Solution of a supersymmetric model of correlated electrons

M. J. Martins
Instituut voor Theoretische Fysica, Universiteit van Amsterdam, Valcknierstraat 65, 1018 XE Amsterdam, The Netherlands and Departamento de Física, Universidade Federal de São Carlos, Caixa Postal 676, 13565-905, São Carlos, Brazil

P. B. Ramos
Departamento de Física, Universidade Federal de São Carlos, Caixa Postal 676, 13565-905, São Carlos, Brazil
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We consider the exact solution of a model of correlated electrons based on the superalgebra Osp(2|2). The corresponding Bethe ansatz equations have an interesting form. We derive an expression for the ground-state energy at half filling. We also present the eigenvalue of the transfer matrix commuting with the Hamiltonian. [S0163-1829(97)07435-3]

Recently, supersymmetric generalizations of the Hubbard model have attracted considerable interest for their possible relevance in describing correlated electron systems. A successful example is the Essler, Korepin, and Schoutens[1] extended Hubbard model due to its superconductive properties. This model is based on the Sl(2|2) superalgebra and is exactly solvable in one dimension by the Bethe ansatz.[1,2] Another interesting system is a supersymmetric free-parameter model based on the continuous representation of the gl(2|1) superalgebra discussed by Bracken *et al.*[3] We recall that, for a special value of the coupling constant, this model was also considered by Karnaukhov.[4] Shortly afterwards, an anisotropic deformation of this model was found[5,6] as well as its solution by the coordinate Bethe ansatz approach.[5,7]

The main feature of these models is that in one dimension they are integrable and therefore they provide us with nonperturbative information concerning physical properties. In fact, they can be derived from supersymmetric solutions of the Yang-Baxter equation invariant either by the Sl(2|2) or by the Osp(2|2) superalgebras.[3,8,9,12] However, at least from the Bethe ansatz point of view, there exists a supersymmetric solution that has not yet been discussed in the literature so far. This solution was found some time ago by Deguchi, Fujii, and Ito[12] in the context of a q-deformed Osp(2|2) superalgebra and very recently has been interpreted as being invariant by the twisted $U_q[\mathrm{Sl}(2|2)^{(2)}]$ superalgebra.[13] It seems that this model exhausts the cases that can be derived by exploiting the Osp(2|2) superalgebra.[9,13] This then provides us with an extra motivation for studying this supersymmetric system. The purpose of this paper is to discuss this solution in the context of an exactly solvable model of correlated electrons by presenting its Bethe ansatz solution.

The one-dimensional lattice Hamiltonian is derived from the logarithmic derivative of the R matrix solving the supersymmetric Yang-Baxter equation. For explicit expressions we refer to Refs. 12, 9, and 13. Its turns out that the corresponding Hamiltonian can be written in terms of fermionic operators $c^{\dagger}_{i,\sigma}$ and $c_{i,\sigma}$ acting on the sites i of a chain of length L and carrying spin index $\sigma=\pm$. We found that the Hamiltonian reads as

$$
\begin{aligned}
H=&\sum_{i=1}^{L}\sum_{\sigma=\pm}[c^{\dagger}_{i,\sigma}c_{i+1,\sigma}+\mathrm{H.c.}][1-n_{i,-\sigma}(1+\sigma V_1)\\
&-n_{i+1,-\sigma}(1-\sigma V_1)]+V_2\sum_{i=1}^{L}[c^{\dagger}_{i,+}c^{\dagger}_{i,-}c_{i+1,-}c_{i+1,+}\\
&-c^{\dagger}_{i,+}c^{\dagger}_{i+1,-}c_{i+1,+}c_{i,-}+\mathrm{H.c.}]\\
&+V_2\sum_{i=1}^{L}[n_{i,+}n_{i,-}+n_{i+1,+}n_{i+1,-}+n_{i,+}n_{i+1,-}\\
&+n_{i,-}n_{i+1,+}-n_i-n_{i+1}+1],
\end{aligned}
\tag{1}
$$

where $n_{i,\sigma}=c^{\dagger}_{i,\sigma}c_{i,\sigma}$ is the number operator for electrons with spin σ on site i and we write $n_i=n_{i,+}+n_{i,-}$. This Hamiltonian conserves the number N_+ and N_- of electrons with spin up and down, respectively. Later on we will use the total number of electrons $N_e=N_++N_-$ and the number N_+ as the good ''quantum'' numbers for characterizing the spectrum of the Hamiltonian. We observe that in the derivation of expression (1) we have used the R matrix of Deguchi, Fujii, and Ito and afterwards performed a convenient canonical transformation. For example, an off-diagonal Hamiltonian element such as $(i/2)(e_{12}\otimes e_{43}-e_{21}\otimes e_{34})$ has been transformed into $\frac{1}{2}(e_{12}\otimes e_{43}+e_{21}\otimes e_{34})$, where e_{ab} stands for the elementary matrix $(e_{ab})_{i,j}=\delta_{i,a}\delta_{j,b}$. This has the advantage of simplifying the form of the hopping terms as well as bringing their coefficients to assume real values. Furthermore, we also have removed from consideration terms (such as the one proportional to n_i-n_{i+1}) that are automatically canceled out when periodic boundary conditions are imposed.

The Hamiltonian (1) presents extra-fine-tuned hopping terms, on-site and off-site Coulomb interactions. We recall that such interactions much resemble those appearing in the model of hole superconductivity proposed by Hirsch,[14] and they can be derived from first principles. This model becomes integrable when the couplings V_1 and V_2 are constrained on the unitary circle, i.e., $V_1^2+V_2^2=1$. The param-

etrization of these couplings in terms of the q-deformed parameter $q=\exp(i\gamma)$ $(0\leqslant\gamma\leqslant\pi)$ of the Osp(2|2) superalgebra can be read as

$$V_1=\sin(\gamma),\quad V_2=\cos(\gamma). \tag{2}$$

From Eq. (1) we see that when $\gamma\to 0$, this Hamiltonian reduces to the model proposed by Essler, Korepin, and Shoutens.[1] This has been noted previously in Ref. 13. It is still possible to reduce the interval for the anisotropy γ if we perform particle-hole transformations. Indeed, the combined set of canonical transformations (I) $c^{\dagger}_{i,\sigma}\leftrightarrow c_{i,\sigma}$ and (II) $c^{\dagger}_{i,\sigma}\leftrightarrow(-1)^i c_{i,\sigma}$ leads us to conclude that the spectra at certain values of γ and $\pi-\gamma$ are related to each other. More precisely, we have the following identity:

$$H(\gamma)=-H(\pi-\gamma), \tag{3}$$

and consequently we can restrict our analysis only for the regime $0\leqslant\gamma\leqslant\pi/2$. This identity indicates that the symmetric point $\gamma=\pi/2$ is somewhat special. In fact, at $\gamma=\pi/2$ we are left with ''almost'' a free-fermion theory. In this case the spectrum splits in two sectors, depending on the parity of the total number of electrons in the lattice. We have two decoupled XY models, with antiperiodic or periodic boundary conditions for N_e even or odd, respectively. We note that $H(\pi/2)$ corresponds also to the limit $U\to\infty$ in the Bariev chain,[15] which is a special case of a generalized XY model proposed long ago by Suzuki.[16] This means that the R matrix of Deguchi, Fujii, and Ito provides also embedding (''covering'' vertex model) for this peculiar limit of the Bariev chain.[17]

We now turn to the diagonalization of Hamiltonian (1) by the coordinate Bethe ansatz formalism. In other words, we would like to solve the eigenvalue problem, $H\Psi=E(L)\Psi$, provided the wave function in the sector of N_e electrons distributed on the positions $1\leqslant x_{Q_1}\leqslant x_{Q_2}\leqslant\cdots\leqslant x_{Q_{N_e}}\leqslant L$ has the following form:

$$\Psi_{\sigma_1,\ldots,\sigma_{N_e}}(x_{Q_1},\ldots,x_{Q_{N_e}})$$

$$=\sum_P \operatorname{sgn}(P)\prod_{j=1}^{N_e}\exp[ip_{P_j}x_{Q_j}]A(P|Q)_{\sigma_1,\ldots,\sigma_{N_e}}, \tag{4}$$

where the P summation extends over all the permutations of the momenta $(P_1,\ldots,P_{N_e})$ and sgn is the sign of the permutation. When electrons are far enough apart $(|x_{Q_i}-x_{Q_j}|\geqslant 2)$ it is direct to derive that the eigenvalues are given by

$$E(L)=\sum_{j=1}^{N_e} 2\cos(p_j)+V_2(L-2N_e). \tag{5}$$

The next step is to consider the matching condition for the Bethe ansatz wave function. Since the Hamiltonian (1) has been derived from a factorizable R matrix, we can restrict ourselves to the discussion of the two-body problem (two electrons in the chain). This is given in terms of the two-body S matrix, which connects the scattering amplitudes between states $\{(p_1,\sigma_1);(p_2,\sigma_2)\}$ and $\{(p_2,\sigma_2');(p_1,\sigma_1')\}$. We found that the non-null two-body S-matrix elements are given by

$$S^{++}_{++}(\lambda)=S^{--}_{--}(\lambda)=1,$$

$$S^{+-}_{+-}(\lambda)=S^{-+}_{-+}(\lambda)=\frac{\sinh(\lambda)}{\sinh(\lambda+2i\gamma)},$$

$$S^{-+}_{+-}(\lambda)=S^{+-}_{-+}(\lambda)=\frac{\sinh(2i\gamma)}{\sinh(\lambda+2i\gamma)}, \tag{6}$$

where $\lambda=\lambda_1-\lambda_2$ and λ_j are the ''dressed'' momenta rapidities that are related to the momenta p_j by the following relation:

$$\exp[ip_j]=\frac{\sinh(\lambda_j/2-i\gamma/2)}{\sinh(\lambda_j/2+i\gamma/2)}. \tag{7}$$

Now we have the basic ingredients to derive the Bethe ansatz equations. The next step is to face the problem of diagonalizing the spin degrees of freedom that are encoded in the S matrix (6). However, since this S matrix is of the six-vertex type, this later problem can be solved by standard algebraic methods.[18] In the course of the solution we have to introduce new spin rapidities μ_j, $j=1,\ldots,N_+$. It turns out that the dressed momenta and the spin variables satisfy the following nested Bethe ansatz equations:

$$\left[\frac{\sinh(\lambda_j/2-i\gamma/2)}{\sinh(\lambda_j/2+i\gamma/2)}\right]^L=\prod_{k=1}^{N_+}\frac{\sinh(\lambda_j-\mu_k-i\gamma)}{\sinh(\lambda_j-\mu_k+i\gamma)},$$

$$j=1,\ldots,N_e, \tag{8}$$

$$\prod_{k=1}^{N_e}\frac{\sinh(\mu_j-\lambda_k-i\gamma)}{\sinh(\mu_j-\lambda_k+i\gamma)}=-\prod_{k=1}^{N_+}\frac{\sinh(\mu_j-\mu_k-2i\gamma)}{\sinh(\mu_j-\mu_k+2i\gamma)},$$

$$j=1,\ldots,N_+, \tag{9}$$

and, in terms of the rapidities λ_j, the eigenvalues $E(L)$ are given by

$$E(L)=\sum_{i=1}^{N_e}\frac{2\sin^2(\gamma)}{\cos(\gamma)-\cosh(\lambda_i)}+\cos(\gamma)L. \tag{10}$$

The Bethe ansatz equations (8) and (9) have the unusual peculiarity that the ''dressed'' momenta variable λ_j enters in different ways in the momenta (as $\lambda_j/2$) and in the bare phase shift [right-hand side of Eq. (8)]. This is a special feature that distinguishes this model from the free-parameter supersymmetric correlated electron system.[3,5,7] Furthermore, near half-filling we have determined that while λ_j are real roots, the spin variables μ_j form strings of the following type:

$$\mu_j=\mu_j+i\,\frac{\pi}{2}. \tag{11}$$

By substituting this ansatz in Eqs. (8) and (9), taking their logarithms, and afterwards performing the thermodynamic limit $L\to\infty$, we are able to obtain the integral equations for the densities $\sigma(\lambda)$ and $\rho(\mu)$ for the variables λ_j and μ_j, respectively. They are given by

$$\sigma(\lambda)=\frac{\psi_1'(\lambda/2,\gamma/2)}{2\pi}+\int_{-\infty}^{+\infty}d\mu\,\psi_2'(\lambda-\mu,\gamma)\rho(\mu), \tag{12}$$

$$2\pi\rho(\mu)=\int_{-\infty}^{+\infty}d\lambda\,\psi_1'(\mu-\lambda,2\gamma)\rho(\lambda)+\int_{-\infty}^{+\infty}d\lambda\,\psi_2'(\mu-\lambda,\gamma)\sigma(\lambda), \tag{13}$$

where $\psi_1(x,\gamma)=2\arctan[\cot(\gamma)\tanh(x)]$ and $\psi_2(x,\gamma)=2\arctan[\tan(\gamma)\tanh(x)]$ and the prime stands for the derivative $\psi_{1,2}'(x/a,\gamma)=[d\psi_{1,2}(x/a,\gamma)]/dx$. These coupled integral equations can be solved by standard Fourier transforms. Considering the thermodynamic limit of $E(L)/L$ and taking into account the expression of $\sigma(\lambda)$, we find that the ground-state energy per particle at half-filling is given by the expression

$$\epsilon_\infty=-4\sin(\gamma)\int_0^\infty d\omega\,\frac{\cosh[\omega(\pi/2-\gamma)]\sinh[\omega(\pi-\gamma)]}{\cosh[\omega\pi/2]\sinh[\omega\pi]}+\cos(\gamma). \tag{14}$$

We would like to conclude with the following remarks. The experience we have gained solving the Hamiltonian (1) by the Bethe ansatz is also helpful if one wants to determine the eigenvalues of the transfer matrix of the underlying classical statistical model. Exact information about these eigenvalues is of considerable interest, because the transfer matrix is the generator of the many conserved currents commuting with the Hamiltonian (1). Considering that the eigenvalues $\Lambda(\lambda,\{\lambda_j,\mu_j\})$ of the transfer matrix are analytical functions of the variables $\{\lambda_j,\mu_j\}$, it is not difficult to start with an ansatz fulfilling this requirement. This is a phenomenological approach that goes by the name of analytical Bethe ansatz. Taking into account the Bethe ansatz equations (8) and (9) (analyticity conditions), as well as the unitary and the crossing properties of the transfer matrix, we can find the following expression for the eigenvalues:

$$\Lambda(\lambda,\{\lambda_i,\mu_j\})=[a(\lambda)]^L\prod_{i=1}^{N_e}\frac{\sinh(\lambda_i/2-\lambda/2-i\gamma/2)}{\sinh(\lambda_i/2-\lambda/2+i\gamma/2)}+[b(\lambda)]^L\prod_{i=1}^{N_e}\frac{\cosh(\lambda/2-\lambda_i/2-i\gamma/2)}{\cosh(\lambda/2-\lambda_i/2+i\gamma/2)}$$
$$-[c(\lambda)]^L\left\{\prod_{i=1}^{N_e}\frac{\sinh(\lambda/2-\lambda_i/2+i\gamma/2)}{\sinh(\lambda/2-\lambda_i/2-i\gamma/2)}\prod_{j=1}^{N_+}\frac{\sinh(\mu_j-\lambda+2i\gamma)}{\sinh(\mu_j-\lambda)}\right.$$
$$\left.+\prod_{i=1}^{N_e}\frac{\cosh(\lambda/2-\lambda_i/2-i\gamma/2)}{\cosh(\lambda/2-\lambda_i/2+i\gamma/2)}\prod_{j=1}^{N_+}\frac{\sinh(\lambda-\mu_j+2i\gamma)}{\sinh(\lambda-\mu_j)}\right\}, \tag{15}$$

where the functions $a(\lambda)$, $b(\lambda)$, and $c(\lambda)$ govern the behavior of the transfer matrix on the pure ferromagnetic pseudovacuum (this is equivalent to the totally empty or full band of electrons). For instance, if we take into account the R-matrix elements of Ref. 12 we find that

$$a(\lambda)=\frac{\sinh(i\gamma-\lambda/2)}{\sinh(i\gamma+\lambda/2)},\quad b(\lambda)=-\frac{\tanh(\lambda/2)}{\tanh(i\gamma+\lambda/2)},$$
$$c(\lambda)=-\frac{\sinh(\lambda/2)}{\sinh(i\gamma+\lambda/2)}. \tag{16}$$

and from Eqs. (15) and (16) it is clear that all the residues of $\Lambda(\lambda,\{\lambda_j,\mu_j\})$ vanish, provided the variables $\{\lambda_j,\mu_j\}$ satisfy the Bethe ansatz equations. We believe that this expression can also be derived more rigorously by using our recent formulation of the quantum inverse scattering method for Hubbard-like models.[19]

In summary, we have solved a supersymmetric model of correlated electrons by the Bethe ansatz approach. The ground-state energy at half-filling as well as the corresponding transfer matrix eigenvalues have been also determined. Preliminary analysis indicates that the excitations in the model are gapless near half-filling. Finally, from the Bethe ansatz point of view, the limit $\gamma\to 0$ needs to be performed taking extra care. Due to the peculiar topology of roots μ_j [see Eq. (11)], the situation is analogous to the Su(3) isotropic limit of the Izergin-Korepin model.[20] We also remark that when both λ_j and μ_j are real roots, one easily recovers the t-J sector present in the Sl(2|2) model.[1,2]

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