# Broken Quantum Symmetry and Confinement Phases in Planar Physics

F. A. Bais,[1] B. J. Schroers,[2] and J. K. Slingerland[1]
[1]*Institute for Theoretical Physics, Valckenierstraat 65, 1018 XE Amsterdam, The Netherlands*
[2]*Department of Mathematics, Heriot-Watt University, Edinburgh EH14 4AS, United Kingdom*
(Received 14 May 2002; published 11 October 2002)

Many two-dimensional physical systems have symmetries which are mathematically described by quantum groups (quasitriangular Hopf algebras). In this Letter we introduce the concept of a spontaneously broken Hopf symmetry and show that it provides an effective tool for analyzing a wide variety of phases exhibiting many distinct confinement phenomena.

DOI: 10.1103/PhysRevLett.89.181601 PACS numbers: 11.15.Ex, 02.20.Uw, 11.10.Kk, 12.38.Aw

*Introduction.*—Planar quantum physics is known to exhibit many surprising properties such as charge fractionalization, spin-charge separation, and fractional and non-Abelian statistics. Important analogies show up between apparently different systems such as fractional quantum Hall systems and rotating bose condensates. Many of the special features are based on a subtle interplay between particles and their duals, e.g., between charges and fluxes, or between particles and vortices. These features are often a consequence of topological interactions among the relevant degrees of freedom. From a mathematical point of view many of these aspects are related to nontrivial realizations of the braid group. The appearance of the braid group is linked quite generically to the presence of an underlying quantum symmetry described by a (quasitriangular) Hopf algebra. Quantum groups naturally provide a framework in which Abelian or non-Abelian representations of the braid group can be constructed explicitly. Moreover, particles and their duals are treated on an equal footing in this framework. As a result, it is possible to give a systematic and a detailed description of the spin and statistics properties of the relevant degrees of freedom.

The generic appearance and therefore importance of Hopf symmetries in two-dimensional systems provide a strong physical motivation for studying what happens to such systems if one of the (bosonic) fields acquires a vacuum expectation value which breaks the Hopf symmetry. How does a phase with broken Hopf symmetry manifest itself physically and how can such phases be characterized? In the case of breaking of ordinary gauge symmetries one usually finds that masses for vector particles are generated and/or massless scalars show up. A further—and equally important—aspect of symmetry breaking is the impact on topological defects: some of them will disappear from the spectrum and new ones may show up depending on the properties of the order parameter [1]. As we will show this is only the simplest case, with other and more complicated situations arising when dual (dis)order parameters—for example, the density of magnetic vortices—come into play [2–4].

In this Letter we report on general results from the study of (dis)order parameters that carry representation labels of a Hopf algebra $A$. The (dis)order parameter breaks the Hopf symmetry to some Hopf-subalgebra $T$. The analysis shows that the representations of $T$ fall into two sets. One contains representations that get confined in the broken phase, while the other contains nonconfined representations. The tensor products of $T$ representations allow one to determine the "hadronic" composites that are not confined. The nonconfined representations together form the representation ring of a smaller algebra $U$, which is the residual symmetry of the effective low energy theory of nonconfined degrees of freedom. We find that both confined and nonconfined representations can be electric, magnetic, or dyonic in nature, depending on the type of (dis)order parameter one assumes is condensed. We have relegated an extensive mathematical treatment of these problems to a separate paper [5], to which we refer the reader for more detailed statements and proofs.

*Hopf symmetry.*—In this section we briefly summarize some essential properties of a Hopf algebra [6], choosing a relatively simple class as an example. This class describes the symmetry that arises if one breaks, for example, a non-Abelian continuous group $G$ to a discrete subgroup $H$, giving rise to what is known as a discrete gauge theory [7–10]. Such a model contains magnetic defects which carry a flux labeled by a group element of $H$. The group $H$ acts on fluxes by conjugation, so that fluxes in the same conjugacy classes form irreducible multiplets. If the group is non-Abelian one finds that the fluxes, when parallel transported around each other, generate non-Abelian Aharonov-Bohm phases. The underlying Hopf symmetry in this case turns out to be the quantum double $A = D(H)$ of the group $H$ [11,12]. This double has more structure than the group $H$ because $D(H) \equiv F(H)\tilde{\otimes}\mathbb{C}H$. Here $F(H)$ are the functions on the group and $\mathbb{C}H$ is the group algebra of $H$ (the linear span of group elements with the given group product). The symbol $\tilde{\otimes}$ indicates that $D(H)$ is the tensor product of $F(H)$ and $\mathbb{C}H$ but that the multiplication of two elements of $D(H)$ is "twisted." Explicitly, the multiplication rule for two elements in $D(H)$ is

  

$$(f_1 \otimes h_1)(f_2 \otimes h_2)(x) = f_1(x) f_2(h_1 x h_1^{-1}) \otimes h_1 h_2, \quad x \in H. \tag{1}$$

Note that the product in the $\mathbb{C}H$ component is the ordinary group multiplication but that the pointwise multiplication of functions is twisted by the conjugation action of $H$. Physically, we think of $H$ as the "electric" gauge group generated by $\{1 \otimes h\}$, while the $F(H)$ component is a "magnetic symmetry" generated by $\{f \otimes e\}$. The unitary irreducible representations of $D(H)$ are denoted by $\Pi^A_\alpha$. Here $A$ is a magnetic (flux) quantum number labeling a conjugacy class of $H$ and $\alpha$ is an electric quantum number labeling a representation $\alpha$ of the centralizer $N_A$ of that conjugacy class. We see that the trivial class $\{e\}$ (consisting of the unit element of $H$) gives the usual representations of $H = N_{\{e\}}$ corresponding to the purely electric states. Conversely the representations with the trivial $\alpha$ representations are the purely magnetic multiplets. At this point one should observe that the labeling of the dyonic (i.e., mixed) states already takes care of a well-known subtlety, namely, the obstruction to defining full $H$ representations in the presence of a non-Abelian magnetic flux. $D(H)$ has a trivial representation $\varepsilon$ (the co-unit) defined by $\varepsilon(f \otimes h) = f(e)$. There is a canonical way in which tensor product representations are defined, leading to a Clebsch-Gordan series:

$$\Pi^A_\alpha \otimes \Pi^B_\beta \cong N^{AB\gamma}_{\alpha\beta C} \Pi^C_\gamma. \tag{2}$$

The final ingredient is the $R$ matrix $R \in D(H) \otimes D(H)$ implementing the braid operation on a two particle state through

$$\mathcal{R} \equiv \sigma \cdot (\Pi^A_\alpha \otimes \Pi^B_\beta)(R), \tag{3}$$

where $\sigma$ is the "flip" operation, interchanging the order of the factors in the tensor product. The $\mathcal{R}^2$ operator yields the monodromy, or generalized Aharonov-Bohm phase factor.

We note that Hopf symmetry plays a role in all planar systems that have a conformal field theory description, such as two-dimensional critical phenomena, fractional quantum Hall states [13,14], and the world sheet picture of string theory. In these systems the tensor product rules of the quantum group are directly related to the fusion rules of the chiral algebra of the conformal field theory. The (quasi)particle excitations carry representations of that quantum group and the same mathematical tools can be used to characterize the Hall plateau states and their excitations.

*Hopf symmetry breaking.*—Let us imagine a condensate forming in a state $|v\rangle$ in the carrier space of some representation $\Pi^A_\alpha$. Then we may define *the maximal Hopf-subalgebra* $T$ of $A$ which leaves $|v\rangle$ invariant. Explicitly, elements $P$ of $T$ satisfy

$$\Pi^A_\alpha(P)|v\rangle = \varepsilon(P)|v\rangle \qquad \forall\ P \in T. \tag{4}$$

Given the original algebra $A$ there is a systematic way of calculating $T$. The most familiar example is the case where $A$ is the group algebra of an ordinary group $H$. In that case one easily checks $T$ is the group algebra of a subgroup of $H$, thus reproducing the well-known form of symmetry breaking. A first nontrivial case is $A = F(H)$. In that case the algebra of functions on the group $H$ gets broken to the algebra of functions on the quotient group $H/K$, where $K$ is some normal subgroup of $H$ (i.e., $HKH^{-1} = K$).

Let us now take a closer look at the situation for $A = D(H)$. If we break by a purely electric condensate $|v\rangle \in V^e_\alpha$, then the magnetic symmetry is unbroken but the electric symmetry $\mathbb{C}H$ is broken to $\mathbb{C}N_v$, with $N_v \subset H$ the stabilizer of $|v\rangle$. In that case we get $T = F(H)\tilde{\otimes}\mathbb{C}N_v$.

We may also break by a gauge invariant purely magnetic state. Interestingly enough one such state exists for each conjugacy class and corresponds to an unweighted sum of the basis vectors representing the group elements in the class: $|v\rangle = \sum_{a\in A}|a\rangle \in V^A_1$. The group action of $H$ leaves this state invariant:

$$\Pi^A_1(1 \otimes h)|v\rangle = \sum_{h\in A}|hah^{-1}\rangle = \sum_{a\in A}|a\rangle = |v\rangle. \tag{5}$$

In this case one may show that the unbroken Hopf algebra is $T = F(H/K_A)\tilde{\otimes}\mathbb{C}H$ with $K_A \subset H$ the subgroup generated by the elements of class $A$. This reduction of the symmetry reflects the physical fact that the fluxes can be defined only up to fusion with the fluxes in the condensate.

As a final example we consider what happens if the condensate corresponds to a single flux state $|v\rangle = |g\rangle$ with $(g \in A)$. Now one finds $T = F(H/K_A)\tilde{\otimes}\mathbb{C}N_g$ with $N_g = \{h \in H \mid hg = gh\}$, showing that both electric and magnetic symmetry are partially broken.

*Confinement.*—Consider now the physical situation after the breaking has taken place. As the ground state has changed we should discuss the fate of the (quasi)particle states belonging to the representations of the residual Hopf algebra $T$. These representations can be constructed [15] and describe the excitations in the broken phase. Furthermore, there is a decomposition of representations of the algebra $A$ into representations $\Omega_j$ of $T \subset A$. Now it may happen that the braiding of the condensed state $|v\rangle \in V_0$ and some (quasi)particle state $|p\rangle$ in a representation $\Omega_j$ is nontrivial. If this happens, the vacuum state is no longer single valued when transported around the (quasi)particle. Consequently, the new ground state does not support a localized excitation of the type $\Omega_j$ and will force it to develop a stringlike singularity, i.e., a domain wall ending on it. Such a wall carries a constant energy per unit length and therefore the particle of type $\Omega_j$ will be confined. The upshot is that we can use braid relations of the $T$ representations $\Omega_j$ with the ground state representation $\Pi_0$ of $A$ to determine whether or not the corresponding particles are confined. Physically speaking this procedure is like imposing a generalized Dirac charge quantization condition to determine the allowed nonconfined excitations in a given phase. In general the determination of these braid relations of the $T$ and $A$ representations is a difficult problem. For detailed calculations we refer to our paper [5]. It is also shown there that all $T$ representations which have trivial braiding with the vacuum representation can survive as localized states in the broken phase.

Consistency requires that the nonconfined representations should form a closed subset under the tensor product for representations of $T$. One may show that this is the case and that the subset of nonconfined representations can in fact be viewed as the representations of yet another Hopf algebra $U$. Mathematically, $U$ is the image of a surjective Hopf map

$$\Gamma : T \to U. \tag{6}$$

The $U$ symmetry characterizes the particlelike representations of the broken phase. Under quite general circumstances $U$ itself is again quasitriangular, implying that it features an $R$ matrix which governs the braid statistics properties of the nonconfined excitations in the broken phase. Returning to $T$, it is clear that the tensor product rules for confined $T$ representations allow one to construct multiparticle composite (hadronic) states which belong to nonconfined representations.

For a complete characterization of the excitations in the broken phase we should comment on the strings attached to confined particles. These are not uniquely characterized by their end points because one can always fuse with nonconfined particles. It turns out that the appropriate mathematical object characterizing the strings is the Hopf kernel $\ker(\Gamma)$ of the Hopf map (6).

To illustrate these concepts we return to the examples mentioned in the previous section. The first example concerned a purely electric condensate which just breaks the electric gauge group to $N_v$ so that, as mentioned, $T \equiv F(H)\tilde{\otimes}\mathbb{C}N_v$. One obtains $U \equiv F(N_v)\tilde{\otimes}\mathbb{C}N_v \equiv D(N_v)$ and $\ker(\Gamma) = F(H/N_v)$. Physically, this means that the only surviving representations are those which have magnetic fluxes corresponding to elements of $N_v$ while the states with fluxes in the set $H - N_v$ get confined. In short, partial electric breaking leads to a partial magnetic confinement. The distinct walls are now in one-to-one correspondence with the $N_v$ cosets in $H - N_v$.

The second example had the gauge invariant magnetic condensate, and we found that $T = F(H/K_A)\tilde{\otimes}\mathbb{C}H$ with $K_A \subset H$ the subgroup generated by the elements of class $A$. In this case we find that $U = D(H/K_A)$ with $\ker(\Gamma) = \mathbb{C}K_A$. Thus, only electric representations which are $K_A$ singlets survive while the others get confined. Partial or complete magnetic breaking will result in partial or complete electric confinement, depending on $K_A$. The walls in this phase are labeled by the representations of $K_A$.

Finally the pure flux condensate $|g\rangle$, which has $T = F(H/K_A)\tilde{\otimes}\mathbb{C}N_g$, leads to a phase for which $U = D(N_g/K_A \cap N_g)$ and $\ker(\Gamma) = F[(H/K_A)/\bar{N}_g]\tilde{\otimes}\mathbb{C}(K_A \cap N_g)$. Here $\bar{N}_g$ is the subgroup of $H/K_A$ which consists of the classes $nK_A$ with $n \in N_g$. In this case we have a breaking of magnetic and electric symmetry leading to a (partial) confinement of both. We do not discuss dyonic condensates here, not because of essential complications but rather because of notational inconveniences. The same analysis can be applied.

*Explicit examples.*—Having discussed our results on a rather general level, it may be useful to be concrete and give some explicit examples. First, we take $H = \mathbb{Z}_3$. This is a case treated by 't Hooft in [4], where the $\mathbb{Z}_3$ arises as the center of the SU(3) color group of QCD. We recover the well-known results for this case. Because $\mathbb{Z}_3$ is Abelian, we have $F(\mathbb{Z}_3) \cong \mathbb{C}\mathbb{Z}_3$ and hence the quantum double $D(\mathbb{Z}_3)$ is isomorphic to the group algebra of $\tilde{\mathbb{Z}}_3 \times \mathbb{Z}_3$. In other words, we have a magnetic group $\tilde{\mathbb{Z}}_3$ and an (isomorphic) electric group $\mathbb{Z}_3$. Denoting the irreps (irreducible representations) of $\mathbb{Z}_3$ by $\beta_i$ with $0 \leq i < 3$, the irreps of $D(\mathbb{Z}_3)$ are simply tensor products $\Pi_q^p \equiv \tilde{\beta}_p \otimes \beta_q$ and describe (quasi)particles with flux $p$ and charge $q$. As everything is Abelian, the $R$ matrix in this case is just given by the usual Aharonov-Bohm phase obtained by taking a particle with flux $p$ and charge $q$ around a particle with flux $r$ and charge $s$, yielding the phase factor $e^{2\pi i(ps+rq)/3}$. Consider now a magnetic condensate, related to a (dis)order parameter which transforms in a nontrivial representation $\Pi_0^p$. It breaks the magnetic $\tilde{\mathbb{Z}}_3$ down to the trivial group but leaves the electric $\mathbb{Z}_3$ unbroken, so that $T \cong \mathbb{C}\mathbb{Z}_3$. This algebra has a trivial representation $\beta_0$ and two nontrivial ones, $\beta_1$ and $\beta_2$, corresponding to the quarks and antiquarks. Particles in nontrivial representations of $T$ will pull strings in the condensate, since their braiding factors with the condensate are nontrivial. Hence these particles are confined. Total color confinement is reflected in the triviality of the algebra $U \cong \mathbb{C}$.

When $H$ is non-Abelian, our methods really come into their own, since in these cases the double $D(H)$ is not a group algebra, but a true quantum group. We give some examples involving the smallest non-Abelian group, $D_3$, the symmetry group of an equilateral triangle. It consists of the unit $e$, 120° and 240° rotations $r$ and $r^2 = r^{-1}$, and three reflections, $s$, $sr$, and $sr^2$. The nontrivial relation between $s$ and $r$ is given by $rs = sr^2$. $D_3$ is the simplest non-Abelian extension of $\mathbb{Z}_3$. It has three conjugacy classes, $[e] := \{e\}$, $[r] := \{r, r^2\}$ and $[s] := \{s, sr, sr^2\}$, whose centralizers are $N_e = D_3$, $N_r = \{e, r, r^2\} \cong \mathbb{Z}_3$, and $N_s = \{e, s\} \cong \mathbb{Z}_2$. $D_3$ also has three irreps, which we label 1, $J$, and $\alpha$. Here 1 is the trivial representation, $J$ is one dimensional and represents the rotations by 1 and the reflections by $-1$, and $\alpha$ is the defining two-dimensional representation in terms of rotations and reflections. The quantum double $D(D_3)$ has eight irreps: the trivial representation, the two electric representations $\alpha$ and $J$, two magnetic representations labeled by the conjugacy classes $[r]$ and $[s]$, and four dyonic representations.

Let us limit ourselves again to magnetic condensates, of which there can be different types within the same $D(D_3)$ irrep. The magnetic irrep labeled by $[r]$ for example, has two basis vectors, $|r\rangle$ and $|r^2\rangle$, labeled by their fluxes. A condensate characterized by the vector $v_1 = |r\rangle$ breaks the gauge group $D_3$ down to the $\mathbb{Z}_3$-centralizer group of $r$, whereas a condensate characterized by the vector $v_2 = |r\rangle + |r^2\rangle$ is gauge invariant and hence leaves the gauge group unbroken. Both condensates break the magnetic $F(D_3)$ down to $F(D_3/\mathbb{Z}_3) \cong F(\mathbb{Z}_2)$. The elements of the $\mathbb{Z}_2$ quotient are the cosets $E := e\mathbb{Z}_3$ and $S := s\mathbb{Z}_3$. These cosets label the flux quantum numbers in the broken phase; fluxes are now determined only up to a power of the condensed flux $r$. The residual symmetry algebras determined by $v_1$ and $v_2$ are $T_1 = F(\mathbb{Z}_2)\tilde{\otimes}\mathbb{C}\mathbb{Z}_3$ and $T_2 = F(\mathbb{Z}_2)\tilde{\otimes}\mathbb{C}D_3$. $T_1$ has six irreps $\Omega_q^{E/S}$, labeled by a $\mathbb{Z}_2$ flux ($E$ or $S$) and a $\mathbb{Z}_3$ charge $0 \leq q < 3$. Only the trivial irrep $\Omega_0^E$ has trivial braiding with the condensate. The other irreps have nontrivial braiding, because the flux $S$ does not commute with $r$ and because $r$ acts nontrivially in the irreps $\beta_1$, $\beta_2$ of $\mathbb{Z}_3$. Hence, we have complete confinement in this case. $T_2$ also has six irreps $\Omega_{1/J/\alpha}^{E/S}$, now labeled by a $\mathbb{Z}_2$ flux and a $D_3$ charge. The braiding between these and the condensate does not depend on the $\mathbb{Z}_2$ flux, since the condensate is gauge invariant, and we need only look at the action of the condensate on the charges $1, J, \alpha$. The $T_2$ irreps that involve $\alpha$ have nontrivial braiding and are hence confined. The four one-dimensional irreps with $D_3$ charges 1 and $J$ have trivial braiding with the condensate, because 1 and $J$ are trivial on $r$ and $r^2$. Thus, we are left with a nontrivial symmetry algebra $U_2 \cong F(\mathbb{Z}_2)\tilde{\otimes}\mathbb{C}\mathbb{Z}_2 \cong D(\mathbb{Z}_2)$ characterizing the nonconfined excitations.

*Conclusion.*—Physical systems on a plane may contain (quasi)particles with nontrivial topological interactions and braid statistics. Such systems often have a hidden quantum symmetry described by a Hopf algebra $A$. Representations of such algebras have the attractive feature that they treat ordinary and topological quantum numbers on an equal footing. In this Letter we investigated what happens when such a Hopf symmetry $A$ gets broken to a Hopf algebra $T$ by a vacuum expectation value of some field carrying a representation of $A$. We showed that generically there is a hierarchy of three Hopf algebras $A$, $T$, and $U$ which play a role in this situation. The representations of $T$ fall into two sets, one set being confined while the other is not. The latter can be interpreted as the representations of the Hopf-subalgebra $U$ which is the residual symmetry in the broken phase. The tensor product rules of $T$ representations tell us also what the nonconfined composites (i.e., the hadronic excitations) will be. The framework described here enables one to analyze a wide variety of phases, each with its specific pattern of (partial) confinement properties, and the way these phases are linked. It is interesting to investigate to what extent similar ideas can be exploited in more than two space dimensions.


[1] F. A. Bais, Phys. Lett. **98B**, 437 (1981).
[2] S. Mandelstam, Phys. Rep. **23**, 245 (1976).
[3] G. 't Hooft, in *Proceedings of the Banff Summer Institute on Particles and Fields, 1977* (Plenum, New York, 1978).
[4] G. 't Hooft, Nucl. Phys. **B138**, 1 (1978).
[5] F. A. Bais, B. J. Schroers, and J. K. Slingerland, hep-th/0205114.
[6] V. Chari and A. Pressley, *A Guide to Quantum Groups* (Cambridge University Press, Cambridge, 1994).
[7] F.A. Bais, Nucl. Phys. **B170**, 3 (1980).
[8] L. M. Krauss and F. Wilczek, Phys. Rev. Lett. **62**, 1221 (1989).
[9] J. Preskill and L. M. Krauss, Nucl. Phys. **B341**, 50 (1990).
[10] M. de Wild Propitius and F. A. Bais, in *Particles and Fields*, edited by G. Semenoff and L. Vinet, CRM Series in Mathematical Physics (Springer-Verlag, New York, 1998), p. 353.
[11] R. Dijkgraaf, V. Pasquier, and P. Roche, Nucl. Phys. (Proc. Suppl.) **B18**, 60 (1990).
[12] F. A. Bais, P. van Driel, and M. de Wild Propitius, Phys. Lett. B **280**, 63 (1992).
[13] G. Moore and N. Read, Nucl. Phys. **B360**, 362 (1991).
[14] J. K. Slingerland and F. A. Bais, Nucl. Phys. **B612**, 229 (2001).
[15] This is hard in general but rather straightforward in the not uncommon case where $T$ is a so-called transformation group algebra [5].