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Separation of spin and charge in paired spin-singlet quantum Hall states

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We propose a series of paired spin-singlet quantum Hall states, which exhibit a separation of spin and charge degrees of freedom. The fundamental excitations over these states, which have filling fraction $\nu = 2/(2m+1)$ with m an odd integer, are spinons (spin-$\frac{1}{2}$ and charge zero) or fractional holons (charge $\pm 1/(2m+1)$ and spin zero). The braid statistics of these excitations are non-Abelian. The mechanism for the separation of spin and charge in these states is topological: spin and charge excitations are liberated by binding to a vortex in a p-wave pairing condensate. We briefly discuss related, Abelian spin-singlet states and possible transitions.

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Strongly correlated electrons in low dimensional systems are known to exhibit physical phenomena that are surprising and, at first sight, counterintuitive. Among these is the remarkable phenomenon of quantum number fractionalization: elementary excitations in strongly interacting many-electron systems can have quantum numbers (for spin and charge) that are fractions of those of the electron. This fractionalization can take the form of a separation of spin and charge, or of a fractionalization of the electric charge of the electron.

In $D=1$ spatial dimension, the separation of spin and charge is well understood. It is seen in explicit solutions of specific integrable model systems (Hubbard and supersymmetric t-J models). The general framework of the Luttinger Liquid has made it clear that in $1+1$ dimensions the separation of spin and charge is a generic feature, which does not require any fine tuning of the interactions among the electrons.

In spatial dimensions $D=2$ or higher, spin and charge tend to confine and a separation of the two is only possible under very special conditions. It has been proposed that the key feature underlying the anomalous behavior of the cuprate high-T_c materials is precisely a separation of spin and charge,[1] and concrete scenarios, based on $\mathbf{Z}_2$ or $U(1)$ gauge theories, have been put forward.[2]

In this paper, we focus on the quantum Hall (QH) regime, which is relevant for two-dimensional (2D) electrons in strong magnetic fields, and for rotating Bose-Einstein condensates.[3] In particular, we discuss the separation of spin and charge in the QH regime. Specifically, we propose a series of paired spin-singlet QH states, of filling fraction $\nu = 2/(2m+1)$, which are generalizations of the Moore-Read or pfaffian states for spin polarized electrons. The fundamental excitations over these states are spinons (with spin-$\frac{1}{2}$ and zero charge) and holons (with zero spin and fractional charge $\pm 1/(2m+1)$, in units of the charge of the electron). The braid statistics of these excitations are non-Abelian, and thereby the paired spin-singlet states fall in the category of "non-Abelian QH states."

It is important to stress that the more conventional Abelian spin-singlet QH states [such as the Halperin states with label $(m+1,m+1,m)$, see below] do not exhibit a separation of spin and charge. The excitations over such states are conveniently analyzed in terms of a "spin-charge decomposition"[4–6] but this is subject to certain gluing conditions (expressing locality of the excitation w.r.t. the electrons), which exclude single spinons or holons from the (bulk) physical spectrum. The essential feature that liberates spin and charge in the paired states proposed here is the presence of the pairing condensate: by binding to a vortex in the pairing condensate, the spin and charge excitations become local with respect to the electrons in the ground state, and they can propagate independently.

It is illustrative to compare the separation of spin and charge in the paired spin-singlet states with the fractionalization of charge in paired, so-called q-pfaffian, spin-polarized states. For the q-pfaffian states, Laughlin's gauge argument gives that the adiabatic insertion of a single flux quantum will produce an excitation of charge $1/q$. However, as in the case of BCS superconductors, the presence of the pairing condensate leads to a reduction of the elementary flux quantum by a factor of 2, and thereby the unit-flux Laughlin quasiparticles are separated into two constituents each carrying a charge $1/(2q)$. In a similar way, conventional quasiparticles (carrying spin and charge) over a paired spin-singlet state are separated into spinons and holons.

Before we present the paired spin-singlet states, we briefly recall some facts about spin-singlet states and paired states in the QH regime. Despite the presence of strong magnetic fields in the QH regime, there is experimental motivation to study states that are not (fully) spin polarized (see, e.g., Ref. 7). In many QH systems, the energy scale for the Zeeman splitting is relatively low, and it can be further suppressed by the application of hydrostatic pressure. Using this technique, combined with a tilted field technique, spin transitions in the QH regime can be studied.[8] The simplest QH states that are singlets w.r.t. the SU(2) spin symmetry are the Halperin states[9]

$$\tilde{\Psi}_{\mathrm{SS}}^{(m+1,m+1,m)}(z_1^{\uparrow},\ldots,z_N^{\uparrow};z_1^{\downarrow},\ldots,z_N^{\downarrow}) = \Pi_{i<j}(z_i^{\uparrow}-z_j^{\uparrow})^{m+1}\Pi_{i<j}(z_i^{\downarrow}-z_j^{\downarrow})^{m+1}\Pi_{i,j}(z_i^{\uparrow}-z_j^{\downarrow})^{m}, \tag{1}$$

where $z_i^{\uparrow}$ and $z_i^{\downarrow}$ are the coordinates of the spin up and spin down electrons, respectively, and m is an even integer. The state Eq. (1) has filling fraction $\nu=2/(2m+1)$. Here and

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below we display reduced QH wave functions $\tilde{\Psi}(x)$, which are related to the actual wave functions $\Psi(x)$ via $\Psi(x) = \tilde{\Psi}(x)\exp(-\Sigma_i(|x_i|^2/4l^2)$ with $x_i = z_i^{\uparrow}, z_i^{\downarrow}$ and $l = \sqrt{\hbar c/eB}$ the magnetic length. Hierarchies of more general (Abelian) spin-singlet states were studied in Refs. 6 and 10–12.

In Ref. 4, Moore and Read introduced the notion of a paired QH state and discussed the so-called q-pfaffian states at filling $\nu = 1/q$ (with q even). It is believed that this state (with $q=2$) is at the origin of the observed QH plateau at filling fraction $\nu = \frac{5}{2}$ (see Ref. 13 for a recent review). The wave function for the q-pfaffian is given by

$$\tilde{\Psi}^{(q)}_{\mathrm{pf}}(z_1, \ldots, z_N) = \mathrm{Pf}\left(\frac{1}{z_i - z_j}\right)\prod_{i<j}(z_i - z_j)^q, \tag{2}$$

where the pfaffian factor for an antisymmetric matrix M_{ij} is defined as $\mathrm{Pf}(M_{ij}) = \mathcal{A}\Pi_{i\ \mathrm{even}} M_{i-1,i}$, with $\mathcal{A}$ denoting antisymmetrization. In Ref. 14, the pfaffian states were generalized to a series of non-Abelian spin-singlet (NASS) states, at filling $\nu = 4/(4M+3)$ with M an odd integer. These states exhibit a pairing of like spins. The excitations over these NASS states have non-Abelian statistics, but there is no separation of spin and charge.

In the paired spin-singlet states that we propose here, the pairing takes place in the charge sector, irrespective of the spin of the electrons. This leads to a wave function

$$\tilde{\Psi}^{(m)}_{\mathrm{paired}}(z_1^{\uparrow}, \ldots, z_N^{\uparrow}; z_1^{\downarrow}, \ldots, z_N^{\downarrow}) = \mathrm{Pf}\left(\frac{1}{x_i - x_j}\right)\tilde{\Psi}^{(m+1,m+1,m)}_{\mathrm{SS}}(z_i^{\uparrow}; z_j^{\downarrow}), \tag{3}$$

where $x_l = z_i^{\uparrow}, z_j^{\downarrow}$, m is now an odd integer and the filling fraction is $\nu = 2/(2m+1)$. There exists a Hamiltonian for which this state is the unique ground state.[15] One way to study the excitations over this state is by using this Hamiltonian. Here we will proceed by analyzing the state Eq. (3) and its excitations using an associated conformal field theory (CFT).

Following the CFT-QH correspondence outlined in Ref. 4, one quickly finds that the CFT associated to the (bosonic) paired spin-singlet state at $m=0$ is the (chiral) CFT based on the affine Kac-Moody algebra $\mathrm{SO}(5)_1$. For this algebra, the eight currents associated to the roots of SO(5) can be written in terms of spin and charge bosons $\varphi_{s,c}$ and a Majorana fermion ψ. [The assignment of spin and charge quantum numbers to the weights and roots of SO(5) is indicated in Fig. 1.] For general m, the "condensate" operators Ψ and Δ are obtained from these currents by the substitution $\varphi_c \to \sqrt{2m+1}\,\varphi_c$,

$$\Psi^{\alpha} = \psi\, e^{i\sqrt{(2m+1)/2}\varphi_c \pm (i/\sqrt{2})\,\varphi_s},$$

$$\overline{\Psi}^{\alpha} = \psi\, e^{-i\sqrt{(2m+1)/2}\varphi_c \pm (i/\sqrt{2})\,\varphi_s}$$

$$\Delta_c = e^{i\sqrt{4m+2}\varphi_c}, \quad \overline{\Delta}_c = e^{-i\sqrt{4m+2}\varphi_c}, \quad \Delta_s^A = e^{\pm i\sqrt{2}\varphi_s}, \tag{4}$$

with $\alpha = \uparrow, \downarrow$ referring to the spin eigenvalue $s_z = \pm\frac{1}{2}$ and $A = \uparrow\uparrow, \downarrow\downarrow$. The quantum numbers q (charge) and s_z are measured by the operators $Q = -i\sqrt{2/(2m+1)}\oint(dz/2\pi i)\,\partial\varphi_c$ and $S_z = i/\sqrt{2}\oint(dz/2\pi i)\,\partial\varphi_s$. The wave function Eq. (3) is obtained as a correlator of N spin-up electrons $\Psi^{\uparrow}$ and N spin-down electrons $\Psi^{\downarrow}$, together with a neutralizing background charge. The CFT description makes it easy to identify the fundamental (quasiparticle) excitations. For $m=0$ they are the operators that generate the spinor (four-dimensional) representation of the $\mathrm{SO}(5)_1$ current algebra. For general m these become

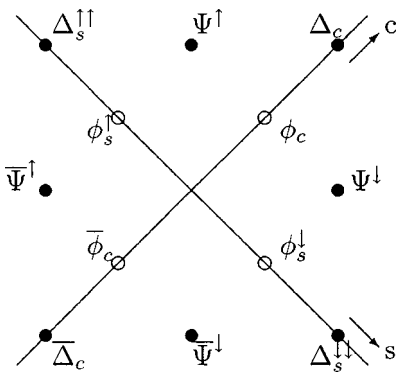

FIG. 1. Roots and weights of the algebra SO(5). The condensate operators Ψ and Δ are associated to the eight roots (filled symbols) and the fundamental excitations $\phi_{s,c}$ correspond to the weights of the spinor representation (open symbols).

$$\phi_c = \sigma e^{(i/\sqrt{4m+2})\,\varphi_c}, \quad \overline{\phi}_c = \sigma e^{-(i/\sqrt{4m+2})\,\varphi_c},$$

$$\phi_s^{\alpha} = \sigma e^{\pm(i/\sqrt{2})\,\varphi_s}, \tag{5}$$

where $\sigma(z)$ is the so-called spin field associated to the Majorana (Ising) fermion $\psi(z)$. Higher excitations, such as those constituting the vector representation, can be generated by bringing together two or more of the fundamental excitations. The expressions Eq. (5) show that the fundamental excitations can be characterized as spinons ϕ_s^{α} (spin-$\frac{1}{2}$ but no charge) and holons ϕ_c, $\overline{\phi}_c$ (of charge $\pm 1/(2m+1)$ and zero spin).

To illustrate the separation of spin and charge, we present explicit wave functions for excited states. We first consider an Abelian excitation, with spin down ($s_z = -\frac{1}{2}$) and charge $1/(2m+1)$, at location w. Its wave function takes the familiar form

$$\prod_i (z_i^{\uparrow} - w)\tilde{\Psi}^{(m)}_{\mathrm{paired}}. \tag{6}$$

The important observation is now that, starting from this wave function, one can separate the locations of the spin and charge parts of this excitation, creating a spinon at position w_s and a holon at w_c. In the corresponding wave function, the pfaffian factor in Eq. (3) is replaced by (compare with Ref. 4)

$$\mathrm{Pf}\left(\frac{\Phi(x_i,x_j;w_c,w_s)}{x_i-x_j}\right)\prod_i (x_i-w_c)^{1/2}\frac{\prod_i (z_i^{\uparrow}-w_s)^{1/2}}{\prod_j (z_j^{\downarrow}-w_s)^{1/2}}, \tag{7}$$

where

$$\Phi(x_i,x_j;w_c,w_s)=\left(\frac{x_i-w_c}{x_j-w_c}\frac{x_j-w_s}{x_i-w_s}\right)^{1/2}+i\leftrightarrow j. \tag{8}$$

That (7) in fact defines a well-behaved electronic wave function can be seen by noting that it is identical to

$$\frac{1}{\prod_j (z_j^{\downarrow}-w_s)}\mathrm{Pf}\left(\frac{(x_i-w_c)(x_j-w_s)+i\leftrightarrow j}{x_i-x_j}\right). \tag{9}$$

In the limit where $w_s,w_c\rightarrow w$, spin and charge recombine and the wave function reduces to Eq. (6). Note that the factor $\Pi_j(z_j^{\downarrow}-w_s)^{-1}$ should be regularized and projected onto the lowest Landau level in the same way as the wave functions for quasiparticles over the Laughlin states.[16]

The charge of the holon excitation equals $\frac{1}{2}\Phi_0\sigma_H$ (with $\Phi_0=h/e$ the flux quantum), showing that the creation of a single holon involves the insertion of a half-quantum of magnetic flux, which is the canonical flux quantum in the presence of a pairing condensate. This flux insertion is accompanied by a vortex in the pairing condensate, and this brings in the factor $\sigma(z)$ in the expressions Eq. (5). The role of the vortices in this discussion is similar to the role of visons in the Senthil-Fisher theory.[17]

An important feature that is implied by the presence of spin-fields $\sigma(z)$ in the expressions Eq. (5) for the spinons and holons, is that the braid statistics of these excitations will be non-Abelian. This feature is analogous to the non-Abelian statistics of the charge $1/(2q)$ excitations over the (spin-polarized) q-pfaffian state, and we refer to the literature for a discussion.[4,18,19]

It is well-known that the q-pfaffian spin-polarized state is closely related to two Abelian states at filling $\nu=1/q$: the two-layer $(q+1,q+1,q-1)$ state and a strong pairing state which is a Laughlin state of strongly paired electrons. Possible transitions among these three states have been discussed in the literature (see, e.g., Refs. 19–21). In the spin-singlet situation, we may similarly identify two series of Abelian spin-singlet states at $\nu=2/(2m+1)$ that allow for a transition into the pfaffian spin-singlet state Eq. (3): a two-layer state associated to SO(6) and a strong pairing state. The wave function for the two-layer state reads as

$$\begin{aligned}\tilde{\Psi}^{(m)}_{\text{2-layer}}(\{z_i^{\uparrow t},z_i^{\downarrow t},z_i^{\uparrow b},z_i^{\downarrow b}\})=&\,\Pi_{i<j}(z_i^{\uparrow t}-z_j^{\uparrow t})^{m+2}\Pi_{i<j}(z_i^{\downarrow t}-z_j^{\downarrow t})^{m+2}\Pi_{i<j}(z_i^{\uparrow b}-z_j^{\uparrow b})^{m+2}\Pi_{i<j}(z_i^{\downarrow b}-z_j^{\downarrow b})^{m+2}\\ &\times\Pi_{i,j}(z_i^{\uparrow t}-z_j^{\downarrow t})^{m+1}\Pi_{i,j}(z_i^{\uparrow b}-z_j^{\downarrow b})^{m+1}\Pi_{i,j}(z_i^{\uparrow t}-z_j^{\uparrow b})^{m}\Pi_{i,j}(z_i^{\downarrow t}-z_j^{\downarrow b})^{m}\\ &\times\Pi_{i,j}(z_i^{\uparrow t}-z_j^{\downarrow b})^{m-1}\Pi_{i,j}(z_i^{\downarrow t}-z_j^{\uparrow b})^{m-1},\end{aligned} \tag{10}$$

where the indices t,b refer to the top and bottom layers. This wave function arises as a correlator of two-layer spinful electron operators which, in the case $m=0$, generate an $\mathrm{SO}(6)_1$ affine Kac-Moody algebra.

The strong pairing state is an Abelian state of strongly bound pairs with quantum numbers $(q=-2,s_z=0)$ and $(q=0,s_z=1)$, which are the operators Δ_c and $\Delta_s^{\uparrow\uparrow}$ in Fig. 1. Spin and charge are decoupled from the start, and (putting $m=0$) we can associate to this state the symmetry SO(4) $\sim\mathrm{SU}(2)_s\times\mathrm{SU}(2)_c$.

There are various ways to understand and describe possible transitions among the three types of paired spin-singlet states at $\nu=2/(2m+1)$. Such transitions are expected when electrons in the two-layer state are subjected to increasing interlayer interactions. A useful framework is that of K matrices describing the topological order of the various states.[22] [For this discussion, we refer to the states via their associated SO(6), SO(5) or SO(4) symmetries.] For the SO(6) states, the naive K matrix for the four electron operators $(\uparrow,t)$, $(\downarrow,t)$, $(\uparrow,b)$, $(\downarrow,b)$ is singular. After a reduction to three independent condensate operators we find the following QH data

$$\mathbf{K}_e=\begin{pmatrix} m+2 & m & 2m+1\\ m & m+2 & 2m+1\\ 2m+1 & 2m+1 & 4m+2\end{pmatrix}, \tag{11}$$

$$\mathbf{q}_e=-(1,1,2),\quad \mathbf{s}_e=(\uparrow,\uparrow,0),\quad \mathbf{l}_e=(t,b,\cdot),$$

where $\mathbf{q}_e$, $\mathbf{s}_e$, and $\mathbf{l}_e$ specify the charge, spin and layer index for an appropriate basis of electron operators, which build the QH condensate. By applying a duality transformation ($\mathbf{K}_\phi=\mathbf{K}_e^{-1}$, $\mathbf{q}_\phi=-\mathbf{K}_\phi\mathbf{q}_e$, etc.) one obtains the topological data for a basis of quasi-hole excitations.[22,23]

Starting from this characterization of the topological order in the SO(6) state, the topological order of the SO(5) and SO(4) states can be obtained in a systematic manner.[21,23] For the SO(5) state, the resulting description employs a so-called pseudoparticle whose role it is to account for the degeneracies that are associated to the non-Abelian braid statistics. Choosing $\Psi^{\uparrow}$, $\Delta_s^{\uparrow\uparrow}$, and Δ_c as the fundamental condensate operators, we find

$$\mathbf{K}_e=\begin{pmatrix} m+2 & 1 & 2m+1\\ 1 & 2 & 0\\ 2m+1 & 0 & 4m+2\end{pmatrix},\quad \begin{aligned}\mathbf{q}_e&=-(1,0,2)\\ \mathbf{s}_e&=(\uparrow,\uparrow\uparrow,0)\end{aligned}$$

$$\mathbf{K}_\phi=\begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{4} & \frac{1}{4} \\ -\frac{1}{2} & \frac{1}{4} & \frac{2m+3}{8m+4} \end{pmatrix}, \quad \mathbf{q}_\phi=\left(0,0,\frac{1}{2m+1}\right) \quad \mathbf{s}_\phi=(0,\downarrow,0). \tag{12}$$

It is the first particle in the ϕ sector that is interpreted as a pseudoparticle, the other two have quantum numbers corresponding to $\phi_s^\downarrow$ and $\overline{\phi_c}$. The matrix $\mathbf{K}_\phi$ is of a general form first proposed in Ref. 24; for the interpretation of K matrices for non-Abelian QH states we refer to Ref. 23. We remark that the ground state degeneracy on the torus is not simply given by $|\det\mathbf{K}_e|$, as is the case for Abelian QH states; the actual value here is $3(2m+1)$.

A further reduction leads to the following QH data for the strong pairing SO(4) state (the data for the ϕ sector is obtained by the duality mentioned above)

$$\mathbf{K}_e=\begin{pmatrix} 2 & 0 \\ 0 & 4m+2 \end{pmatrix}, \quad \mathbf{q}_e=-(0,2), \quad \mathbf{s}_e=(\uparrow\uparrow,0). \tag{13}$$

This same set of QH data can be obtained by starting from the SO(6) data Eq. (11) and condensing quasiparticle-quasihole pairs, following Ref. 21.

The simplest filling fraction where the paired spin-singlet states that we propose are possible is $\nu=\frac{2}{3}$. At that same filling fraction, there exists an Abelian spin-singlet state, described by composite fermions with antiparallel flux attachment.[11] To distinguish the different states, one may consider the exponents for various tunneling processes. For the paired spin-singlet state the scaling dimensions for electrons, holons and spinons are $g_{\rm el}=m+2$, $g_{\rm hol}=(2m+5)/(16m+8)$, and $g_{\rm sp}=\frac{5}{8}$, respectively. Thus, for tunneling through the bulk, the holon is the most relevant particle (for $m\geqslant 1$), while the I-V for tunneling electrons from a Fermi-liquid into the edge is $I\sim V^{g_e}=V^{m+2}$. According to Ref. 25, the scaling dimensions for the composite fermion spin-singlet state at $\nu=\frac{2}{3}$ are $g_{\rm el}=2$, $g_{\rm qp}=\frac{2}{3}$. They give rise to a quadratic I-V for electron tunneling, in contrast to the cubic I-V for the paired state. Another way to distinguish the two states is via the spin-Hall conductance, which has opposite sign as compared to the ordinary Hall conductance for the Abelian state. For the paired spin-singlet state both conductances have the same sign.

There are two ways in which the paired state Eq. (3) can be relevant in a double-layer geometry. First, as already mentioned, there is the possibility of a transition from a double-layer state for spin-full electrons, Eq. (10), into a single-layer paired state. A second possibility is a realization of the paired state as a double-layer state for spin-polarized electrons, with the layer index playing the role of the spin index.

As is the case for the pfaffian and the NASS states, these states can be generalized to states which show clustering instead of pairing. Starting from an $\mathrm{SO}(5)_k$ symmetry structure, one derives states that allow clusters of up to $2k$ particles of equal spin, with filling fractions given by $\nu=2k/(2km+1)$.

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[1] P.W. Anderson, Science **235**, 1196 (1987).
[2] For a recent discussion, see T. Senthil and M.P.A. Fisher, J. Phys. A **34**, L119 (2001).
[3] N.K. Wilkin and J.M.F. Gunn, Phys. Rev. Lett. **84**, 6 (2000).
[4] G. Moore and N. Read, Nucl. Phys. B **B360**, 362 (1991).
[5] A.V. Balatsky and E. Fradkin, Phys. Rev. B **43**, 10 622 (1992).
[6] M. Milovanovic and N. Read, Phys. Rev. B **56**, 1461 (1997).
[7] For a review, see J.P. Eisenstein, in *Perspectives in Quantum Hall Effects*, edited by S. Das Sarma and A. Pinczuk (Wiley, New York, 1997).
[8] W. Kang *et al.*, Phys. Rev. B **56**, 12 776 (1997); H. Cho *et al.*, Phys. Rev. Lett. **81**, 2522 (1998).
[9] B. Halperin, Helv. Phys. Acta **56**, 75 (1983).
[10] E.H. Rezayi, Phys. Rev. B **39**, 13 541 (1989).
[11] X.G. Wu, G. Dev, and J.K. Jain, Phys. Rev. Lett. **71**, 153 (1993).
[12] A. Lopez and E. Fradkin, Phys. Rev. B **51**, 4347 (1995).
[13] N. Read, Physica B **298**, 121 (2001).
[14] E. Ardonne and K. Schoutens, Phys. Rev. Lett. **82**, 5096 (1999); E. Ardonne, N. Read, E. Rezayi, and K. Schoutens, Nucl. Phys. **B607**, 549 (2001).
[15] E. Rezayi, private communication.
[16] R.B. Laughlin, Phys. Rev. Lett. **50**, 1395 (1983).
[17] T. Senthil and M.P.A. Fisher, Phys. Rev. B **62**, 7850 (2000).
[18] N. Read and E. Rezayi, Phys. Rev. B **54**, 16 864 (1996).
[19] C. Nayak and F. Wilczek, Nucl. Phys. **B479**, 529 (1996).
[20] N. Read and D. Green, Phys. Rev. B **61**, 10 267 (2000).
[21] X.-G. Wen, Phys. Rev. Lett. **84**, 3950 (2000).
[22] X.-G. Wen, Adv. Phys. **44**, 405 (1995).
[23] E. Ardonne, P. Bouwknegt, S. Guruswamy, and K. Schoutens, Phys. Rev. B **61**, 10 298 (2000); E. Ardonne, P. Bouwknegt, and K. Schoutens, J. Stat. Phys. **102**, 421 (2001).
[24] S. Guruswamy and K. Schoutens, Nucl. Phys. **B556**, 530 (1999); P. Bouwknegt, L.-H. Chim, and D. Ridout, *ibid.* **B572**, 547 (2000).
[25] A. Lopez and E. Fradkin, Phys. Rev. B **63**, 085306 (2001); **64**, 049903(E) (2001).