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ON THE EQUIVALENCE OF CONVERGENT KINETIC EQUATIONS FOR HOT DILUTE PLASMAS

III. COLLISION TERMS WITH EFFECTIVE INTERACTIONS

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A general class of approximate expressions for the pair correlation function of a classical one-component plasma with small plasma parameter ϵ is discussed. The collision terms which follow from these approximate correlation functions are characterized by effective potentials that differ from the bare Coulomb potential by modulation factors depending on the interparticle distance. It is shown that for small ϵ the transport coefficients are independent of the precise form of the modulation factors.

1. Introduction

In this series of papers convergent kinetic equations for classical one-component plasmas with a small plasma parameter are studied. In particular, in papers I and II[1,2]) we discussed the consequences of a collision term put forward recently by Kleinsmith[3]) and Mondt[4]). To obtain their result these authors introduced as an essential ingredient a particular approximation for the non-equilibrium pair correlation function. The linearized version of their collision term contains Boltzmann-, Balescu–Guernsey–Lenard- and Landau-like contributions, which are each separately convergent. This convergence is guaranteed by the occurrence of effective interactions that differ from the bare Coulomb interaction by additional 'modulation' factors depending on the interparticle distance; these factors are determined by the equilibrium pair correlation function. In papers I and II we have shown that the dominant terms of the ensuing transport coefficients for small plasma parameter are the same as those following from an earlier type of convergent kinetic equation[5–7]). It remained unclear, however, whether this equivalence is a special feature of the particular approximation for the pair correlation function used in refs. 1 and 2.

To settle this question we shall investigate in the present paper a larger class of convergent kinetic equations. This class arises by weakening the assumptions that lead to the approximate expression for the pair correlation function. As a consequence the linearized collision term is again the sum of Boltzmann-, BGL- and Landau-like expressions with effective interactions. However, the modulation factors are now left unspecified; only certain scaling properties are required. In fact, in the Boltzmann case the modulation factor varies appreciably only over distances of the order of the Debye length r_D; in the BGL case the modulation is present only for distances of the order of the Landau length r_L.

The paper is organized as follows. In section 2 the general properties of the modulation factors are established. To that end a general approximation scheme for the pair correlation function of a non-equilibrium plasma is developed. Subsequently, in section 3, the asymptotic form of the generating function for the collision brackets is calculated with the use of the techniques described in the preceding papers of this series.

2. Approximate expressions for the pair correlation function

For a spatially homogeneous plasma the single-particle distribution function f_1 satisfies the equation:

$$\frac{\partial f_1(\boldsymbol{r}_1, t)}{\partial t} = \frac{1}{m}\frac{\partial}{\partial \boldsymbol{v}} \cdot \int \mathrm{d}\boldsymbol{r}_2\, \mathrm{d}\boldsymbol{v}_2 [\nabla_{r_1}\varphi(\boldsymbol{r}_1 - \boldsymbol{r}_2)] g(\boldsymbol{r}_1 - \boldsymbol{r}_2, \boldsymbol{v}_1, \boldsymbol{v}_2, t), \tag{2.1}$$

where $g(\boldsymbol{r}_1 - \boldsymbol{r}_2, \boldsymbol{v}_1, \boldsymbol{v}_2, t) = f_2(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{v}_1, \boldsymbol{v}_2, t) - f_1(\boldsymbol{r}_1, \boldsymbol{v}_1, t) f_1(\boldsymbol{r}_2, \boldsymbol{v}_2, t)$ is the pair correlation function and $\varphi(r) = e^2/4\pi r$ the Coulomb potential. For a hot dilute plasma, with plasma parameter $\epsilon \ll 1$, the three-particle correlations are negligible. Then the pair correlation function is the solution of

$$\begin{aligned}\frac{\partial}{\partial t} g(\boldsymbol{r}_1 - \boldsymbol{r}_2, \boldsymbol{v}_1, \boldsymbol{v}_2, t) &= -\boldsymbol{v}_1 \cdot \nabla_{r_1} g(\boldsymbol{r}_1 - \boldsymbol{r}_2, \boldsymbol{v}_1, \boldsymbol{v}_2, t) \\ &+ \frac{1}{m}\left[\nabla_{r_1}\varphi(\boldsymbol{r}_1 - \boldsymbol{r}_2)\right] \cdot \frac{\partial}{\partial \boldsymbol{v}_1}[f_1(\boldsymbol{r}_1, \boldsymbol{v}_1, t) f_1(\boldsymbol{r}_2, \boldsymbol{v}_2, t) + g(\boldsymbol{r}_1 - \boldsymbol{r}_2, \boldsymbol{v}_1, \boldsymbol{v}_2, t)] \\ &+ \frac{1}{m}\frac{\partial}{\partial \boldsymbol{v}_1} \cdot \int \mathrm{d}\boldsymbol{r}_3\, \mathrm{d}\boldsymbol{v}_3 [\nabla_{r_1}\varphi(\boldsymbol{r}_1 - \boldsymbol{r}_3)] f_1(\boldsymbol{r}_1, \boldsymbol{v}_1, t) g(\boldsymbol{r}_2 - \boldsymbol{r}_3, \boldsymbol{v}_2, \boldsymbol{v}_3, t) + (1 \leftrightarrow 2),\end{aligned} \tag{2.2}$$

where $(1 \leftrightarrow 2)$ stands for the preceding terms with the indices 1 and 2 interchanged. For a hot dilute plasma the pair correlation function relaxes much faster than the single-particle distribution function. Hence, the pair correlation function depends on the time only through the time dependence of the single-particle distribution function. The eq. (2.2) then determines the pair correlation function as a functional of the single-particle distribution function.

Substitution of this functional in the right-hand side of (2.1) yields the kinetic equation.

Eq. (2.2) is too complicated to be solved for g. However, as is well known[8]), approximate solutions can be found by considering the scaling properties of the various terms in the right-hand side. To that end we introduce the dimensionless quantities r/r_0, v/v_0, and t/t_0, where r_0 is either the Landau length r_L or the Debye length $r_D = \epsilon^{-1} r_L$; furthermore $v_0 = (\beta m)^{-1/2}$ and $t_0 = r_0/v_0$. In the case $r_0 = r_L$ the integral in (2.2) turns out to be negligible, since it is of order ϵ^2 with respect to the remaining terms. The solution of the ensuing equation is the Boltzmann expression g_B for the pair correlation function. It depends (apart from the velocities) on r/r_L; for $r \gg r_L$ it shows no structure. Such functions will be called of *short-range* type in the sequel. On the other hand, if $r_0 = r_D$ the second term within the brackets in (2.2) is of relative order ϵ and may be omitted. Then the Balescu–Guernsey–Lenard expression g_{BGL} is obtained for g. It varies appreciably only over distances of the order of r_D and will hence be called a function of *long-range* type. Under suitable circumstances both the Boltzmann- and BGL-expressions for g reduce to the Landau form g_L.

By using scaling arguments an approximate solution of (2.2), which holds for all values of the interparticle distance r, has been obtained by Kleinsmith[3]) and Mondt[4]), independently. We give a succinct version of their argument. Since $g = g_B$ for $r \lesssim r_L$, one may write

$$g = g_B g_1, \quad 0 \leq r < \infty, \tag{2.3}$$

where $g_1 = 1$ for $r \lesssim r_L$. Similarly, since $g = g_{BGL}$ for $r \gtrsim r_D$, one may write

$$g = g_{BGL} g_2, \quad 0 \leq r < \infty, \tag{2.4}$$

where $g_2 = 1$ for $r \gtrsim r_D$. One can combine (2.3) and (2.4) as follows:

$$g = g_B g_{BGL} g_3, \quad 0 \leq r < \infty. \tag{2.5}$$

Then $g_{BGL} g_3 = 1$ for $r \lesssim r_L$ and $g_B g_3 = 1$ for $r \gtrsim r_D$. Since $g_{BGL} = g_L$ for $r \lesssim r_L$ and $g_B = g_L$ for $r \gtrsim r_D$, we have

$$g_3 = g_L^{-1}, \quad r \lesssim r_L, \quad r \gtrsim r_D. \tag{2.6}$$

As the crucial step in the derivation one now assumes that (2.6) holds for $r_L \lesssim r \lesssim r_D$ as well. Then one concludes

$$g = \frac{g_B g_{BGL}}{g_L}, \quad 0 \leq r < \infty. \tag{2.7}$$

The solution (2.7) is not unique. The particular choice (2.3), (2.4) can be replaced by

$$g = g_B + g_1, \tag{2.8}$$

$$g = g_{BGL} + g_2, \tag{2.9}$$

or equivalently,

$$g = g_B + g_{BGL} + g_3. \tag{2.10}$$

Now g_3 must satisfy

$$g_3 = -g_L, \quad r \leqslant r_L, \quad r \geqslant r_D. \tag{2.11}$$

The assumption $g_3 = -g_L$ for $r_L \leqslant r \leqslant r_D$ yields

$$g = g_B + g_{BGL} - g_L. \tag{2.12}$$

The kinetic equation which one obtains from (2.12) has been put forward by Hubbard[9]).

The argument leading to (2.7) and (2.12) can be generalized in the following way. We put for all $\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2$

$$\begin{aligned} g(\boldsymbol{r}, \boldsymbol{v}_1, \boldsymbol{v}_2, t) &= G_1[g_B(\boldsymbol{r}, \boldsymbol{v}_1, \boldsymbol{v}_2, t) \mid \boldsymbol{r}, \boldsymbol{v}_1, \boldsymbol{v}_2], \\ g(\boldsymbol{r}, \boldsymbol{v}_1, \boldsymbol{v}_2, t) &= G_2[g_{BGL}(\boldsymbol{r}, \boldsymbol{v}_1, \boldsymbol{v}_2, t) \mid \boldsymbol{r}, \boldsymbol{v}_1, \boldsymbol{v}_2], \end{aligned} \tag{2.13}$$

or equivalently

$$g = G_3(g_B, g_{BGL} \mid \boldsymbol{r}, \boldsymbol{v}_1, \boldsymbol{v}_2). \tag{2.14}$$

Then we have

$$\begin{aligned} &G_3(g_B, g_L \mid \boldsymbol{r}, \boldsymbol{v}_1, \boldsymbol{v}_2) = g_B, \quad r \leqslant r_L, \\ &G_3(g_L, g_{BGL} \mid \boldsymbol{r}, \boldsymbol{v}_1, \boldsymbol{v}_2) = g_{BGL}, \quad r \geqslant r_D. \end{aligned} \tag{2.15}$$

Hence the $(\boldsymbol{r}, \boldsymbol{v}_1, \boldsymbol{v}_2)$-dependence, which is written explicitly in G_3 is implicit via g_B and g_L for $r \leqslant r_L$, and implicit via g_{BGL} and g_L for $r \geqslant r_D$. We now assume that the $(\boldsymbol{r}, \boldsymbol{v}_1, \boldsymbol{v}_2)$-dependence is governed by g_B, g_{BGL} and g_L for all r. Then g gets the form

$$g = G(g_B, g_{BGL}, g_L), \tag{2.16}$$

for some as yet unknown function G.

Since both the arguments of G, and G itself, are pair correlation functions the function G must be homogeneously linear. We use this property to write

$$G(g_B, g_{BGL}, g_L) = g_L G\left(\frac{g_B}{g_L}, \frac{g_{BGL}}{g_L}, 1\right). \tag{2.17}$$

With the variables

$$x = g_B/g_L, \quad y = g_{BGL}/g_L \tag{2.18}$$

and the abbreviation

$$\bar{G}(x, y) = G(x, y, 1), \tag{2.19}$$

we then have from (2.17)

$$g = g_{\mathrm{L}}\bar{G}(x, y). \tag{2.20}$$

From (2.15) we deduce the following properties of the function G:

$$\begin{aligned} G(g_{\mathrm{B}}, g_{\mathrm{L}}, g_{\mathrm{L}}) &= g_{\mathrm{B}}, \\ G(g_{\mathrm{L}}, g_{\mathrm{BGL}}, g_{\mathrm{L}}) &= g_{\mathrm{BGL}}. \end{aligned} \tag{2.21}$$

Since the arguments of G can be regarded as independent variables these relations are valid for all values of g_i. The corresponding properties of the function $\bar{G}$ read

$$\bar{G}(x, 1) = x, \quad \bar{G}(1, y) = y. \tag{2.22}$$

In the special cases (2.7) and (2.12) we have

$$\bar{G}(x, y) = xy \tag{2.23a}$$

and

$$\bar{G}(x, y) = x + y - 1, \tag{2.23b}$$

respectively. In both cases the relations (2.22) are satisfied.

It is clear, that the relations (2.22) do not determine $\bar{G}$ uniquely. One might expect that different choices of $\bar{G}$, which correspond to different approximate solutions of (2.2), would lead to alternative expressions for the transport coefficients. However, the dominant terms for small ϵ in the transport coefficients will turn out to depend only on some general properties rather than on the specific form of $\bar{G}$.

To obtain the transport coefficients we have to linearize (2.1) about local thermodynamic equilibrium. To that end we need a linearized expression for the pair correlation function, which can be obtained from (2.20), if we assume $\bar{G}$ to possess sufficient differentiability properties. With the notation $\Delta g_i = g_i - g_i^{(0)}$ and $\bar{G}^{(0)} = \bar{G}(x^{(0)}, y^{(0)})$, we have:

$$\Delta g = \left[\frac{\delta \bar{G}}{\delta x}\right]^{(0)} \Delta g_{\mathrm{B}} + \left[\frac{\delta \bar{G}}{\delta y}\right]^{(0)} \Delta g_{\mathrm{BGL}} + \left[\bar{G} - x\frac{\delta \bar{G}}{\delta x} - y\frac{\delta \bar{G}}{\delta y}\right]^{(0)} \Delta g_{\mathrm{L}}. \tag{2.24}$$

We illustrate this relation for the case (2.23a):

$$\Delta g = y^{(0)}\Delta g_{\mathrm{B}} + x^{(0)}\Delta g_{\mathrm{BGL}} - x^{(0)}y^{(0)}\Delta g_{\mathrm{L}}. \tag{2.25}$$

From (I.3.8)–(I.3.10) one has

$$x^{(0)} = \frac{g_{\mathrm{B}}^{(0)}}{g_{\mathrm{L}}^{(0)}} = \frac{1 - \exp(-r_{\mathrm{L}}/r)}{r_{\mathrm{L}}/r},$$

$$y^{(0)} = \frac{g_{\mathrm{BGL}}^{(0)}}{g_{\mathrm{L}}^{(0)}} = \exp(-r/r_{\mathrm{D}}). \tag{2.26}$$

Clearly, the coefficient $y^{(0)}$ of Δg_{B} in (2.25) is of long-range type, while the coefficient $x^{(0)}$ of Δg_{BGL} is of short-range character.

As a generalization we shall now consider functions $\bar{G}$ such that the coefficients $[\delta\bar{G}/\delta x]^{(0)}$ and $[\delta\bar{G}/\delta y]^{(0)}$ in (2.24) have the same properties with respect to their range as in (2.25), (2.26), without being specified in any more detail. In other words, we shall write

$$\begin{aligned}
\left[\frac{\delta\bar{G}}{\delta x}\right]^{(0)} &= R^{(\mathrm{l})}\left(\frac{r}{r_{\mathrm{D}}}\right),\\
\left[\frac{\delta\bar{G}}{\delta y}\right]^{(0)} &= R^{(\mathrm{s})}\left(\frac{r}{r_{\mathrm{L}}}\right),\\
\left[\bar{G} - x\frac{\delta\bar{G}}{\delta x} - y\frac{\delta\bar{G}}{\delta y}\right]^{(0)} &= R^{(\mathrm{ls})}\left(\frac{r}{r_{\mathrm{D}}}, \frac{r}{r_{\mathrm{L}}}\right),
\end{aligned} \tag{2.27}$$

with labels l, s and ls indicating long range, short range and mixed range, respectively.

The assumptions (2.27) determine the pair correlation function nearly completely. In fact one has

$$g = \alpha\frac{g_{\mathrm{B}}g_{\mathrm{BGL}}}{g_{\mathrm{L}}} + (1-\alpha)(g_{\mathrm{B}} + g_{\mathrm{BGL}} - g_{\mathrm{L}}), \tag{2.28}$$

for any real number α. The proof is as follows. Since $\bar{G}^{(0)}$ depends on $x^{(0)}$ and $y^{(0)}$ only one may conclude from (2.27)

$$\left[\frac{\delta\bar{G}}{\delta x}\right]^{(0)} = \varphi(y^{(0)}), \quad \left[\frac{\delta\bar{G}}{\delta y}\right]^{(0)} = \psi(x^{(0)}), \tag{2.29}$$

for some functions φ and ψ. These relations retain their form outside equilibrium, as $\bar{G}$ depends only on x and y. Hence we have:

$$\frac{\delta^2\bar{G}}{\delta x^2} = 0. \tag{2.30}$$

Similarly

$$\frac{\delta^2\bar{G}}{\delta y^2} = 0. \tag{2.31}$$

Consequently $\bar{G}$ must be some linear combination of xy, x, y and 1. Finally (2.22) with (2.20) implies that g is the convex combination (2.28).

From (2.27) and (2.28) we are able to identify

$$R^{(l)} = \alpha \frac{g_{BGL}^{(0)}}{g_L^{(0)}} + (1-\alpha),$$

$$R^{(s)} = \alpha \frac{g_B^{(0)}}{g_L^{(0)}} + (1-\alpha), \tag{2.32}$$

$$R^{(ls)} = -\alpha \frac{g_B^{(0)} g_{BGL}^{(0)}}{g_L^{(0)2}} - (1-\alpha).$$

According to (2.28) g is a convex combination of the two particular approximations of the pair correlation function which we have discussed before. These approximations are known to lead, for small ϵ, to the same values of the collision brackets, and hence of the transport coefficients[1,2,10]). Since the collision brackets depend linearly on the pair correlation function it is clear, that the kinetic equation corresponding to the approximation (2.28) for g yields again the same values for the transport coefficients in the limit of small ϵ.

Summarizing, we have found that, if the pair correlation function can be approximated by a function of the general form (2.16) or (2.20) and if moreover, its functional derivatives have the scaling properties embodied in (2.27), then g has the form of the convex combination (2.28); the ensuing transport coefficients are then uniquely determined for small ϵ.

Let us consider now a generalization of the basic assumption (2.16). In fact, we shall allow the function G to depend not only on g_B, g_{BGL} and g_L, but also explicitly on the interparticle distance r. To be able to distinguish short-range and long-range dependences it is convenient to introduce two arguments, viz. r/r_D and r/r_L. Using the homogeneity of the first degree in g_i we can write

$$g = g_L \bar{G}\left(x, y \,\middle|\, \frac{r}{r_D}, \frac{r}{r_L}\right). \tag{2.33}$$

where x and y have been defined in (2.18). Since g must coincide with the Boltzmann expression for $r \lesssim r_L$, we have

$$\bar{G}\left(x, 1 \,\middle|\, \frac{r}{r_D}, \frac{r}{r_L}\right) = x, \quad r \lesssim r_L. \tag{2.34}$$

Similarly, demanding g to reduce to the BGL-expression for $r \gtrsim r_D$, we find

$$\bar{G}\left(1, y \,\middle|\, \frac{r}{r_D}, \frac{r}{r_L}\right) = y, \quad r \gtrsim r_D. \tag{2.35}$$

When we linearize g about local thermodynamic equilibrium we obtain again (2.24). From (2.35) we have, since $x = 1$ for $r \gtrsim r_D$

$$y\frac{\delta\bar{G}}{\delta y} = y = \bar{G}, \quad r \gtrsim r_D. \tag{2.36}$$

As a consequence we get

$$\frac{\delta\bar{G}}{\delta x} = -\left(\bar{G} - x\frac{\delta\bar{G}}{\delta x} - y\frac{\delta\bar{G}}{\delta y}\right), \quad r \gtrsim r_D. \tag{2.37}$$

For $r \lesssim r_L$, so that $y = 1$, we obtain a similar relation as (2.37), but now with x and y interchanged. As before we assume the coefficients of Δg_B and Δg_{BGL} in (2.24) to be of long-range and short-range type, respectively, so that we may introduce the notations (2.27). The relation (2.37) and its counterpart for $r \lesssim r_L$ then yield

$$R^{(ls)}\left(\frac{r}{r_D}, \frac{r}{r_D}\right) = \begin{cases} -R^{(l)}\left(\frac{r}{r_D}\right), & r \gtrsim r_D, \\ -R^{(s)}\left(\frac{r}{r_L}\right), & r \lesssim r_L. \end{cases} \tag{2.38}$$

Furthermore, we have from (2.36)

$$R^{(s)}\left(\frac{r}{r_L}\right) = 1, \quad r \gtrsim r_D, \tag{2.39}$$

and analogously

$$R^{(l)}\left(\frac{r}{r_D}\right) = 1, \quad r \lesssim r_L. \tag{2.40}$$

In the next section we show that, if (2.38)–(2.40) hold, the ensuing transport coefficients are independent of the precise form of $R^{(l)}$, $R^{(s)}$ and $R^{(ls)}$, at least for small values of the plasma parameter.

3. Uniqueness of the transport coefficients

For a plasma close to thermodynamical equilibrium the single-particle distribution function satisfies a linearized kinetic equation with a collision term that follows from the right-hand side of (2.1) by replacing g by $\Delta g = g - g^{(0)}$. Under fairly general assumptions Δg reads according to (2.24) and (2.27):

$$\Delta g = R^{(l)}\Delta g_B + R^{(s)}\Delta g_{BGL} + R^{(ls)}\Delta g_L, \tag{3.1}$$

with functions $R^{(i)}$ satisfying the relations (2.38)–(2.40). The ensuing linear collision operator I can then be written as

$$I = I_{\hat{B}} + I_{B\hat{G}L} + I_{\hat{L}}. \tag{3.2}$$

The operators I_i are analogous to those discussed in paper I. In particular, $I_{\hat{B}}$ has the form (I. 3.20). In the present case the effective potential $\varphi_{\hat{B}}$ is defined through:

$$\nabla\varphi_{\hat{B}}(r) = R^{(l)}\left(\frac{r}{r_D}\right)\nabla\left(\frac{e^2}{4\pi r}\right). \tag{3.3}$$

Furthermore, $I_{B\hat{G}L}$ has been given in (I. 3.24). The effective potential $\varphi_{B\hat{G}L}$ now follows from

$$\nabla\varphi_{B\hat{G}L}(r) = R^{(s)}\left(\frac{r}{r_L}\right)\nabla\left(\frac{e^2}{4\pi r}\right); \tag{3.4}$$

its Fourier transform is (cf. (I. 3.23)):

$$\hat{\varphi}_{B\hat{G}L}(q) = \frac{e^2}{q}\int_0^\infty \mathrm{d}r R^{(s)}\left(\frac{r}{r_D}\right) j_1(qr). \tag{3.5}$$

Finally, $I_{\hat{L}}$ has the same form as (I. 3.25); the effective potential $\varphi_{\hat{L}}$ is defined by means of the relation

$$\nabla\varphi_{\hat{L}}(r) = R^{(ls)}\left(\frac{r}{r_D}, \frac{r}{r_L}\right)\nabla\left(\frac{e^2}{4\pi r}\right), \tag{3.6}$$

or equivalently,

$$\hat{\varphi}_{\hat{L}}(q) = \frac{e^2}{q}\int_0^\infty \mathrm{d}r R^{(ls)}\left(\frac{r}{r_D}, \frac{r}{r_L}\right) j_1(qr). \tag{3.7}$$

The transport coefficients are determined by the collision brackets $[h, k]$, which are the matrix elements of the linear collision operator I. As in paper II, the general properties of the collision brackets can be studied by making use of a generating function, which we have defined as

$$\mathcal{B}(\boldsymbol{x}, \boldsymbol{y}) = [\mathrm{e}^{\boldsymbol{x}\cdot(\bar{\boldsymbol{v}}-\bar{\boldsymbol{V}})}, \mathrm{e}^{\boldsymbol{y}\cdot(\bar{\boldsymbol{v}}-\bar{\boldsymbol{V}})}], \tag{3.8}$$

with $\bar{\boldsymbol{v}} = (\frac{1}{2}\beta m)^{1/2}\boldsymbol{v}$ the dimensionless velocity of an individual particle and $\bar{\boldsymbol{V}}$ the dimensionless hydrodynamical velocity. From the generating function $\mathcal{B}$ all brackets determining the transport coefficients can easily be derived with the help of (II. 2.4), (II. 2.10) and (II. 2.19).

The generating function for the Boltzmann-type brackets can be discussed along the lines of section 4 of paper II. It has the general form

$$\mathcal{B}_{\hat{B}}(\boldsymbol{x}, \boldsymbol{y}) = 2\mathrm{e}^{(x+y)^2/8}\sum_l{}' (2l+1)P_l(\hat{\boldsymbol{x}}\cdot\hat{\boldsymbol{y}})I^l_{\hat{B}}(x, y), \tag{3.9}$$

with

$$I^l_{\mathrm{B}}(x,y)=\tfrac{1}{2}\pi^{-3/2}(\beta m)^{-1/2}\int \mathrm{d}\bar{u}\,\mathrm{d}r\,\exp\Big(-\tfrac{1}{2}\bar{u}^2-\frac{r_L}{r}\Big)$$
$$\times\,\bar{u}\cdot\nabla_r\Big[R^{(l)}\Big(\frac{r}{r_{\mathrm{D}}}\Big)\Big]i_l\big(\tfrac{1}{2}x\bar{u}\big)i_l\big(\tfrac{1}{2}y\bar{w}\big)P_l(\hat{u}\cdot\hat{w}) \qquad (3.10)$$

(cf. (II. 3.5) and (II. 4.3)). Here i_l are modified spherical Bessel functions and P_l Legendre polynomials. Furthermore, $\bar{w}$ is the dimensionless relative velocity in the infinite past of two particles that have a dimensionless relative velocity $\bar{u}$ and a relative position r at a finite time; its explicit expression has been given in (I. 3.16). Upon introducing new integration variables in (3.10) and expanding the Bessel functions we obtain (cf. (II. 4.4) with (II. C.1) and (II. C.2)):

$$I^l_{\mathrm{B}}(x,y)=-\frac{e^4\beta^{3/2}}{8\pi^{1/2}m^{1/2}\epsilon}\sum_{m,n=0}^{\infty}\frac{c^l_{mn}}{(l+m+n-1)!}J^l_{mn}(\epsilon)\big(\tfrac{1}{8}x^2\big)^{l/2+m}\big(\tfrac{1}{8}y^2\big)^{l/2+n}, \qquad (3.11)$$

with numerical coefficients c^l_{mn} that have been defined in (II. 3.16) and with

$$J^l_{mn}(\epsilon)=-\int_0^\infty \mathrm{d}\xi\int_0^1 \mathrm{d}\eta\,\xi^{l+m+n-2}(1-\eta)^{l/2+m-1}\eta^{-2}\,\mathrm{e}^{-\xi}F_l(\eta)\Big[\mathrm{d}/\mathrm{d}\Big(\frac{\epsilon}{\eta\xi}\Big)\Big]R^{(l)}\Big(\frac{\epsilon}{\eta\xi}\Big), \qquad (3.12)$$

which is a generalization of (II. C.3). The dominant contributions for small ϵ may be found by dividing the ξ-integration domain into two parts, viz. $[0,\epsilon^a]$ and $[\epsilon^a,\infty)$. The choice of a will depend on the behaviour of $R^{(l)}$ for small values of its argument. According to (2.40) $R^{(l)}(t)$ tends to 1 for small t. If one has for small t

$$R^{(l)}(t)=1+\mathcal{O}(t^p), \qquad (3.13)$$

with $p>0$, the dominant contributions to J^l_{mn} are found to be

$$J^l_{mn}(\epsilon)=\frac{l(l+1)}{4\epsilon}(l+m+n-1)!\Big[-\int_{\epsilon^{1-a}}^{\infty}\mathrm{d}\xi'\frac{R^{(l)}(\xi')}{\xi'}+a\log\epsilon$$
$$+\gamma-2\log 2-S_{l+m+n-1}+\frac{1}{2}S_{l/2}+T_{l/2}+\mathcal{O}(\epsilon^q)\Big], \qquad (3.14)$$

with $q=\min[a,(1-a)p]$. The numbers S_l and T_l have been defined in (II. 3.10). An alternative form for J^l_{mn} is obtained by using (3.13) to write

$$\int_{\epsilon}^{\epsilon^{1-a}}\mathrm{d}\xi'\frac{R^{(l)}(\xi')}{\xi'}=-a\log\epsilon+\mathcal{O}(\epsilon^{(1-a)p}). \qquad (3.15)$$

Then (3.14) becomes

$$J^{l}_{mn}(\epsilon) = \frac{l(l+1)}{4\epsilon}(l+m+n-1)!\Big[-\int_{\epsilon}^{\infty} d\xi' \frac{R^{(l)}(\xi')}{\xi'}$$
$$+ \gamma - 2\log 2 - S_{l+m+n-1} + \frac{1}{2}S_{l/2} + T_{l/2} + \mathcal{O}(\epsilon^{q})\Big]. \tag{3.16}$$

The smallest relative error is obtained by choosing a such that q is as large as possible. This is achieved by putting $a = p/(1+p) = q$.

The BGL-type collision brackets are generated by the function $\mathcal{B}_{\mathrm{B\hat{G}L}}$ which reads according to (II. 5.2)

$$\mathcal{B}_{\mathrm{B\hat{G}L}}(\boldsymbol{x}, \boldsymbol{y}) = \frac{\sqrt{2}e^{4}\beta^{3/2}}{8\pi^{2}m^{1/2}} \int \frac{d\hat{\boldsymbol{q}}}{4\pi} x_{\|}y_{\|}\, e^{(x_{\perp}^{2}+y_{\perp}^{2})/4}(e^{\boldsymbol{x}_{\perp}\cdot \boldsymbol{y}_{\perp}/2} - 1) I_{\mathrm{B\hat{G}L}}(x_{\|} + y_{\|}). \tag{3.17}$$

The lables $\|$ and $\perp$ denote components parallel with and perpendicular to the unit vector $\hat{\boldsymbol{q}}$. The integral $I_{\mathrm{B\hat{G}L}}$ is defined as

$$I_{\mathrm{B\hat{G}L}}(z) = \epsilon \int_{-\infty}^{\infty} d\zeta\, e^{-2\zeta^{2}+\zeta z} K(\zeta, \epsilon), \tag{3.18}$$

with

$$K(\zeta, \epsilon) = \int_{0}^{\infty} d\xi \int_{0}^{\infty} d\eta R^{(s)}(\xi) \frac{\eta^{4} j_{1}(\epsilon\eta\xi)}{[F_{1}(\zeta) + \eta^{2}]^{2} + F_{2}(\zeta)^{2}} \tag{3.19}$$

(cf. (II.5.6) and (I. B.1)); the functions $F_i(\zeta)$ appearing here have been given in (I. 5.20). The dominant contributions of (3.19) for small ϵ can be obtained by a reasoning which we developed in appendix B of paper I. Both the ξ- and η-integration domains are divided into two intervals, with boundaries depending on ϵ and on the behaviour of $R^{(s)}(t)$ for large t. We shall assume, in accordance with (2.39), that for large t the asymptotic form of $R^{(s)}$ is given by

$$R^{(s)}(t) = 1 + \alpha t^{-r} + \mathcal{O}(t^{-r'}), \tag{3.20}$$

with exponents $r' > r > 0$ and with an arbitrary constant α. The η-domain is split now at the point ϵ^{-b}, and the ξ-interval at ϵ^{-c} for $\eta \in [\epsilon^{-b}, \infty)$ and at $\epsilon^{-c'}$ for $\eta \in [0, \epsilon^{-b}]$, with $b, c, c' \in [0, 1]$ as yet undetermined. Then the dominant terms of $K(\zeta, \epsilon)$ for small ϵ are found as

$$K(\zeta, \epsilon) = \epsilon^{-1}\Big[\int_{0}^{\epsilon^{-c}} d\xi \frac{R^{(s)}(\xi)}{\xi} - (1-c)\log\epsilon - \gamma + 1 + \frac{\alpha}{r}\epsilon^{cr} - G(\zeta) + \mathcal{O}(\epsilon^{s})\Big]. \tag{3.21}$$

The function $G(\zeta)$ has been defined in (I.5.24). Furthermore, $s = \min[2b,$

$1-b-c$, $2(1-b-c')$, cr', $c'r$]. With the use of the auxiliary relation

$$\int_{\epsilon^{-c}}^{\epsilon^{-1}} \mathrm{d}\xi \frac{R^{(s)}(\xi)}{\xi} = -(1-c)\log\epsilon - \frac{\alpha}{r}(\epsilon^{r} - \epsilon^{cr}) + \mathcal{O}(\epsilon^{cr'}), \tag{3.22}$$

one arrives at the simpler expression for $K(\zeta, \epsilon)$:

$$K(\zeta, \epsilon) = \epsilon^{-1}\Big[\int_0^{\epsilon^{-1}} \mathrm{d}\xi \frac{R^{(s)}(\xi)}{\xi} - \gamma + 1 - G(\zeta) + \mathcal{O}(\epsilon^{s})\Big]. \tag{3.23}$$

The choices of c and c' that lead to the best estimate of the error term are therefore $c = (1-b)/(r'+1)$ and $c' = 2(1-b)/(r+2)$. For b we take $b = r'/(3r'+2)$ if $r'(2-r) \leq 2r$ and $b = r/(2r+2)$ elsewhere; in both cases $s = 2b$.

Finally we consider the generating function for the Landau-type collision brackets. The function $\mathscr{B}_{\mathrm{L}}$ is similar to $\mathscr{B}_{\mathrm{B\bar{G}L}}$, as given by (3.17) with (3.18). Instead of (3.19) we now have:

$$K^{0}(\epsilon) = \int_0^{\infty} \mathrm{d}\xi \int_0^{\infty} \mathrm{d}\eta R^{(\mathrm{ls})}(\xi, \epsilon\xi) j_1(\epsilon\eta\xi). \tag{3.24}$$

By introducing the variable $\eta' = \epsilon\eta\xi$ and performing the η'-integral we get

$$K^{0}(\epsilon) = \epsilon^{-1} \int_0^{\infty} \mathrm{d}\xi \frac{R^{(\mathrm{ls})}(\xi, \epsilon\xi)}{\xi}. \tag{3.25}$$

We have to use now the properties (2.38) of $R^{(\mathrm{ls})}$. These will be specified in more detail, as follows. For $\xi \gg 1$ we assume

$$R^{(\mathrm{ls})}(\xi, \epsilon\xi) = -R^{(\mathrm{l})}(\epsilon\xi) + \xi^{-t} Q^{(\mathrm{l})}(\epsilon\xi), \tag{3.26}$$

where $Q^{(\mathrm{l})}$ is bounded and t a positive constant. Likewise, for $\epsilon\xi \ll 1$ $R^{(\mathrm{ls})}$ is supposed to have the asymptotic property:

$$R^{(\mathrm{ls})}(\xi, \epsilon\xi) = -R^{(\mathrm{s})}(\xi) + (\epsilon\xi)^{u} Q^{(\mathrm{s})}(\xi), \tag{3.27}$$

with $Q^{(\mathrm{s})}$ bounded and u a positive exponent. To determine the dominant terms of (3.25) for small ϵ the integration domain is again split, this time at the point ϵ^{-d}, with $0 \leq d \leq 1$ as yet unspecified. The contribution from the interval $[\epsilon^{-d}, \infty)$ becomes upon insertion of (3.26) and introducing the variable $\xi' = \epsilon\xi$:

$$K_1^{0}(\epsilon) = \epsilon^{-1}\Big[-\int_{\epsilon^{1-d}}^{\infty} \mathrm{d}\xi' \frac{R^{(\mathrm{l})}(\xi')}{\xi'} + \mathcal{O}(\epsilon^{dt})\Big], \tag{3.28}$$

or, by changing the lower bound of the integral, as in (3.15):

$$K_1^0(\epsilon) = \epsilon^{-1}\Big[-\int_{\epsilon}^{\infty} \mathrm{d}\xi' \frac{R^{(l)}(\xi')}{\xi'} - d \log \epsilon + \mathcal{O}(\epsilon^{dt}) + \mathcal{O}(\epsilon^{(1-d)p})\Big]. \tag{3.29}$$

Similarly, the contribution from $[0, \epsilon^{-d}]$ is found upon substitution of (3.27) and with the use of (3.22)

$$K_2^0(\epsilon) = \epsilon^{-1}\Big[-\int_{0}^{\epsilon^{-1}} \mathrm{d}\xi \frac{R^{(s)}(\xi)}{\xi} - (1-d) \log \epsilon + \mathcal{O}(\epsilon^{dr}) + \mathcal{O}(\epsilon^{(1-d)u})\Big]. \tag{3.30}$$

Adding (3.29) and (3.30) and choosing $d = v/(v+w)$, with $v = \min(p, u)$, $w = \min(r, t)$ we finally obtain

$$K^0(\epsilon) = \epsilon^{-1}\Big[-\int_{\epsilon}^{\infty} \mathrm{d}\xi' \frac{R^{(l)}(\xi')}{\xi'} - \int_{0}^{\epsilon^{-1}} \mathrm{d}\xi \frac{R^{(s)}(\xi)}{\xi} - \log \epsilon + \mathcal{O}(\epsilon^{vw/(v+w)})\Big]. \tag{3.31}$$

The brackets corresponding to the complete collision operator I (3.2) are generated by the sum of the partial generating functions $\mathcal{B}_i (i = \tilde{B}, B\tilde{G}L, \tilde{L})$ for which explicit expressions valid for small ϵ have now been obtained. These expressions still depend on the unknown functions $R^{(l)}$ and $R^{(s)}$. If we write the contributions to the generating functions that contain $R^{(l)}$ or $R^{(s)}$ as $\Delta\mathcal{B}_i$, with

$$\Delta\mathcal{B}_i = \frac{e^4 \beta^{3/2}}{16\pi^{3/2} m^{1/2}} e^{(x+y)^2/8}\overline{\Delta\mathcal{B}_i}, \tag{3.32}$$

we find for $i = \tilde{B}$ from (3.9) with (3.11) and (3.16)

$$\overline{\Delta\mathcal{B}_{\tilde{B}}} \simeq \pi\Big[\int_{\epsilon}^{\infty} \mathrm{d}\xi' \frac{R^{(l)}(\xi')}{\xi'}\Big] \sum_l{}' (2l+1) l(l+1) P_l(\hat{x}\cdot\hat{y}) \times \sum_{m,n=0}^{\infty} c^l_{mn} (\tfrac{1}{8}x^2)^{l/2+m} (\tfrac{1}{8}y^2)^{l/2+n}. \tag{3.33}$$

Likewise (3.17) with (3.18) and (3.23) yield

$$\overline{\Delta\mathcal{B}_{B\tilde{G}L}} \simeq 2\Big[\int_{0}^{\epsilon^{-1}} \mathrm{d}\xi \frac{R^{(s)}(\xi)}{\xi}\Big] \int \frac{\mathrm{d}\hat{q}}{4\pi} x_{\parallel} y_{\parallel} [e^{(x_\perp + y_\perp)^2/8} - e^{(x_\perp - y_\perp)^2/8}]. \tag{3.34}$$

Finally, (3.17) with (3.18) and (3.31) give rise to:

$$\overline{\Delta\mathcal{B}_{\hat{\mathrm{L}}}} \simeq 2\left[-\int_{\epsilon}^{\infty} \mathrm{d}\xi' \frac{R^{(\mathrm{l})}(\xi')}{\xi'} - \int_{0}^{\epsilon^{-1}} \mathrm{d}\xi \frac{R^{(\mathrm{s})}(\xi)}{\xi}\right] \int \frac{\mathrm{d}\hat{q}}{4\pi}\, x_{\|} y_{\|} [\mathrm{e}^{(x_{\perp}+y_{\perp})^2/8} - \mathrm{e}^{(x_{\perp}-y_{\perp})^2/8}]. \tag{3.35}$$

It has been proved in paper II (see (II. 6.6), (II. 6.7) and (II. 6.22)) that the factors which multiply the integrals in (3.33) and (3.34) or (3.35) are equal. Hence, it follows that

$$\Delta\mathcal{B}_{\hat{\mathrm{B}}} + \Delta\mathcal{B}_{\mathrm{B}\hat{\mathrm{G}}\mathrm{L}} + \Delta\mathcal{B}_{\hat{\mathrm{L}}} \cong 0, \tag{3.36}$$

with an error following from (3.16), (3.23) and (3.31).

From (3.36) we conclude that the dominant terms for small ϵ in the generating function $\mathcal{B}$ which determines the collision brackets do not depend on the precise form of the functions $R^{(\mathrm{l})}$ and $R^{(\mathrm{s})}$. The same applies to the transport coefficients which follow directly from the collision brackets. Hence, it turns out that knowledge of the range character of $R^{(\mathrm{l})}$, $R^{(\mathrm{s})}$, $R^{(\mathrm{ls})}$ and of the properties (2.38)–(2.40) is sufficient to prove the uniqueness of the transport coefficients for a hot dilute plasma, with small plasma parameter.

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