt{2}+1) - 1\} - (\sqrt{2}-1) \cdot \ln 2}{2 \cdot \ln(\sqrt{2}+1) - 4 \cdot (\sqrt{2}-1)} = 0.9445 \\
b &= \frac{\pi/\sqrt{2} \cdot \{\ln(\sqrt{2}+1) - 1\} + (2-\sqrt{2}) \cdot \ln 2}{2 \cdot \ln(\sqrt{2}+1) - 4 \cdot (\sqrt{2}-1)} = 1.3459
\end{aligned}
\tag{7}
$$

and for the three parameter estimates:

$$a = 0.980, \qquad b = 1.406, \qquad c = 2.204. \tag{8}$$

The differences between (7) and (5) and between (8) and (6) are small. Since it is claimed in [2] that (5) and (6) are optimal values they have a theoretical meaning. For practical purposes the conclusions in [2] still hold.

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