File ID	uvapub:129
Filename	0305.pdf
Version	unknown

SOURCE (OR PART OF THE FOLLOWING SOURCE):

Type	report
Title	Bartlett correction in stationary VARs
Author(s)	Pieter Omtzigt
Faculty	UvA: Universiteitsbibliotheek
Year	2003

FULL BIBLIOGRAPHIC DETAILS:
http://hdl.handle.net/11245/1.418442

UvA ECONOMETRICS

Discussion Paper: 2003/05

Bartlett correction in stationary VARs

Pieter Omtzigt

www.fee.uva.nl/ke/UvA-Econometrics

Department of Quantitative Economics
Faculty of Economics and Econometrics
Universiteit van Amsterdam
Roetersstraat 11
1018 WB AMSTERDAM
The Netherlands

UvA UNIVERSITEIT VAN AMSTERDAM

Bartlett corrections in stationary VARs

Pieter Omtzigt *

July 17, 2003

Abstract

We derive the Bartlett correction for a simple hypothesis on the regression parameters in a multivariate stationary autoregressive process.

Three applications illustrate the use of the correction: the test for absence of autocorrelation of any order, a simple hypothesis on the autoregressive parameters and two tests for weak exogeneity in the cointegrated VAR model. In the first of these tests, the cointegration space is known, in the second it is not.

The Bartlett correction performs well in all simulation studies, except in the one of the last test, that is a test for weak exogeneity in the cointegrated VAR with an unknown cointegration space.

1 Introduction

Vector Autoregressive Models (VAR) are widely applied both in macroeconomics and econometrics. Estimation of these models is often done by means of maximum likelihood methods. For almost every test statistics only asymptotic results are available regarding the distribution of the statistic under the null hypothesis. In small samples, the size distortion can be particularly large if large models (in terms of number of variables and lags) are used for relatively short spans of data series. A Bartlett correction (Bartlett, 1937) to a likelihood ratio test is one method to correct for the size distortion.

In this paper we consider the following multivariate model:

$$Y_t = AX_t + \eta_{2t}$$

where

$$\begin{aligned} X_t &= Q(L)\eta_{t-1} = Q_0\eta_{t-1} + Q_1\eta_{t-2} + Q_2\eta_{t-3} + \dots \\ \eta_t &= \begin{bmatrix} \eta_{1t}' & \eta_{2t}' \end{bmatrix}' \sim MIIDN(0, \Omega) \end{aligned}$$

under the assumption that $Q(L)$ is an exponentially decreasing polynomial and we derive the Bartlett correction for a simple hypothesis on A $\mathcal{H} : A = A_0$ both when $var(\eta_t)$ is known (theorem 2) and when it is unknown (theorem 1).

*University of Amsterdam, Faculty of Economics and Econometrics, Roetersstraat 11, 1018WB Amsterdam, The Netherlands, Email: P.H.Omtzigt@uva.nl

After a short introduction into Bartlett corrections and the two main theorems, we consider three specific applications. In section 4 we consider likelihood ratio tests for the absence of autocorrelation in a VAR model and in section 5 we consider a more general hypothesis on the autoregressive parameters of the VAR. Section 6 contains the Bartlett correction for two different tests of no long run feedback in the cointegrated VAR model. These last three sections all contain Monte Carlo studies of the derived results.

Conclusions are drawn in section 7. The longest section, the proof of the two main theorems and two other theorems, is given in the only appendix of this chapter, section A.

2 Bartlett corrections

Let $l_T(\theta), \theta = (\theta_1, \theta_2)$ denote the log likelihood function of T observations. Then the log likelihood ratio (W_T) test statistic for the null hypothesis $\mathcal{H}_0 : \theta_1 = \theta_1^0$ equals

$$-2\ln LR\left[\theta_1 = \theta_1^0 | \theta\right] = W_T = -2\left(\max_{\theta_2} l_T\left(\theta_1^0, \theta_2\right) - \max_{\theta_1,\theta_2} l_T(\theta_1, \theta_2)\right)$$

Under a number of regularity conditions, this test statistic converges in distribution. In many cases this is the χ^2-distribution, but it can also also be a different distribution; The rank test in cointegration analysis (Johansen, 1988, 1991) for instance, converges to a stochastic integral.

In small samples, the asymptotic distribution does not necessarily provide a good approximation to the actual one. The idea of the Bartlett correction (Bartlett, 1937) is to expand the expectation of the LR-statistic:

$$E[W_T] = f\left(1 + \frac{B(\theta)}{T} + O\left(T^{-2}\right)\right)$$

where $f = \lim_{T\to\infty} E_\theta[W_T]$ and then to define the Bartlett adjusted likelihood ratio statistic W_T^{BC} as:

$$W_T^{BC} = W_T / (1 + B(\theta)/T)$$

The term $B(\theta)/T$ shall be referred to as the Bartlett Factor (BF). It generally depends on the parameters of the model. When substituting values, it will sometimes make a difference whether we take the true values from the data generating process, the restricted estimates (that is the maximum likelihood estimates under the null hypothesis), or the unrestricted estimates.

Lawley (1956) proves that for stationary series and under a number of stochastic order conditions that the Bartlett Correction (BC) not only corrects the first moment up to $O(T^{-2})$, but also all higher moments. Barndorff-Nielsen and Hall (1988) prove the same result elegantly and demonstrate that is holds when $B\left(\hat{\theta}\right)$ replaces $B(\theta)$, where $\hat{\theta}$ is a $\sqrt{n}$-consistent estimator of θ. Often small sample corrections are referred to as Bartlett correction only if the result of Lawley holds. We shall however also refer to any division of the likelihood ratio test statistic by its expectation as a Bartlett correction.

Nielsen (1997) and Johansen (2000, 2002a,b) show that a Bartlett correction can be useful in models with unit roots. Jensen and Wood (1997) show by means of calculation of the first two moments that the result of Lawley does not hold for the Dickey-Fuller distribution. More precisely they show that $E\left[W_T\right] = f\left(1+\frac{b_1}{T}\right) + O\left(T^{-2}\right)$ and $E\left[W_T^2\right] = 2f\left(1+\frac{2b_2}{T}\right) + O\left(T^{-2}\right)$, but that $b_1 \neq b_2$.

General overviews of Bartlett and related corrections can be found in Jensen (1993) and Cribari-Neto and Cordeiro (1996) .

A large number of Bartlett correction concern univariate models, but Attfield (1995, 1998) derives a number of Bartlett corrections for simultaneous systems with fixed exogenous regressors. In this paper we consider multivariate models with lagged endogenous regressors.

3 The model and main results

Let us consider the following statistical model $\mathcal{K}_1$:

$$Y_t = AX_t + \eta_{2t} \tag{1}$$

where

$$\begin{aligned} & X_t = Q(L)\eta_{t-1} = Q_0\eta_{t-1} + Q_1\eta_{t-2} + Q_2\eta_{t-3} + \dots \\ & \eta_t = \left[\begin{array}{cc} \eta_{1t}' & \eta_{2t}' \end{array}\right]' \sim MIIDN(0, \Omega) \\ & A \in \mathcal{R}^{q\times n}, \Omega \in \mathcal{S}_{p\times p} \end{aligned}$$

and the null hypothesis

$$\mathcal{H}_0 : A = A_0$$

$\mathcal{R}$ is the space of real numbers $\mathcal{S}_{p\times p}$ the space of positive definite matrices of dimension $p \times p$. The process η_t is of dimension p and η_{2t} is of dimension $q(\leq p)$. The independent variable X_t $(1 \times n)$ is a moving average process. The innovations η_{2t} of the dependent variable Y_t are a subset of the innovations η_t, which constitute the moving average process X_t. This model allows for the possibility that X_t contains not only past values of Y_t, but also past value of exogenous variables, but not present values of exogenous variables. The model does not contain any deterministic terms.

We shall make the explicit assumption that the process Y_t is stationary.

Define

$$C_i = Q_i \Omega^{\frac{1}{2}}, \qquad i = 0, 1, 2, \dots \tag{2}$$

such that $X_t = C(L)\varepsilon_{t-1}$ and $\varepsilon_t = \Omega^{-\frac{1}{2}}\eta_t$ is distributed $MIIDN(0, I_p)$

Define the jth autocovariance matrix of X_t as $\Gamma_j = E\left[X_t X_{t-j}'\right] = \sum_{\alpha=0}^{\infty} Q_{\alpha+j}\Omega Q_\alpha' = \sum_{\alpha=0}^{\infty} C_{\alpha+j}C_\alpha'$ and its variance $\Phi = \Gamma_0$.

In the examples, it will be clarified how seemingly more general situations, like multiple lags, are in fact special cases of the following theorem, which concerns a simple hypothesis on the parameter A :

Theorem 1 *For the statistical model $\mathcal{K}_1$, the expected value of the likelihood ratio test of the null hypothesis that $\mathcal{H}_0 : A = A_0$ equals:*

$$E\left[W_T\right] \overset{1}{=} nq + \frac{1}{T}\left(\sqsupset(n,q) + \urcorner\left(n,q,\{C_i\}\right)\right) \tag{3}$$

where:

$$\beth = \frac{1}{2}\left(-4q + qn + q^2 n + qn^2\right) \tag{4}$$

$$\daleth = \sum_{\beta,\kappa=0}^{\infty} tr\left\{\left[C'_\kappa \Phi^{-1} \Gamma'_{\kappa+1} \Phi^{-1} \Gamma_{\beta+1} \Phi^{-1} C_\beta\right]_{22}\right\} \tag{t1}$$

$$+ 2\sum_{\beta,\kappa=0}^{\infty} tr\left\{\left[C'_\kappa \Phi^{-1} \Gamma'_{\kappa+1} \Phi^{-1} C_\beta\right]_{22}\right\} tr\left\{\Gamma'_{\beta+1} \Phi^{-1}\right\} \tag{t2}$$

$$+ \sum_{\beta,\kappa=0}^{\infty} tr\left\{\left[C'_\kappa \Phi^{-1} C_\beta\right]_{22}\right\} tr\left\{\Gamma'_{\kappa+1} \Phi^{-1}\right\} tr\left\{\Gamma'_{\beta+1} \Phi^{-1}\right\} \tag{t3}$$

$$+ \sum_{\beta,\kappa=0}^{\infty} tr\left\{\left[C'_\beta \Phi^{-1} \Gamma'_{\kappa+1} \Phi^{-1} \Gamma_{\beta+1} \Phi^{-1} C_\kappa\right]_{22}\right\} \tag{t4}$$

$$+ 2\sum_{\beta,\kappa=0}^{\infty} tr\left\{\left[C'_\beta \Phi^{-1} \Gamma'_{\kappa+1} \Phi^{-1} \Gamma'_{\beta+1} \Phi^{-1} C_\kappa\right]_{22}\right\} \tag{t5}$$

$$+ \sum_{\beta,\kappa=0}^{\infty} tr\left\{\left[C'_\beta \Phi^{-1} C_\kappa\right]_{22}\right\} tr\left\{\Gamma_{\beta+1} \Phi^{-1} \Gamma_{\kappa+1} \Phi^{-1}\right\} \tag{t6}$$

$$- 2\sum_{\beta,\kappa=0}^{\infty} tr\left\{\left[C'_\kappa \Phi^{-1} C_\beta\right]_{22}\right\} tr\left\{\Gamma'_{\kappa+\beta+2} \Phi^{-1}\right\} \tag{t7}$$

$$- 2\sum_{\beta,\kappa=0}^{\infty} tr\left\{\left[C'_\kappa \Phi^{-1} \Gamma'_{\kappa+\beta+2} \Phi^{-1} C_\beta\right]_{22}\right\} \tag{t8}$$

$$- 2\sum_{\beta,\kappa=0}^{\infty} tr\left\{\left[C'_\kappa \Phi^{-1} \Gamma_{\beta+1} \Phi^{-1} C_{\beta+\kappa+1}\right]_{22}\right\} \tag{t9}$$

$$- 2\sum_{\beta,\kappa=0}^{\infty} tr\left\{\left[C'_\kappa \Phi^{-1} \Gamma'_{\beta+1} \Phi^{-1} C_{\beta+\kappa+1}\right]_{22}\right\} \tag{t10}$$

Proof. See the appendix ■

With $[M]_{22}$ we indicate the lower right hand block of dimension $q \times q$ in the matrix M, which itself is of dimension $p \times p$. Thus $tr\left\{[M]_{22}\right\}$ is the sum of the last q elements on the main diagonal of the matrix M.

The expression $\daleth(n, q, C(L))$ looks complicated, but it should be borne it mind that it needs to be programmed only once and is programmed and computed relatively quickly. Furthermore it simplifies considerably in most cases. The version in the theorem has been written down with an eye on programming: it contains only two loops. The loops in the theorem go to infinity, but in all the examples and corollaries contained in this paper, the expression for $\daleth$ simplifies, such that only finite loops remain. The following expression for $\daleth(n, q, C(L))$ is useful in the corollaries and examples that will follow (we just substitute $\sum_{\alpha=0}^{\infty} C_{\alpha+\eta} C'_\eta$ for Γ_α):

$$\daleth = \sum\nolimits_{\beta,\eta,\kappa,\zeta=0}^{\infty} tr\left\{\left[C_\kappa'\Phi^{-1}C_\zeta C_{\kappa+\zeta+1}'\Phi^{-1}C_{\beta+\eta+1}C_\eta'\Phi^{-1}C_\beta\right]_{22}\right\} \quad \text{(t1')}$$
$$+\, 2\sum\nolimits_{\alpha,\eta,\kappa,\zeta=0}^{\infty} tr\left\{\left[C_\kappa'\Phi^{-1}C_\zeta C_{\kappa+\zeta+1}'\Phi^{-1}C_\alpha\right]_{22}\right\} tr\left\{C_{\alpha+\eta+1}'\Phi^{-1}C_\eta\right\} \quad \text{(t2')}$$
$$+ \sum\nolimits_{\alpha,\eta,\lambda,\zeta=0}^{\infty} tr\left\{\left[C_\lambda'\Phi^{-1}C_\alpha\right]_{22}\right\} tr\left\{C_\zeta'\Phi^{-1}C_{\lambda+\zeta+1}\right\} tr\left\{C_{\alpha+\eta+1}'\Phi^{-1}C_\eta\right\} \quad \text{(t3')}$$
$$+ \sum\nolimits_{\beta,\eta,\kappa,\zeta=0}^{\infty} tr\left\{\left[C_\beta'\Phi^{-1}C_\eta C_{\kappa+\eta+1}'\Phi^{-1}C_{\beta+\zeta+1}C_\zeta'\Phi^{-1}C_\kappa\right]_{22}\right\} \quad \text{(t4')}$$
$$+\, 2\sum\nolimits_{\beta,\eta,\lambda,\zeta=0}^{\infty} tr\left\{\left[C_\beta'\Phi^{-1}C_\eta C_{\lambda+\eta+1}'\Phi^{-1}C_\zeta C_{\beta+\zeta+1}'\Phi^{-1}C_\lambda\right]_{22}\right\} \quad \text{(t5')}$$
$$+ \sum\nolimits_{\alpha,\eta,\lambda,\zeta=0}^{\infty} tr\left\{\left[C_\alpha'\Phi^{-1}C_\lambda\right]_{22}\right\} tr\left\{C_\zeta'\Phi^{-1}C_{\lambda+\eta+1}C_\eta'\Phi^{-1}C_{\alpha+\zeta+1}\right\} \quad \text{(t6')}$$
$$-\, 2\sum\nolimits_{\zeta,\eta,\kappa=0}^{\infty} tr\left\{\left[C_\kappa'\Phi^{-1}C_\zeta\right]_{22}\right\} tr\left\{C_{\kappa+\zeta+\eta+2}'\Phi^{-1}C_\eta\right\} \quad \text{(t7')}$$
$$-\, 2\sum\nolimits_{\zeta,\eta,\lambda=0}^{\infty} tr\left\{\left[C_\lambda'\Phi^{-1}C_\eta C_{\lambda+\zeta+\eta+2}'\Phi^{-1}C_\zeta\right]_{22}\right\} \quad \text{(t8')}$$
$$-\, 2\sum\nolimits_{\kappa,\eta,\alpha=0}^{\infty} tr\left\{\left[C_\kappa'\Phi^{-1}C_{\alpha+\eta+1}C_\eta'\Phi^{-1}C_{\alpha+\kappa+1}\right]_{22}\right\} \quad \text{(t9')}$$
$$-\, 2\sum\nolimits_{\kappa,\zeta,\alpha=0}^{\infty} tr\left\{\left[C_{\alpha+\kappa+1}'\Phi^{-1}C_{\zeta+\alpha+1}C_\zeta'\Phi^{-1}C_\kappa\right]_{22}\right\} \quad \text{(t10')}$$

In most applications the variance of η_t is unknown. There is however little difference in deriving the main result for known and unknown variance. In section 5 we shall encounter one instance of a result in the literature which deals with known variance. We thus include the version of the main theorem with known variance in this paper to make results comparable. Consider the following statistical model $\mathcal{K}_2$:

$$Y_t = AX_t + \varepsilon_{2t} \quad (5)$$

where

$$\begin{aligned} X_t &= C(L)\varepsilon_{t-1} = C_0\varepsilon_{t-1} + C_1\varepsilon_{t-2} + C_2\varepsilon_{t-3} + \ldots \\ \varepsilon_t &= \left[\begin{array}{cc} \varepsilon_{1t}' & \varepsilon_{2t}' \end{array}\right]' \sim MIIDN(0, I_p) \\ A &\in \mathcal{R}^{q\times n} \end{aligned}$$

and the null hypothesis

$$\mathcal{H}_0 : A = A_0$$

Theorem 2 *For the statistical model $\mathcal{K}_2$, the expected value of the likelihood ratio test of the null hypothesis that $\mathcal{H}_0 : A = A_0$ equals:*

$$E\left[W_T\right] \stackrel{1}{=} nq + \frac{1}{T}\left(\beth_2\left(n,q\right) + \daleth\left(n,q,\{C_i\}\right)\right) \quad (6)$$

where:

$$\beth_2 = -2\sum_{\zeta=0}^{\infty} tr\left\{\left[C_\zeta'\Phi^{-1}C_\zeta\right]_{22}\right\}$$

and $\daleth\left(n,q,\{C_i\}\right)$ is given in theorem 1.

Proof. See the appendix, section A.11. ■

All corollaries that follow will be of theorem 1. The only exceptions is corollary 7, which follows from theorem 2.

The following three sections carry examples of increasing complexity, of the main result. Each section contains at least one simulation study to see how useful the correction is in practice.

4 Autocorrelation

4.1 First order autocorrelation

A first illustration of the theory is the test that a certain p-dimensional process u_t is white noise versus the alternative that it contains first order autocorrelation. The model is

$$\begin{aligned} u_t &= B_1 u_{t-1} + \eta_t \\ \eta_t &\sim MIIDN(0, \Omega) \end{aligned} \tag{7}$$

and the hypothesis: $\mathcal{H}_0 : B_1 = 0$. Note that Maximum Likelihood and Ordinary Least Squares coincide in this case (see also lemma 13 in the appendix) and that $q = n = p$. The last equality implies that $tr\{M_{22}\} = tr\{M\}$. Under the null hypothesis (7) collapses to $u_{t-1} = \eta_{t-1}$, such that we find that $Q_0 = I_p$ and $C_i = \Gamma_i = 0$ for all $i \geq 1$. This implies $C_0 = \Omega^{\frac{1}{2}}$ and $\Phi_0 = \Omega$. Now each of the terms t1-t10 in $\daleth$ has at least one term, whose summation starts at $t = 1$, for instance $\Gamma'_{\kappa+1}$ or $C_{\beta+\kappa+1}$. Therefore each term in all 10 summations is zero and thus $\daleth = 0$. So we obtain:

Corollary 3 *The likelihood ratio test that $B_1 = 0$ in model (7) has the following expected value*

$$E[W_T] \stackrel{1}{=} p^2 + \frac{1}{2T}\left(p^2 + 2p^3 - 4p\right)$$

The Bartlett correction here does not depend on the parameters of the model, that is $B(\theta) = B$. The correction only depends on p, the dimension of the system. In this simple example we therefore do not encounter any problem as to which estimate for the parameters we should take.

By means of a Monte Carlo Study we investigate how well the Bartlett correction performs. As parameters of choice we take $\Omega = I_n, n \in \{1, 2, \ldots, 8\}$ and $T \in \{25, 50, 100\}$. The results are reported by means of QQ-plots for half of the experiments, that is for $n \in \{1, 3, 5, 7\}, T \in \{25, 50, 100\}$ whereas all the results are reported in table 1 and are based on 10^6 Monte Carlo replications each.

For each experiment we report $E[W_T]$, $E\left[W_T^{BC}\right]$, the Bartlett Factor and the empirical rejection probabilities at the nominal 10%, 5% and 1% level of both the asymptotic and Bartlett adjusted test statistic. We note that the Bartlett corrections brings the rejection probability close to the nominal one, except for the area $T \in \{25\}, n \in \{5, 6, 7, 8\}$ where at the 5% nominal rejection probability the empirical rejection probability is still above 8% after the correction. Yet it does come down from values as high as 81% to at most 25%.

The QQ-plots show that W_T is a straight line, which makes it ideally suited for the Bartlett correction. A Bartlett correction, which does not depend on the estimated parameters, rotates the QQ-plot around the origin. If it is negative (as it is for $p = 1$) it rotates the line anti-clockwise and if it is positive it rotates it clockwise. Success is measured in how well the rotated line coincides with the 45-degree line. In the QQ-plots in figure 1, we see that with the possible exceptions of subfigures 1(j),1(g) and 1(h) the rotated line is virtually indistinguishable from the 45-degree line.

4.2 Fourth order autocorrelation

A second illustration is a test that fourth order autocorrelation is absent

$$\begin{aligned} u_t &= B_4 u_{t-4} + \eta_t \\ \eta_t &\sim MIIDN(0, \Omega) \end{aligned} \tag{8}$$

Now our null hypothesis is $\mathcal{H}_0 : B_4 = 0$. We find that $Q_3 = I$ and $Q_i = 0$ for $i \in \{0, 1, 2, 4, 5, \ldots\}$. As a consequence $\Phi = \Omega$. Let us now define $F_{\alpha\beta} = (C'_\alpha \Phi^{-1} C_\beta) = \left(\Omega^{\frac{1}{2}} Q'_\alpha \Phi^{-1} Q_\beta \Omega^{\frac{1}{2}}\right)$. It is immediately clear that in this example $F_{\alpha,\beta} = I_p$ iff $\alpha = \beta = 3$ and $F_{\alpha,\beta} = 0$ otherwise. Now rewrite $t'_1 = \sum_{\beta,\eta,\kappa,\zeta=0}^{\infty} tr \left\{F_{\kappa,\zeta} F_{\kappa+\zeta+1,\beta+\eta+1} F_{\eta,\beta}\right\}$. For any of the terms in this summation to be different from zero, we need $\kappa = \zeta = \kappa + \zeta + 1 = 3$, such that we conclude that $t'_1 = 0$. In similar fashion we see that all other nine terms $t2' - t10'$ equal zero as well, such $\daleth = 0$ and we obtain the same expression as in the last parapraph:

Corollary 4 *The likelihood ratio test that $B_4 = 0$ in model (8) has the following expected value*

$$E\left[W_T\right] \stackrel{1}{=} p^2 + \frac{1}{2T}\left(p^2 + 2p^3 - 4p\right)$$

which once again does not depend on the parameters of the model.

4.3 First to kth order autocorrelation

Third we test whether there is no first up to kth order autocorrelation:

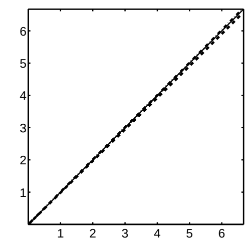
$$\begin{aligned} u_t &= B_1 u_{t-1} + \ldots + B_k u_{t-k} + \eta_t \\ \eta_t &\sim MIIDN(0, \Omega) \end{aligned} \tag{9}$$

The null hypothesis is thus $\mathcal{H}_0 : B_1 = \ldots = B_k = 0$. We see that the regressors in the model u_{t-1} are all independently identically distributed with mean 0 and variance-covariance matrix Ω. The polynomial matrices Q are of dimension $pk \times p$ and read:

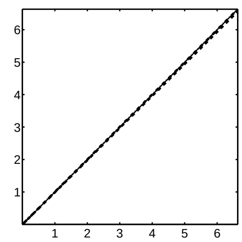
$$\begin{aligned} Q_0 &= [I_p : 0 : 0 : \cdots : 0 : 0]' \\ Q_1 &= [0 : I_p : 0 : \cdots : 0 : 0]' \\ &\vdots \\ Q_{k-1} &= [0 : 0 : 0 : \cdots : 0 : I_p]' \\ Q_j &= [0 : 0 : 0 : \cdots : 0 : 0]' \text{ for } j \geq k \end{aligned}$$

T	25		50		100	
	W_T	W_T^{BC}	W_T	W_T^{BC}	W_T	W_T^{BC}
$p=1$						
$E[LR]$	0.9836	1.0036	0.9906	1.0006	0.9938	0.9988
BF	-0.0200		-0.0100		-0.0050	
10%	9.70	10.04	9.85	10.02	9.92	10.00
5%	4.78	5.01	4.90	5.02	4.94	4.99
1%	0.95	1.02	0.96	1.02	0.99	1.00
$p=2$						
$E[LR]$	4.2749	4.0329	4.1322	4.0118	4.0652	4.0051
BF	0.0600		0.0300		0.0150	
10%	12.20	10.26	11.03	10.09	10.48	10.01
5%	6.44	5.18	5.65	5.06	5.31	5.01
1%	1.44	1.05	1.19	1.01	1.08	1.00
$p=3$						
$E[LR]$	10.2294	9.1881	9.5609	9.0482	9.2612	9.0060
BF	0.1133		0.0567		0.0283	
10%	16.64	10.98	12.90	10.28	11.30	10.05
5%	9.45	5.65	6.86	5.16	5.83	5.02
1%	2.50	1.20	1.37	1.05	1.24	1.00
$p=4$						
$E[LR]$	19.2543	16.5986	17.4245	16.1338	16.6750	16.0336
BF	0.1600		0.0800		0.0400	
10%	24.10	12.28	15.65	10.53	12.55	10.16
5%	14.87	6.47	8.67	5.31	6.60	5.09
1%	4.69	1.44	2.16	1.09	1.48	1.02
$p=5$						
$E[LR]$	31.8993	26.4944	27.9155	25.3317	26.3618	25.0826
BF	0.2040		0.1020		0.0510	
10%	35.88	14.66	19.61	10.94	14.12	10.23
5%	24.40	8.02	11.40	5.59	7.63	5.14
1%	9.44	1.96	3.15	1.17	1.81	1.06
$p=6$						
$E[LR]$	48.9253	39.2449	41.2386	36.7109	38.4106	36.1796
BF	0.2467		0.1223		0.0617	
10%	52.68	19.46	25.41	11.73	16.29	10.44
5%	39.88	10.96	15.67	6.07	9.06	5.26
1%	19.46	3.00	4.88	1.30	2.27	1.07
$p=7$						
$E[LR]$	71.1650	55.2278	57.5895	50.3279	52.8800	49.3218
BF	0.2886		0.1443		0.0721	
10%	71.97	25.92	33.05	12.78	19.00	10.66
5%	60.51	16.14	21.74	6.72	10.92	5.41
1%	37.35	5.16	7.69	1.50	2.92	1.11
$p=8$						
$E[LR]$	100.0981	75.2617	77.2226	66.2855	69.8832	64.5572
BF	0.3300		0.1650		0.0825	
10%	88.69	37.26	42.92	14.31	22.48	10.97
5%	81.58	25.49	30.18	7.72	13.38	5.59
1%	62.77	9.95	12.42	1.83	3.86	1.17

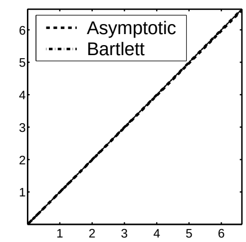
Table 1: Bartlett corrections for the test of absence of first order autocorrelation



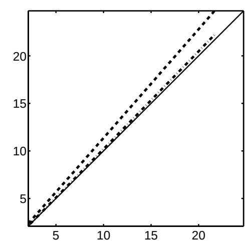
(a) $p = 1, T = 25$



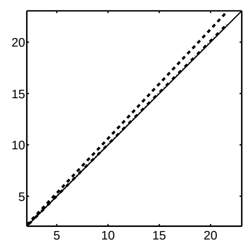
(b) $p = 1, T = 50$



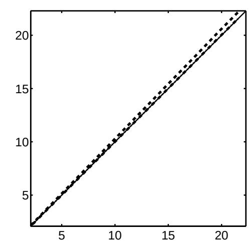
(c) $p = 1, T = 100$



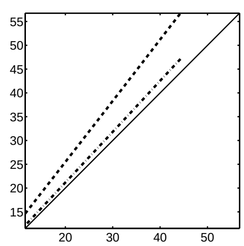
(d) $p = 3, T = 25$



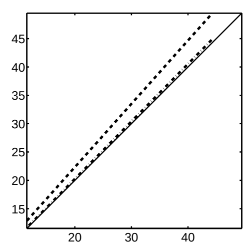
(e) $p = 3, T = 50$



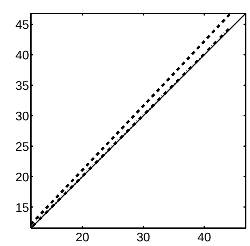
(f) $p = 3, T = 100$



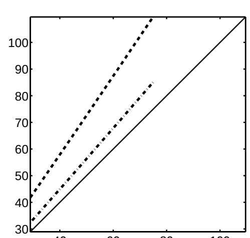
(g) $p = 5, T = 25$



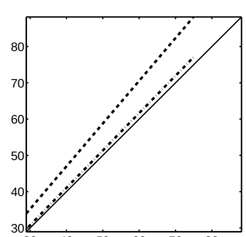
(h) $p = 5, T = 50$



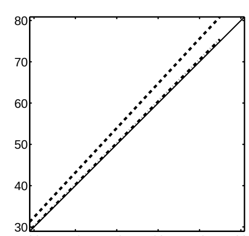
(i) $p = 5, T = 100$



(j) $p = 7, T = 25$



(k) $p = 7, T = 50$



(l) $p = 7, T = 100$

Figure 1: QQ-plots of LR-tests (asymptotic and Bartlett corrected) for residual autocorrelation

This implies $\Phi = (I_k \otimes \Omega)$. Realizing that $Q_i'Q_j = I_p$ iff $i = j, i \leq k-1$ and 0 otherwise, we check each of the ten terms in turn to find out which ones are non-zero. As in the last paragraph we define $F_{\alpha,\beta} = (C_\alpha' \Phi^{-1} C_\beta) = \left(\Omega^{\frac{1}{2}} Q_\alpha' \Phi^{-1} Q_\beta \Omega^{\frac{1}{2}}\right)$ and see that $F_{\alpha,\beta} = I_p$ if $Q_i'Q_j = I_p$ and equals zero when $Q_i'Q_j = 0$.

For the first term $t_1' = \sum_{\beta,\eta,\kappa,\zeta=0}^{\infty} tr\left\{F_{\kappa,\zeta} F_{\kappa+\zeta+1,\beta+\eta+1} F_{\eta,\beta}\right\}$ we see that each time $\kappa = \zeta, \kappa+\zeta+1 = \beta+\eta+1 \leq k-1$ and $\beta = \eta$ simultaneously, this term equals p. In all other cases it equals zero. Thus we look for how many combination there are for which $\kappa = \zeta \geq 0$ and $\kappa + \zeta + 1 \leq k - 1$ hold true. There are $\left\lfloor \frac{k}{2} \right\rfloor$ such that this term equals $p\left\lfloor \frac{k}{2} \right\rfloor$.

The second term is $t_2' = \sum_{\alpha,\eta,\kappa,\zeta=0}^{\infty} tr\left\{F_{\kappa,\zeta} F_{\kappa+\zeta+1,\alpha}\right\} tr\left\{F_{\alpha+\eta+1,\eta}\right\}$. By definition $\alpha + \eta + 1 \neq \eta$, leading to the conclusion that this term is zero. Similarly we see that t_3', t_5', t_6', t_7', t_8' and t_{10}' are zero.

For any of the terms in the summation $t_4' = \sum_{\beta,\eta,\kappa,\zeta=0}^{\infty} tr\left\{F_{\beta,\eta} F_{\kappa+\eta+1,\beta+\zeta+1} F_{\zeta,\beta}\right\}$ to be different from zero we need $\kappa = \zeta$ and $\eta = \beta$ and $\kappa + \eta + 1 = \beta + \zeta + 1 \leq k-1$. There are $\sum_{i=1}^{k-1} i$ such combination, giving a contribution of $\frac{1}{2}pk\,(k-1)$. Similarly $t_{10}' = -\sum_{\kappa,\zeta,\alpha=0}^{\infty} tr\left\{\left(C_{\alpha+\kappa+1}' C_{\zeta+\alpha+1}\right)\left(C_\zeta' C_\kappa\right)\right\}$ equals $-p$ iff $\kappa = \zeta$ and $\alpha + \kappa + 1 = \zeta + \alpha + 1 \leq k - 1$ which is possible in $\frac{1}{2}k\,(k-1)$ ways.

For the problem at hand we see that $q = p$ and $n = pk$. Substituting all these terms in the expression in theorem 1 we obtain the following result:

Corollary 5 *The likelihood ratio test that $\mathcal{H}_0 : B_1 = \ldots = B_k = 0$ in model (9) has the following expected value*

$$E\left[W_T\right] \stackrel{1}{=} kp^2 + \frac{1}{2T}\left(p^2k + p^3k^2 + p^3k - 4p\right) + \frac{1}{T}\left(p\left\lfloor \frac{k}{2} \right\rfloor - \frac{1}{2}pk\,(k-1)\right)$$

Once more we notice that the Bartlett factor does not depend on any of the parameters.

5 Multivariate AR(1) process

Let us once more consider the p-dimensional AR(1) model and denote it by $\mathcal{L}_1$:

$$\begin{aligned} X_t &= BX_{t-1} + \eta_t \\ \eta_t &\sim MIIDN(0, \Omega) \end{aligned} \tag{10}$$

The parameters of this model are $\theta = (B, \Omega) \in (\mathcal{R}^{p\times p}, \mathcal{S}_{p\times p})$ and we test the hypothesis $\mathcal{H}_0 : B = \rho_0 I$, where $|\rho_0| < 1$. Under $\mathcal{H}_0$ the dependent variable X_{t-1} has the following moving average representation:

$$X_{t-1} = \sum_{i=0}^{\infty} \rho_0^i I \eta_{t-1-i}$$

from which we see directly that $Q_i = I\rho_0^i$ for $i \geq 0$. Then $C_i = \Omega^{\frac{1}{2}}\left(I\rho_0^i\right)$ and $\Phi = \sum_{i=0}^{\infty}\left(I\rho_0^{2i}\right)\Omega = \left(I\frac{1}{1-\rho_0^2}\right)\Omega$. Now take the first term

$$
\begin{aligned}
t_1' &= \sum\nolimits_{\beta,\eta,\kappa,\zeta=0}^{\infty} tr\left\{C_\kappa'\Phi^{-1}C_\zeta C_{\kappa+\zeta+1}'\Phi^{-1}C_{\beta+\eta+1}C_\eta'\Phi^{-1}C_\beta\right\} \\
&= \sum\nolimits_{\beta,\eta,\kappa,\zeta=0}^{\infty} tr\left\{\rho_0^k\left(1-\rho_0^2\right)\rho_0^\zeta\rho_0^{\kappa+\zeta+1}\left(1-\rho_0^2\right)\rho_0^{\beta+\eta+1}\rho_0^\eta\left(1-\rho_0^2\right)\rho_0^\beta I_p\right\} \\
&= \sum\nolimits_{\beta,\eta,\kappa,\zeta=0}^{\infty} tr\left\{\rho_0^2\left(1-\rho_0^2\right)^3\rho_0^{2\zeta}\rho_0^{2\kappa}\rho_0^{2\beta}\rho_0^{2\eta}I_p\right\} \\
&= tr\left\{\rho_0^2\left(1-\rho_0^2\right)^{-1}I_p\right\} \\
&= \frac{\rho_0^2}{\left(1-\rho_0^2\right)}p
\end{aligned}
$$

The third term is derived in the following way:

$$
\begin{aligned}
t_3' &= \sum\nolimits_{\alpha,\eta,\lambda,\zeta=0}^{\infty} tr\left\{C_\lambda'\Phi^{-1}C_\alpha\right\} tr\left\{C_\zeta'\Phi^{-1}C_{\lambda+\zeta+1}\right\} tr\left\{C_{\alpha+\eta+1}'\Phi^{-1}C_\eta\right\} \\
&= \sum\nolimits_{\alpha,\eta,\lambda,\zeta=0}^{\infty} tr\left\{\rho_0^\lambda\left(1-\rho_0^2\right)\rho_0^\alpha I\right\} tr\left\{\rho_0^\zeta\left(1-\rho_0^2\right)\rho_0^{\lambda+\zeta+1}\right\} tr\left\{\rho_0^{\alpha+\eta+1}\left(1-\rho_0^2\right)\rho_0^\eta\right\} \\
&= \sum\nolimits_{\alpha,\eta,\lambda,\zeta=0}^{\infty} p^3\left(\rho_0^2\left(1-\rho_0^2\right)^3\rho_0^{2\alpha}\rho_0^{2\eta}\rho_0^{2\lambda}\rho_0^{2\zeta}\right) \\
&= \frac{\rho_0^2}{\left(1-\rho_0^2\right)}p^3
\end{aligned}
$$

The other 8 terms are derived in an entirely analogous manner. In fact each of them gives a contribution equal to $\left(\frac{\rho_0^2}{\left(1-\rho_0^2\right)}\right)p^s$ where s is the number of different traces in the expression. We obtain the following result:

Corollary 6 *The likelihood ratio test that $\mathcal{H}_0 : B = \rho_0 I$ in model $\mathcal{L}_1$ (10) has the following expected value*

$$
E\left[W_T\right] \stackrel{1}{=} p^2 + \frac{1}{2T}\left(p^2+2p^3-4p\right) + \frac{1}{T}\left(p^3+p^2-2p\right)\left(\frac{\rho_0^2}{\left(1-\rho_0^2\right)}\right) \qquad (11)
$$

The expected value of the likelihood depends on the parameters θ_1 (B in this case) but not on the parameters θ_2 (Ω). This means that when using this correction, no estimated parameters have to be substituted in the Bartlett correction.

We could have substituted the maximum likelihood estimate for B $\hat{B}_{ML} = \left(\sum_{t=1}^{T}X_{t-1}X_{t-1}'\right)^{-1}\left(\sum_{t=1}^{T}X_{t-1}X_t'\right)$, which is $\sqrt{n}$-consistent, and used it in the Bartlett correction, instead of $\rho_0 I$. Both methods are valid in this case. If we did however use $\hat{B}_{ML}$, the expression in corollary 6 will be considerably more complicated.

Now consider model $\mathcal{L}_2$:

$$
\begin{aligned}
X_t &= BX_{t-1} + \varepsilon_t \\
\varepsilon_t &\sim MIIDN(0, I_p)
\end{aligned}
\qquad (12)
$$

with the parameters $\theta = B \in \mathcal{R}^{p\times p}$.

Taniguchi (1988, 1991) derives Bartlett corrections for univariate ARMA-processes and in the special case of an AR(1) process with known variance finds that the expected value of the likelihood ratio equals $1 - \frac{2}{T}$. We thus also state the corollary for model $\mathcal{L}_2$ which is based on theorem 2:

Corollary 7 *The likelihood ratio test that $\mathcal{H}_0 : B = \rho_0 I$ in model $\mathcal{L}_2$ (12) has the following expected value*

$$E\left[W_T\right] \stackrel{1}{=} p^2 - \frac{2p}{T} + \frac{1}{T}\left(p^3 + p^2 - 2p\right)\left(\frac{\rho_0^2}{(1-\rho_0^2)}\right) \tag{13}$$

and conclude that the result of Taniguchi is a special case of (13) with $p = 1$.

Both expectations, (11) and (13) have a pole for $|\rho_0| = 1$. Even though the Bartlett correction is only valid when $|\rho_0| < 1$, it is of interest how close to the pole the Bartlett correction is still of practical use. We thus perform a Monte Carlo study for both corollary 6 and 7.

The DPG is

$$\begin{aligned} X_t &= \left(\rho I_p\right) X_{t-1} + \varepsilon_t \\ \varepsilon_t &\sim MIIDN(0, I_p) \end{aligned} \tag{14}$$

and the parameters of choice are $T = \{100\}$, $\rho = \{-0.9, -0.6, -0.3, 0, 0.3, 0.6, 0.9\}$, $p = \{1, 5\}$ and we test the hypothesis $\mathcal{H}_0 : B = \rho_0 I$ both when Ω is unknown and when it is known. The results are reported in table 2 and are based on 10^5 replications. The Bartlett factor for the case of a one-dimensional process does not depend on any of the parameters and is thus constant over the choice of ρ. For the 5-dimensional VAR, we see that when $|\rho|$ approaches unity, the uncorrected test becomes severely oversized. The Bartlett correction does however somewhat overcorrect, which is what we expected with the pole in the expression. Overall the Bartlett corrected test is closer to the nominal size of the test than the uncorrected one in 69 out of 84 cases.

6 No level feedback in the cointegrated VAR

Let us consider the cointegrated VAR model in the Equilibrium Correction form:

$$\begin{aligned} \Delta X_t &= \Pi X_{t-1} + \sum_{i=1}^{k-1} \Gamma_i \Delta X_{t-i} + \eta_t \\ \eta_t &\sim MIIDN(0, \Omega) \end{aligned} \tag{15}$$

with the following assumptions:

1. Every root z of the characteristic polynomial of X_t satisfies $z = 1$ or $|z| > 1$.
2. $\Pi := -A(1) = \alpha\beta'$, where α and β are $p \times r$ matrices of full rank $r < p$.
3. $\alpha'_\perp \Gamma \beta_\perp$ has full rank $p - r$, where $\Gamma := I - \sum_{i=1}^{k-1} \Gamma_i$.

$T = 100$	Ω unknown (corollary 6)				Ω known (corollary 7)			
	$p = 1$		$p = 5$		$p = 1$		$p = 5$	
	W_T	W_T^{BC}	W_T	W_T^{BC}	W_T	W_T^{BC}	W_T	W_T^{BC}
$\rho = -0.9$								
$E[LR]$	0.997	1.002	31.31	24.28	0.983	1.003	29.33	23.75
BF	-0.005		0.290		-0.020		0.235	
10%	9.95	10.03	32.85	6.96	9.61	9.96	24.23	5.72
5%	4.95	5.00	21.04	3.06	4.76	4.99	14.21	2.36
1%	1.00	1.02	7.00	0.43	0.98	1.05	3.88	0.33
$\rho = -0.6$								
$E[LR]$	0.996	1.001	27.10	25.04	0.981	1.001	25.57	24.88
BF	-0.005		0.083		-0.020		0.028	
10%	9.95	10.03	16.61	10.15	9.65	10.01	10.62	9.64
5%	5.00	5.06	9.32	5.16	4.80	5.02	5.95	4.76
1%	1.00	1.01	2.34	1.03	0.94	1.01	1.22	0.87
$\rho = -0.3$								
$E[LR]$	0.992	0.997	26.47	25.05	0.977	0.997	25.00	24.96
BF	-0.005		0.057		-0.020		0.002	
10%	9.86	9.94	14.51	10.15	9.62	9.97	10.04	9.3
5%	4.93	5.00	7.96	5.16	4.77	4.99	4.95	4.88
1%	0.97	0.99	1.88	1.03	0.93	0.99	0.99	0.97
$\rho = 0$								
$E[LR]$	0.990	0.995	26.33	25.05	0.975	0.995	24.88	24.98
BF	-0.005		0.051		-0.020		-0.004	
10%	9.90	10.00	14.19	10.29	9.64	9.99	9.64	9.93
5%	4.87	4.92	7.58	5.12	4.73	4.97	4.76	4.92
1%	0.96	0.97	1.79	1.04	0.90	0.98	0.93	0.99
$\rho = 0.3$								
$E[LR]$	0.990	0.995	26.48	25.06	0.976	0.996	25.01	24.97
BF	-0.005		0.057		-0.020		0.002	
10%	9.91	9.98	14.54	10.15	9.66	10.03	9.95	9.85
5%	4.90	4.95	7.83	5.11	4.72	4.94	4.99	4.92
1%	0.94	0.96	1.88	1.03	0.88	0.94	0.96	0.94
$\rho = 0.6$								
$E[LR]$	0.993	0.998	27.13	25.06	0.978	0.998	25.60	24.91
BF	-0.005		0.083		-0.020		0.028	
10%	10.04	10.13	16.71	10.08	9.78	10.09	11.64	9.58
5%	5.00	5.05	9.26	4.97	4.83	5.06	5.91	4.65
1%	0.98	1.00	2.34	1.02	0.92	0.99	1.21	0.91
$\rho = 0.9$								
$E[LR]$	0.999	1.004	31.34	24.30	0.984	1.004	29.37	23.79
BF	-0.005		0.290		-0.020		0.235	
10%	9.93	10.02	33.01	6.94	9.67	9.99	24.35	5.74
5%	5.00	5.06	21.09	3.06	4.83	5.04	14.40	2.47
1%	1.05	1.07	6.99	0.44	0.99	1.06	3.98	0.34

Table 2: Bartlett corrections of tests on the autoregressive parameters in the multivariate AR(1) model with unknown and known variance

We consider maximum likelihood estimation as proposed by Johansen (1988).

Divide the variable-vector X_t in two, X_{1t} of dimension $p-s$ and X_{2t} of dimension s $(\leq p-r)$ and the parameters α and Γ_i conformably, that is $\alpha = [\alpha_1', \alpha_2']'$. We then obtain the following system of equations:

$$\Delta X_{1t} = \alpha_1 \beta' X_{t-1} + \sum_{i=1}^{k-1} \Gamma_{1i} \Delta X_{t-i} + \eta_{1t} \tag{16}$$

$$\Delta X_{2t} = \alpha_2 \beta' X_{t-1} + \sum_{i=1}^{k-1} \Gamma_{2i} \Delta X_{t-i} + \eta_{2t} \tag{17}$$

$$\eta_t = \begin{bmatrix} \eta_{1t} \\ \eta_{2t} \end{bmatrix} \sim MIIDN(0, \Omega), \quad \Omega = \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{bmatrix}$$

Conditioning on ΔX_{2t} in equation (16) we obtain the following system.

$$\Delta X_{1t} = \omega \Delta X_{2t} + (\alpha_1 - \omega\alpha_2)\, \beta' X_{t-1} + \sum_{i=1}^{k-1} (\Gamma_{1i} - \omega\Gamma_{2i})\, \Delta X_{t-i} + \tilde{\eta}_{1t} \tag{18}$$

$$\Delta X_{2t} = \alpha_2 \beta' X_{t-1} + \sum_{i=1}^{k-1} \Gamma_{2i} \Delta X_{t-i} + \eta_{2t} \tag{19}$$

$$\tilde{\eta}_t = \begin{bmatrix} \tilde{\eta}_{1t} \\ \eta_{2t} \end{bmatrix} \sim MIIDN(0, \tilde{\Omega}), \quad \tilde{\Omega} = \begin{bmatrix} \Omega_{11} - \omega\Omega_{21} & 0 \\ 0 & \Omega_{22} \end{bmatrix}$$

where we have defined $\omega = \Omega_{12}\Omega_{22}^{-1}$. Furthermore define $\Psi_1^* = (\Gamma_{11} - \omega\Gamma_{21}, \ldots, \Gamma_{1k-1} - \omega\Gamma_{2k-1})$ and $\Psi_2 = (\Gamma_{21}, \ldots, \Gamma_{2k-1})$. The parameters in the conditional equation (18) are $\theta_{con} = (\alpha_1 - \omega\alpha_2, \beta, \Psi_1^*, \omega, \Omega_{11} - \omega\Omega_{21})$ and those in the marginal model (19) read $\theta_{mar} = (\alpha_2, \beta, \Psi_2, \Omega_{22})$. θ_{con} and θ_{mar} do not vary in a product space, such that for inference the whole system (15) needs to be analyzed.

The following concept will offer a way to analyze partial systems:

Definition 8 *There is No Level Feedback (NLF) from the cointegration relations $\beta' X_{t-1}$ to ΔX_{2t} , when ΔX_{2t} does not react to a disequilibrium in the cointegration relations $\beta' X_{t-1}$ that is when $\alpha_2 = 0$.*

This means that the differences ΔX_{2t} do not react directly to a disequilibrium in the cointegration relation. Of course they may still react to past changes in the differences as under NLF Ψ_2 does not necessarily equal zero.

If NLF holds, then the parameters in the marginal equation become $\theta_{mar}^2 = (\Psi_2, \Omega_{22})$. Johansen(1996, theorem 8.1) proves that if $\alpha_2 = 0$, that is NLF from $\beta' X_{t-1}$ to ΔX_{2t}, then the maximum likelihood estimates of β (and α_1) are obtained from the conditional equation (18) only, as θ_{mar}^2 and θ_{con} do vary in a product space.

There are two moments, one can test for NLF: before and after determination of the cointegration space. Even though both tests have the same asymptotic distribution under the null, namely $\chi^2_{s(p-r)}$ they do not have the same small sample properties.

The first test is the one proposed by Harbo et al. (1998) as an ex-post misspecification test after analyzing a conditional system. The second one is a test on the adjustment parameters α before inference on β is made. If the test does not reject conditional inference can be made afterwards. First we shall outline each of these tests in turn and their Bartlett correction. A Monte Carlo simulation study will illustrate the use of the Bartlett correction in each case and show remarkable differences between the two tests.

6.1 Testing NLF after determination of the cointegration space

Harbo et al. (1998) propose to use economic arguments to determine which $s\,(\leq r)$ variables ΔX_{2t} do not react to disequilibria in the cointegration relations. Having assumed NLF from $\beta' X_{t-1}$ to ΔX_{2t} they suggest estimating the rank from the conditional model (18), as this is maximum likelihood estimator if NLF holds. They then go on and and restrict the cointegration space, still using only the conditional model.

After this they propose to do a misspecification test to check whether the initial assumption of NLF was correct. Defining $Z_t = \beta' X_t$ this is done by testing $\mathcal{H}_0 : \alpha_2 = 0$ in

$$\Delta X_{2t} = \alpha_2 Z_{t-1} + \sum_{i=1}^{k-1} \Gamma_{2i} \Delta X_{t-i} + \eta_{2t} \tag{20}$$

by means of a likelihood ratio test. The parameter space in this model is $\theta^3_{mar} = (\alpha_2, \Psi_2, \Omega_{22})$. The null hypothesis only concerns α_2 and not Ψ_2 such that we cannot apply theorem 1 directly. We can however write the expectation of the desired test as the difference between two tests, that are each special cases of theorem 1.

Define the following three models, which successively restrict the parameter space in the marginal model (20):

1. $\mathcal{M}_1$: unrestricted parameters α_2, Ψ_2 and Ω_{22}.
2. $\mathcal{M}_2 : \alpha_2 = 0$, but Ψ_2 and Ω_{22} unrestricted.
3. $\mathcal{M}_3 : \alpha_2 = 0$, $\Psi_2 = \Psi_{20}$ and Ω_{22} unrestricted.

Let $\left(\tilde{\alpha}_2, \tilde{\Psi}_2\right)$ be the maximum likelihood estimators of $\mathcal{M}_1$ and $\hat{\Psi}_2$ those of $\mathcal{M}_2$. Then the test that $\alpha_2 = 0$ in $\mathcal{M}_1$, that is $\mathcal{M}_2$ in $\mathcal{M}_1$ can be written as:

$$\begin{aligned} LR\left(\mathcal{M}_2|\mathcal{M}_1\right) &= \frac{L\left(\alpha_2 = 0, \hat{\Psi}\right)}{L\left(\tilde{\alpha}_2, \tilde{\Psi}\right)} \\ &= \frac{L\left(\alpha_2 = 0, \hat{\Psi}\right)}{L\left(\alpha_2 = 0, \Psi = \Psi_0\right)} \times \frac{L\left(\alpha_2 = 0, \Psi = \Psi_0\right)}{L\left(\tilde{\alpha}_2, \hat{\Psi}\right)} \end{aligned}$$

This means that the log-likelihood ratio test can be written as the difference between two log-likelihood ratio tests:

$$-2\ln LR\left(\mathcal{M}_2|\mathcal{M}_1\right) = -2\ln LR\left(\mathcal{M}_3|\mathcal{M}_1\right) + 2\ln LR\left(\mathcal{M}_3|\mathcal{M}_2\right)$$

such that to get the Bartlett correction, we just have to take the difference between the two expectations. To see how these tests are both special cases of theorem 1, rewrite the stationary part of the cointegrated VAR model (15) in the following moving average form:

$$\begin{bmatrix} \Delta X_t \\ \Delta X_{t-1} \\ \vdots \\ \Delta X_{t-k+3} \\ \Delta X_{t-k+2} \\ Z_t \end{bmatrix} = \begin{bmatrix} \Gamma_1 & \Gamma_2 & \dots & \dots & \Gamma_{k-1} & \alpha \\ I_p & 0 & \dots & \dots & \dots & 0 \\ 0 & \ddots & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ 0 & \dots & 0 & I_p & 0 & 0 \\ \beta'\Gamma_1 & \dots & \dots & \dots & \beta'\Gamma_{k-1} & \beta'\alpha + I_r \end{bmatrix} \begin{bmatrix} \Delta X_{t-1} \\ \Delta X_{t-2} \\ \vdots \\ \Delta X_{t-k+2} \\ \Delta X_{t-k+1} \\ Z_{t-1} \end{bmatrix} + \begin{bmatrix} I_p \\ 0 \\ \vdots \\ \vdots \\ 0 \\ \beta' \end{bmatrix} \eta_t \tag{21}$$

$$Y_t = DY_{t-1} + E\eta_t \tag{22}$$

$$\eta_t \sim MIIDN(0, \Omega) \tag{23}$$

The regressors in $\mathcal{M}_1$ are Y_{t-1}. These can be written in terms of the $MIIDN(0, \Omega)$ process η_t as $Y_{t-1} = \sum_{i=0}^{\infty} G_i \eta_{t-1-i}$ where

$$G_i = D^i E \qquad \text{for } i = 0, 1, \dots \tag{24}$$

$$H_i = D^i E \Omega^{\frac{1}{2}} = D^i F \qquad \text{for } i = 0, 1, \dots \tag{25}$$

In the last line we defined $\{H_i\}$ by postmultiplying $\{G_i\}$by $\Omega^{\frac{1}{2}}$, just as we postmultiplied $\{Q_i\}$ to obtain $\{C_i\}$ and then expressed the theorems in terms of $\{C_i\}$. Next define the matrix S which selects the first differences and the lagged first differences, but not the cointegration relationships from Y_{t-1} as:

$$S = \left[I_{p(k-1)}, 0_{p(k-1)\times r} \right]' \tag{26}$$

such that $S'Y_{t-1}$ are the regressors in $\mathcal{M}_2$ and we obtain the following expressions for its polynomial

$$N_i = S'D^i E \qquad \text{for } i = 0, 1, \dots \tag{27}$$

$$O_i = S'D^i E \Omega^{\frac{1}{2}} = S'D^i F \qquad \text{for } i = 0, 1, \dots \tag{28}$$

For future reference we also define the variance of the process Y as Σ_{yy}:

$$\Sigma_{yy} = var\left(Y_t\right) \tag{29}$$

In $\mathcal{M}_1$ the dimension of the coefficient matrix is $s \times ((k-1)\, p + r)$, whereas in $\mathcal{M}_2$ it is $s \times (k-1)\, p$. The null hypothesis is $\mathcal{H}_0 : \alpha_2 = 0$. Consequently the Bartlett factor can be used and the expectation of the likelihood ratio is given in the following corollary:

Corollary 9 *The likelihood ratio for $\mathcal{H}_0 : \alpha_2 = 0$ in (19) has the following expected value:*

$$E\left[-2\ln LR\left(\mathcal{M}_2|\mathcal{M}_1\right)\right] \overset{1}{=} sr + \frac{1}{2T}\left(sr + s^2 r + sr^2 + 2rsp(k-1)\right)$$
$$+ \frac{1}{T}\urcorner\left(\left(k-1\right)p + r, s, \{H_i\}\right) - \frac{1}{T}\urcorner\left(\left(k-1\right)p, s, \{O_i\}\right)$$

where H_i and O_i are defined in (25) and (28) respectively and $\urcorner$ is defined in theorem 1.

$\Psi = \sum_{i=0}^{\infty} D^i F F' D^{i\prime}$	$A_3 = V' S' \Phi^{-1} S V \Lambda$	$A_7 = \left(I - \Lambda^2\right)^{-1}$
$P = S' \left(S \Psi S'\right)^{-1} S \Psi$	$A_4 = V^{-1} P' V$	$A_8 = \left(l l' - \Lambda^{co} \Lambda^{ro}\right)$
$A_1 = V^{-1} F I_{22} F' V^{-1\prime}$	$A_5 = V' S' \Phi^{-1} S V$	$A_{9i} = \left(I_n - \upsilon_i \Lambda\right)^{-1}$
$A_2 = V' P V^{-1\prime} \Lambda$	$A_6 = V' \Phi^{-1} V$	$A_{9j} = \left(I_n - \upsilon_j \Lambda\right)^{-1}$

Table 3: Definition of a number of terms for theorems 10 and 11

The two expression for $\daleth$ in corollary 9 contain infinite loops, but due to their structure $\{H_i\}$ and $\{O_i\}$ in equations (25) and (28) can be simplified, such that the expressions can be computed exactly.

Let

$$\upsilon_1, \ldots, \upsilon_n \tag{30}$$

be the (possibly complex) eigenvalues of D and $w_1, \ldots, w_n$ the corresponding eigenvectors. Then define:

$$V = \left[\begin{array}{ccc} w_1 & \ldots & w_n \end{array}\right] \tag{31}$$

$$\Lambda = \left[\begin{array}{ccc} \upsilon_1 & & \\ & \ddots & \\ & & \upsilon_n \end{array}\right] \tag{32}$$

$$\Lambda^{ro} = \left[\begin{array}{ccc} \upsilon_1 & \ldots & \upsilon_n \end{array}\right] \tag{33}$$

$$\Lambda^{co} = \left[\begin{array}{c} \upsilon_1 \\ \vdots \\ \upsilon_n \end{array}\right] \tag{34}$$

$$l_{n\times 1} = \left[\begin{array}{ccc} 1 & \ldots & 1 \end{array}\right]' \tag{35}$$

A number of terms, which are expressed in terms of υ_i,V,Λ,$\Lambda^{ro}$and Λ^{co} are given in table 3. They are used in the following two theorems.

Theorem 10 *If $C_i = S' D^i F$ for $i \geq 0$ then the expression for $\daleth$ in theorem 1 simplifies* *to:*

$$
\begin{aligned}
\daleth &= tr\left\{A_1\left(A_2 \oslash A_8\right) A_3\left(A_4 \oslash A_8\right)\right\} \\
&+ 2\sum_{i=1}^{n}\left(A_2\right)_{ii} tr\left\{A_1\left(A_2 \oslash A_8\right) A_5 A_{9i}\right\} \\
&+ \sum_{i,j=1}^{n}\left(A_2\right)_{ii}\left(A_2\right)_{jj} tr\left\{A_1 A_{9i} A_5 A_{9j}\right\} \\
&+ \sum_{i,j,k,m=1}^{n} \frac{\left(A_1\right)_{ij}\left(A_2\right)_{jk}\left(A_3\right)_{km}\left(A_4\right)_{mi}}{(1-\upsilon_j\upsilon_m)(1-\upsilon_i\upsilon_k)} \\
&+ 2\sum_{i,j,k,m=1}^{n} \frac{\left(A_1\right)_{ij}\left(A_2\right)_{jk}\left(A_2\right)_{km}\left(A_5\right)_{mi}}{(1-\upsilon_j\upsilon_m)(1-\upsilon_i\upsilon_k)} \\
&+ \sum_{i,j=1}^{n}\left(A_2\right)_{ji}\left(A_2\right)_{ij} tr\left\{A_1 A_{9i} A_5 A_{9j}\right\} \\
&- 2\sum_{i=1}^{n}\left(A_4\right)_{ii} \upsilon_i^2 tr\left\{A_1 A_{9i} A_5 A_{9i}\right\} \\
&- 2tr\left\{A_1\left(A_4' \oslash A_8\right)\Lambda^2\left(A_5 \oslash A_8\right)\right\} \\
&- 2tr\left\{\left(A_1 \oslash A_8\right) A_3\left(A_4 \oslash A_8\right)\Lambda\right\} \\
&- 2tr\left\{\left(A_1 \oslash A_8\right)\Lambda\left(A_3 \oslash A_8\right) A_4\right\}
\end{aligned}
$$

where relevant definitions are given in equations (30)-(34) and in table 3. $\oslash$ denotes Hadamard division. For three matrices A, B and C of equal dimension $C = A \oslash B$ is the matrix with entries $c_{ij} = a_{ij}/b_{ij}$.

Proof. see section A.12 ■

Theorem 11 *If $C_i = D^i F$ for $i \geq 0$ then the expression for $\daleth$ in theorem 1 simplifies to:*

$$
\begin{aligned}
\daleth &= tr\left\{A_1 \Lambda A_7 A_6 \Lambda A_7\right\} \\
&+ 2\sum_{i=1}^{n} \upsilon_i tr\left\{A_1 \Lambda A_7 A_6 A_{9i}\right\} \\
&+ \sum_{i,j=1}^{n} \upsilon_i \upsilon_j tr\left\{A_1 A_{9i} A_6 A_{9j}\right\} \\
&- tr\left\{\left(\left(\Lambda A_1 \Lambda\right) \oslash A_8\right)\left(A_6 \oslash A_8\right)\right\} \\
&- 2tr\left\{A_1 A_7 \Lambda^2\left(A_6 \oslash A_8\right)\right\} \\
&- \sum_{i=1}^{n} \upsilon_i^2 tr\left\{A_1 A_{9i} A_6 A_{9i}\right\}
\end{aligned}
$$

where relevant definitions are given in equations (30)-(34) and and in table 3.

Proof. see section A.13. ■

Both expressions are quickly programmed and as they contain only finite loops[1], the first order expansion of the expectation of the likelihood ratio test statistic can be calculated exactly.

6.2 Testing NLF before determination of the cointegration space

Under the assumption of NLF from $\beta' X_{t-1}$ to ΔX_{2t} the parameters of the conditional model (18) θ_{con} and those in the marginal model (19) θ^2_{mar} vary in a product space, such

[1]Note that $\Psi = V\left(\left(V^{-1}FF'V^{-1\prime}\right) \oslash \left(ll' - \Lambda^{co}\Lambda^{ro}\right)\right)V'$ such that only finite loops remain for the expression in table 3.

that ΔX_{2t} is weakly exogenous for β. The aim of the test for NLF is thus to be able to do inference on β in the conditional model only.

We can find the Bartlett correction for that test, but once again we need to take differences between likelihood ratio tests to be able to find a first order approximation to the expectation of the test of interest. Define the following models:

1. $\mathcal{N}_{-1}$: matrix Π is of full rank p.
2. $\mathcal{N}_0$: unrestricted parameters in the cointegrated VAR, equation (15)
3. $\mathcal{N}_1 : \beta = \beta_0 \phi$
4. $\mathcal{N}_{1a} : \alpha = \alpha_0 \psi$
5. $\mathcal{N}_2 : \beta = \beta_0, \alpha = \alpha_0$

where ϕ and ψ are $(r \times r)$ matrices of full rank.

The difference between $\mathcal{N}_2$ and $\mathcal{M}_2$ is that in $\mathcal{M}_2$ s $(\leq p-r)$ rows equal zero and the others are estimated freely. In $\mathcal{N}_2$ the whole column space of α is fixed. This implies that $LR\left(\mathcal{N}_2|\mathcal{N}_1\right)$ is a special case of $LR\left(\mathcal{M}_2|\mathcal{M}_1\right)$.

Our interest focuses on $LR\left(\mathcal{N}_{1a}|\mathcal{N}_0\right)$ which can be written as:

$$LR\left(\mathcal{N}_{1a}|\mathcal{N}_0\right) = \frac{L\left(\mathcal{N}_{1a}\right)}{L\left(\mathcal{N}_2\right)} \times \frac{L\left(\mathcal{N}_2\right)}{L\left(\mathcal{N}_1\right)} \times \frac{L\left(\mathcal{N}_1\right)}{L\left(\mathcal{N}_0\right)} \tag{36}$$

such that we find:

$$-2\ln LR\left(\mathcal{N}_{1a}|\mathcal{N}_0\right) = +2\ln LR\left(\mathcal{N}_2|\mathcal{N}_{1a}\right) - 2\ln LR\left(\mathcal{N}_2|\mathcal{N}_1\right) - 2\ln LR\left(\mathcal{N}_1|\mathcal{N}_0\right)$$

In this section we have already derived the first order approximation to the expectation of $-2\ln LR\left(\mathcal{N}_2|\mathcal{N}_1\right)$, whereas Johansen (2000) derives that of $-2\ln LR\left(\mathcal{N}_1|\mathcal{N}_0\right)$ and Johansen (2002a) contains the one for $-2\ln LR\left(\mathcal{N}_2|\mathcal{N}_{1a}\right)$. We can simply add up the three expectations of these terms to find the Bartlett correction of the test for $-2\ln LR\left(\mathcal{N}_{1a}|\mathcal{N}_0\right)$.

All three tests concern the whole system of equations, namely (15), but $-2\ln LR\left(\mathcal{N}_2|\mathcal{N}_1\right)$ is done in the marginal equation only, as we saw in the last paragraph. Adding up the three expressions we obtain:

Corollary 12 *For unknown cointegration parameter β the likelihood ratio for $\mathcal{H}_0 : \alpha_2 = 0$ in (19) has the following expected value:*

$$\begin{aligned} E\left[-2\ln LR\left(\mathcal{N}_{1a}|\mathcal{N}_0\right)\right] &\stackrel{1}{=} r\left(p-r\right) + \frac{1}{2T}\left(r^2 + 2r^3 + 2r^2 p(k-1)\right) \\ &+ \frac{1}{T}\urcorner\left(\left(k-1\right)p + r, r, \{H_i\}\right) - \frac{1}{T}\urcorner\left(\left(k-1\right)p, r, \{O_i\}\right) \\ &+ \frac{1}{T} r^2 tr\left\{\left(\alpha'\Omega^{-1}\alpha\right)^{-1} S_\perp' \Sigma_{yy}^{-1} S_\perp\right\} \end{aligned}$$

where H_i, O_i, S and Σ_{yy} are defined in $(25)-(29)$ and $\urcorner$ is defined in theorem 1.

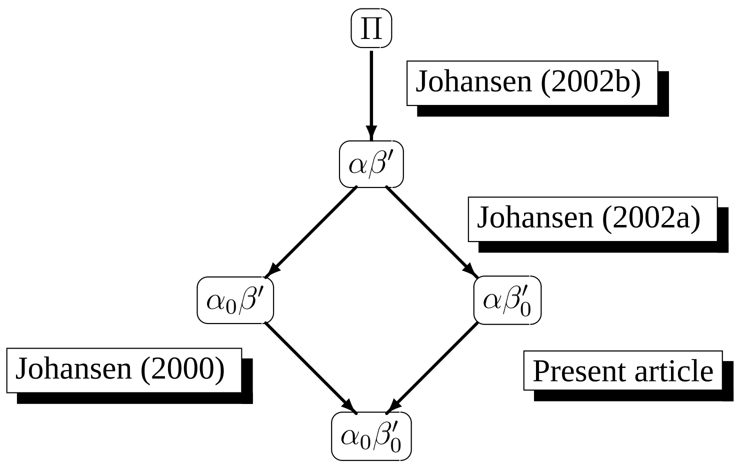


Figure 2: Overview of Bartlett corrections in the cointegrated VAR

For completeness we state that Johansen (2002b) derives the Bartlett correction for the rank test, that is for $LR\left(\mathcal{N}_0|\mathcal{N}_{-1}\right)$ and graphically represent this information in figure 2.

Equation (36) shows that we are able to Bartlett correct the one test in the diagram, for which the Bartlett correction has not been derived explicitly. We do stress that whereas the Bartlett corrections in Johansen (2000, 2002a,b) allow for certain deterministic terms, the one in this paper does not and is therefore somewhat less general.

6.3 A Monte Carlo study of the test for NLF

We perform a Monte Carlo study of the two tests for no long run feedback and use the following 5-dimensional Data Generating Process:

$$\begin{aligned} \phi_1(L) X_{1t} &= \varepsilon_{1t} \\ \phi_2(L) X_{2t} &= \varepsilon_{2t} \\ g(L) X_{it} &= \varepsilon_{it} \qquad \text{for } i = 3,4,5 \\ \varepsilon_t &\sim MIIDN(0, I_n) \end{aligned}$$

where

$$\begin{array}{lll} \phi_1(L) = \prod_{i=1}^{k}(1-\phi_{1i}L) & \varphi_1 = \left[\begin{array}{ccc} \phi_{11} & \ldots & \phi_{1k} \end{array}\right] & max\left(|\phi_{1i}|\right) < 1 \\ \phi_2(L) = \prod_{i=1}^{k}(1-\phi_{2i}L) & \varphi_2 = \left[\begin{array}{ccc} \phi_{21} & \ldots & \phi_{2k} \end{array}\right] & max\left(|\phi_{2i}|\right) < 1 \\ g(L) = \prod_{i=1}^{k}(1-g_iL) & \gamma = \left[\begin{array}{ccc} g_1 & \ldots & g_k \end{array}\right] & max\left(|g_i|\right) = 1 \end{array}$$

The first two variables are stationary, whereas the last three each contain exactly one unit root. As the calculation of the Bartlett correction is computer-intensive (in a simulation framework) and in order to keep the size of this experiment under control, we have opted for a benchmark case and then varied one or two aspects of the benchmark DGP.

When we rewrite the model in the equilibrium correction form (15), then α and β take

		β known (corollary 9)				β unknown (corollary 12)			
		W_T	W_T^{BC}			W_T	W_T^{BC}		
			θ	$\hat{\theta}_r$	$\hat{\theta}_u$		θ	$\hat{\theta}_r$	$\hat{\theta}_u$
Experiment 1	$E[LR]$	6.71	6.07	6.08	6.10	11.49	7.68	8.65	9.21
$\varphi_1 = [0.8, 0.6]$	BF	0.106				0.496			
$\varphi_2 = [0.8, 0.6]$	10%	14.8	10.4	10.4	10.5	51.0	19.8	27.9	32.6
$\gamma = [1, \epsilon]$	5%	8.0	5.2	5.2	5.2	37.6	10.5	16.4	21.0
$T = 100$	1%	1.8	0.7	0.7	0.8	16.5	1.8	4.2	6.7
Experiment 2	$E[LR]$	6.28	5.96	5.97	5.97	8.53	6.83	7.08	7.28
$\varphi_1 = [0.8, 0.6]$	BF	0.053				0.248			
$\varphi_2 = [0.8, 0.6]$	10%	11.7	9.8	9.8	9.8	27.0	14.7	16.35	17.9
$\gamma = [1, \epsilon]$	5%	6.3	5.3	5.3	5.3	17.2	8.6	9.5	10.8
$\mathbf{T = 200}$	1%	2.0	1.4	1.4	1.5	7.2	2.4	2.8	3.3
Experiment 3	$E[LR]$	6.09	5.93	5.93	5.93	6.91	6.15	6.21	6.27
$\varphi_1 = [0.8, 0.6]$	BF	0.026				0.124			
$\varphi_2 = [0.8, 0.6]$	10%	11.0	10.0	10.0	10.0	16.2	11.0	11.4	11.8
$\gamma = [1, \epsilon]$	5%	5.4	4.8	4.8	4.8	8.7	5.6	5.6	6.1
$\mathbf{T = 400}$	1%	1.0	0.8	0.8	0.8	2.6	1.3	1.4	1.6
Experiment 4	$E[LR]$	7.46	6.11	6.25	6.31	12.59	7.82	9.08	9.72
$\varphi_1 = [0.8, 0.6]$	BF	0.221				0.611			
$\varphi_2 = [0.8, 0.6]$	10%	21.2	11.6	11.6	12.2	59.0	20.1	31.6	36.6
$\gamma = [1, 0.\mathbf{6}]$	5%	12.0	5.8	5.8	6.3	45.3	10.0	18.4	24.4
$T = 100$	1%	3.1	0.9	0.9	1.0	21.6	2.3	5.5	7.7
Experiment 5	$E[LR]$	6.74	6.06	6.07	6.09	13.03	7.19	9.54	10.29
$\varphi_1 = [0.8, \mathbf{0.8}]$	BF	0.112				0.812			
$\varphi_2 = [0.8, \mathbf{0.8}]$	10%	14.8	10.3	10.3	10.4	61.7	15.0	35.6	42.1
$\gamma = [1, \epsilon]$	5%	8.2	5.4	5.4	5.6	47.3	6.7	23.1	28.2
$T = 100$	1%	1.8	0.9	0.9	0.9	24.8	0.9	5.9	10.2
Experiment 6	$E[LR]$	6.48	6.08	6.06	6.06	9.78	7.28	7.80	8.13
$\varphi_1 = [0.8, \mathbf{-0.6}]$	BF	0.066				0.343			
$\varphi_2 = [0.8, \mathbf{-0.6}]$	10%	13.1	10.2	10.2	10.2	37.9	17.0	21.7	24.0
$\gamma = [1, \epsilon]$	5%	6.8	5.3	5.3	5.3	25.2	9.3	12.2	14.9
$T = 100$	1%	1.8	1.3	1.3	1.3	9.6	2.0	2.9	4.3
Experiment 7	$E[LR]$	6.45	6.02	6.01	6.01	7.22	6.28	6.34	6.39
$\varphi_1 = [\mathbf{0.6}, \mathbf{-0.6}]$	BF	0.072				0.150			
$\varphi_2 = [\mathbf{0.6i}, \mathbf{-0.6i}]$	10%	13.4	10.7	10.7	10.7	18.8	12.4	12.9	13.6
$\gamma = [1, \epsilon]$	5%	7.4	5.4	5.4	5.3	11.2	6.0	6.2	6.4
$T = 100$	1%	1.2	0.8	0.8	0.8	2.5	1.2	1.3	1.4
Experiment 8	$E[LR]$	7.92	6.34	6.41	6.45	14.37	8.53	9.87	10.65
$\varphi_1 = [0.8, 0.6, \mathbf{0.2}, \mathbf{0.2}]$	BF	0.250				0.684			
$\varphi_2 = [0.8, 0.6, \mathbf{0.2}, \mathbf{0.2}]$	10%	23.5	12.5	12.5	12.7	67.6	27.2	38.1	43.8
$\gamma = [1, \epsilon, \epsilon, \epsilon]$	5%	14.7	6.7	6.7	7.0	54.7	16.2	25.6	31.2
$T = 100$	1%	4.7	1.2	1.2	1.2	31.8	4.1	8.45	12.6

Table 4: Bartlett corrections for two tests of no level feedback in the cointegrated VAR. The variations with respect to Experiment 1 are given in bold face. $\epsilon = 10^{-4}$ (If ϵ were equal to zero, Φ would be of reduced rank in the DGP)

the following values:

$$\alpha' = \begin{bmatrix} \alpha_{11} & 0 & 0 & 0 & 0 \\ 0 & \alpha_{22} & 0 & 0 & 0 \end{bmatrix}, \quad \beta' = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\alpha_{11} = \sum_{i=1}^{k} \phi_{1i} - 1$$

$$\alpha_{22} = \sum_{i=1}^{k} \phi_{2i} - 1$$

We vary the following aspects of the DGP: T (the number of observations), k (the number of lags) and φ_1, φ_2 and γ. For each experiment we report two tests (in their uncorrected and corrected versions): the test that the last three rows of the adjustment parameters α are zero for known cointegration space β and for unknown β. Under $\mathcal{H}_0$ both tests asymptotically have a χ^2-distribution with six degrees of freedom and the Bartlett correction for the first test is given in corollary 9.Corrollary 12 provides the expression for the second test.

Each of these Bartlett corrections depends on the parameters of the model. We calculate the Bartlett correction based on

1. The true (DGP) value of the parameters, θ.
2. The maximum likelihood estimates of the parameters under $\mathcal{H}_0, \hat{\theta}_r$.
3. The maximum likelihood estimates of the parameters under the alternative, $\hat{\theta}_u$.

Omtzigt and Fachin (2002) argue that for the test of corollary 12 one needs to use $\hat{\theta}_u$ as $lim_{T\to\infty}tr\left\{(\alpha'\Omega^{-1}\alpha)^{-1} S_\perp' \Sigma_{yy}^{-1} S_\perp\right\}$ is not defined under the alternative. Their point does not apply to the test in corollary 9.

The simulation is based on 2000 replications and for each test we report the expected value of the likelihood ratio test, as well as the expected value of the Bartlett corrected test based on $\theta, \hat{\theta}_r$ and $\hat{\theta}_u$.We also give the Bartlett factor based on θ. As before we report the empirical rejection probabilities at the nominal 10%, 5% and 1% level.

In the benchmark model (experiment 1), both stationary variables, X_{1t} and X_{2t} have relatively large residual roots at 0.6 and 0.8. The other three series are pure random walks[2] and we have 100 observations. The first block-row of table 4 shows that the Bartlett correction in the test for known β performs well: at the 5% nominal level, it corrects from 8.0% to 5.2% for all three Bartlett corrected tests. For unknown β the results are different. The original size distortion is considerably larger, as the empirical size of the asymptotic test at the nominal 5% level is 37.6%. The Bartlett correction based on the true value brings this down to 10.5%, but those based on the restricted and unrestricted estimates only bring it down to 16.4% and 21.0% respectively. Even for T=200 (experiment 2) the corrected test remains size distorted. Four hundred observations (experiment 3) are needed for the corrected test to reach a rejection probability close to 5%. In experiments 4 and 5, the smallest residual roots in the non-stationary and

[2]There are one or three very small extra small roots in the polynomial, which are $\epsilon = 10^{-4}$. They serve no other purpose than to ensure invertibility of the matrix Φ.

stationary variables respectively are raised. The Bartlett correction for the test based on known β continues to perform well, but the one based on unknown β does even worse than in the benchmark case. If the roots are more scattered on the real line (experiment 6) or inside the unit circle (experiment 7), the performance of the Bartlett corrected test with unknown β is more acceptable. The size corrections perform worse with a longer lag length (experiment 8), which is in line with the findings in Omtzigt and Fachin (2002) .

Overall the Bartlett correction performs better when β is known than when it is unknown, though this may be specific to the Monte Carlo design chosen and the larger size distortion of the non-corrected test.

In figure 3 we give the QQ-plots of the uncorrected and corrected test in experiment 1, based on 20000 replications. We observe that the plots on the left hand side, which correspond to corollary 9 are straight and that all three corrected test virtually coincide with the 45 degree line, showing the effectiveness of the Bartlett correction. The plots on the right hand side correspond to the case where β is unknown and in none of the four plots does the empirical QQ-plot coincide with the 45 degree line. However all four plots are almost straight lines. (In the bottom two rows, the Bartlett correction depends on the estimated parameters, such that the Bartlett correction does not just rotate the QQ-plot. Potentially it can also change the curvature). The relatively straight line and the fact that the correction functions with 400 observations are consistent with the view that a higher order expansion of the expectation of the likelihood ratio test is needed in this case. Nielsen (1997) and Johansen (2002b) provide examples of Bartlett corrections in which higher order terms are needed to make the Bartlett correction function.

7 Conclusions

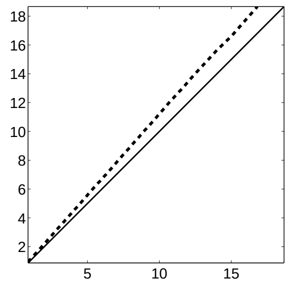
We have derived the Bartlett correction for a simple hypothesis on the regression parameters in a multivariate stationary autoregressive process. Three applications illustrate the use of the correction: the test for absence of autocorrelation of any order, a simple hypothesis on the autoregressive parameters and two tests for no long run feedback in the cointegrated VAR model. In the first of these last two tests, the cointegration space is known, in the second it is not. In all sections explicit expressions for the Bartlett correction are given.

The Bartlett correction performs well in all simulation studies, except in the one of the last test, that is a test for weak exogeneity in the cointegrated VAR with an unknown cointegration space. In that particular case a second order expansion might improve the Bartlett correction.

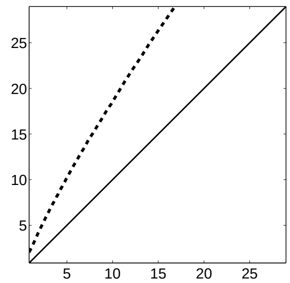
References

Attfield, C. (1995). A Bartlett adjustment to the likelihood ratio test for a system of equations. *Journal of Econometrics 66*, 207–223.

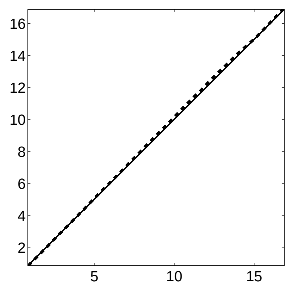
Attfield, C. (1998). Bartlett adjustments for systems of linear equations with linear restrictions. *Economic Letters 60*, 277–283.



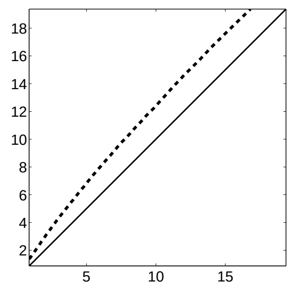
(a) β known, not corrected



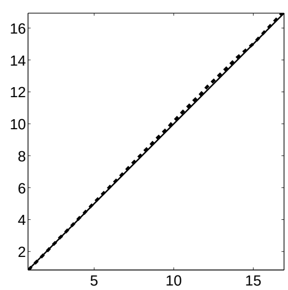
(b) β unknown, not corrected



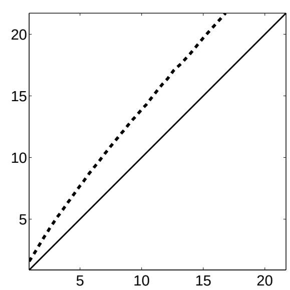
(c) β known, BC based on θ



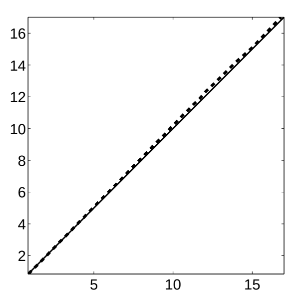
(d) β unknown, BC based on θ



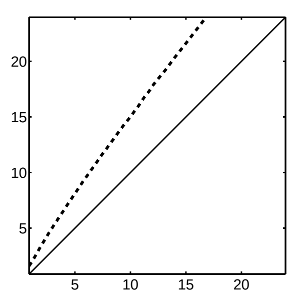
(e) β known, BC based on $\hat{\theta}_r$



(f) β unknown, BC based on $\hat{\theta}_r$



(g) β known, BC based on $\hat{\theta}_u$



(h) β unknown, BC based on $\hat{\theta}_u$

Figure 3: QQ-plots of LR-tests (asymptotic and Bartlett corrected) for 'no level feedback' in the cointegrated VAR model

Barndorff-Nielsen, O. and P. Hall (1988). On the level-error after Bartlett adjustment of the likelihood ratio statistic. *Biometrika 75*, 374–378.

Bartlett, M. (1937). Properties of sufficiency and statistical tests. *Proceedings of the Royal Society of London Series A 160*, 268–282.

Cribari-Neto, F. and G. Cordeiro (1996). On Bartlett and Bartlett-type corrections. *Econometric Reviews 15*, 339–367.

Harbo, I., S. Johansen, B. Nielsen, and A. Rahbek (1998). Test for cointegrating rank in partial systems. *Journal of Business and Economic Statistics 16*, 388–399.

Jensen, J. (1993). A historical sketch and some new results on the improved log likelihood ratio statistic. *Scandinavian Journal of Statistics 20*, 1–15.

Jensen, J. and A. Wood (1997). On the non-existence of a Bartlett correction for unit root tests. *Statistics and Probability Letters 35*, 181–187.

Johansen, S. (1988). Statistical analysis of cointegration vectors. *Journal of Economic Dynamics and Control 12*, 231–254.

Johansen, S. (1991). Estimation and hypothesis testing of cointegration vectors in Gaussian vector autoregressive processes. *Econometrica 59*, 1551–1580.

Johansen, S. (2000). A Bartlett correction factor for tests on the cointegrating relations. *Econometric Theory 16*, 740–778.

Johansen, S. (2002a). A small sample correction for tests of hypotheses on the cointegrating vectors. *Journal of Econometrics 111*, 195–221.

Johansen, S. (2002b). A small sample correction for the test of cointegrating rank in the vector autoregressive model. *Econometrica 70*, 1929–1961.

Lawley, D. (1956). A general method for approximating the distribution for the likelihood ratio criteria. *Biometrika 43*, 293–303.

Nielsen, B. (1997). Bartlett correction of the unit root test in autoregressive models. *Biometrika 84*, 500–504.

Omtzigt, P. and S. Fachin (2002). Bootstrapping and Bartlett corrections in the cointegrated VAR model. Technical Report 2002/15, UvA-Econometrics, University of Amsterdam, Amsterdam. Available at http://www.fee.uva.nl/ke/Uva-Econometrics. (Chapter four of this thesis).

Taniguchi, M. (1988). Asymptotic expansions of the distributions of some test for Gaussian ARMA processes. *Journal of Multivariate Analysis 27*, 494–511.

Taniguchi, M. (1991). *Higher Order Asymptotic Theory for Time Series Analysis*. Berlin: Springer-Verlag.

A Derivation of the main results

In this appendix we prove theorem 1 and 2. In the first subsection we derive a number of useful lemma's, which will be applied over and again in the theorems. Then theorem 1 is derived. Theorem 2 is derived in subsection A.11: the short proof is in some way a special case of theorem 1. Theorems 11 and 10 are derived in subsections A.13 and A.12 respectively.

A.1 Lemma's

To prove the two theorems and their corollaries, we shall state a few useful lemma's. The first one states that in all the estimation problems we consider in this paper, the Ordinary Least Squares (OLS) estimator and Maximum Likelihood (ML) estimators coincide:

Lemma 13 *If A varies unrestricted in a product space, that is $A \in R^{n\times(q+r)}$ in the model:*

$$\begin{bmatrix} Y_{1t} \\ Y_{2t} \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} X_t + \begin{bmatrix} \varepsilon_t \\ 0 \end{bmatrix} \tag{37}$$
$$\varepsilon_{1t} \sim MIIDN(0,\Omega)$$

then the maximum likelihood estimator of A, $\tilde{A}$ and the OLS-estimator of $A, \hat{A}$ coincide. Furthermore $\hat{A}_2 = A_2$

Proof. In the first sub model $Y_{1t} = A_1X_t + \varepsilon_{1t}$, $\hat{A}_1 = (X'X)^{-1} X'Y_1$ whereas in the full model (37) $\hat{A} = (X'X)^{-1} X'Y$ which implies $\hat{A}_1 = (I,0)(X'X)^{-1} X'Y = (X'X)^{-1} X'Y_1$. Therefore the OLS estimators in the two small submodels coincide with the OLS estimator of the large model (37)
The variance-covariance matrix of $\begin{bmatrix} \varepsilon_{1t} & 0 \end{bmatrix}'$ is trivially block-diagonal with Ω and 0 as diagonal elements. Therefore maximization of the likelihood function of (37) is the same as the separate maximization of the likelihood functions of the two submodels.
In the second sub model $Y_{2t} = A_2X_t$,$\hat{A}_2 = (X'X)^{-1} X'Y_2 = A_2$ as we are estimating an identity. This estimator trivially equals the maximum likelihood estimator. The ML-estimator of the first submodel equals the OLS-estimator as $A_1 \in R^{n\times q}$ ■

Next we state two standard result on the products of the errors in the multivariate normal distribution:

Lemma 14 *Let $\varepsilon_i = [\varepsilon_{1i}', \varepsilon_{2i}']', i = 1, ..., T$ be $(n \times 1)$ vectors, distributed i.i.d.$N(0, I_n)$ and let ε_{2i} be of dimension $q \leq n$. Further let M be an $(n \times n)$ matrix.*
Then:

$$E[\varepsilon_i' M \varepsilon_j] = \begin{cases} tr\{M\} & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

$$E[D] = E\left[\varepsilon_{2i}\varepsilon_j' M \varepsilon_k \varepsilon_{2l}'\right] = \begin{cases} M_{22} + M_{22}' + I_q tr\{M\} & \text{if } i = j = k = l \\ I_q tr\{M\} & \text{if } i = l \neq j = k \\ M_{22}' & \text{if } i = k \neq j = l \\ M_{22} & \text{if } i = j \neq k = l \\ 0 & \text{otherwise} \end{cases}$$

Proof. First let $\varepsilon_i = \begin{bmatrix} \epsilon_i^1 & \epsilon_i^2 & \cdots & \epsilon_i^n \end{bmatrix}'$, $\varepsilon_{2i} = \begin{bmatrix} \epsilon_i^{n-q+1} & \cdots & \epsilon_i^n \end{bmatrix}'$ and denote the element in row a and column b of matrix M as m^{ab}. Then let $L_{q\times n} = [0, I_q]$. Throughout we use the fact that the first and third moment of this normal distribution are zero and that $E\left[\epsilon_i^a \epsilon_j^b\right] = 1$ iff $a = b$ and $i = j$.

- $E\left[\varepsilon_i' M \varepsilon_i\right] = E\left[\sum_{a=1}^n \sum_{b=1}^n \epsilon_i^a m^{ab} \epsilon_i^b\right] = E\left[\sum_{a=1}^n \epsilon_i^a m^{aa} \epsilon_i^a\right] = tr\left\{M\right\}$
- If $i = j \neq k = l$, then $E\left[\varepsilon_{2i}\varepsilon_j' M \varepsilon_k \varepsilon_{2l}'\right] = E\left[\varepsilon_{2i}\varepsilon_i'\right] M E\left[\varepsilon_k \varepsilon_{2k}'\right] = LML' = M_{22}$
- If $i = l \neq j = k$, then $E\left[\varepsilon_{2i}\varepsilon_j' M \varepsilon_k \varepsilon_{2l}'\right] = E\left[\varepsilon_{2i}\varepsilon_{2i}'\right] tr\left\{M\right\} = I_q tr\left\{M\right\}$
- If $i = k \neq j = l$, then $E\left[\varepsilon_{2i}\varepsilon_j' M \varepsilon_k \varepsilon_{2l}'\right] = E\left[\varepsilon_{2i}\varepsilon_i'\right] M' E\left[\varepsilon_j \varepsilon_{2j}'\right] = LM'L' = M_{22}'$
- If $i = j = k = l$, Consider $D^* = L'DL$. Then only the entries in the lower right hand part of the matrix are non zero.Let δ be the Kronecker delta, such that $\delta_{\alpha\beta} = 1$ iff $\alpha = \beta$ and zero otherwise to find: $\left[d^{*ab}\right] = (1-\delta_{ab}) E\left[\left(\epsilon_i^a\right)^2 \left(m^{ab} + m^{ba}\right)\left(\epsilon_i^b\right)^2\right] +$
  $\delta_{ab} E\left[\left(\epsilon_i^a\right)^2 \left(\epsilon_i^b\right)^2 \sum_{\substack{b=1 \\ b\neq a}}^n m^{bb} + \left(\epsilon_i^a\right)^4 m^{aa}\right]$
  $= (1-\delta_{ab})\left(m^{ab} + m^{ba}\right) + \delta_{ab} \sum_{\substack{b=1 \\ b\neq a}}^n m^{bb} + \delta_{ab} 3 m^{aa}$ for $a, b \geq n-q+1$
  otherwise $E\left[d^{*ab}\right] = 0$
  We thus find that $E\left[D^*\right] = L'ML + L'M'L + L'L \times tr\left\{M\right\}.$
  $E\left[D\right] = M + M' + I_q \times tr\left\{M\right\}.$
- If we have a $\varepsilon-$vector, whose index does not coincide with the index of another $\varepsilon-$vector, then by independence the expectation of the whole expression becomes zero.

■

Lemma 15 *Let* $\varepsilon_i = [\varepsilon_{1i}, \varepsilon_{2i}], i = 1, ..., T$ *be* $(n \times 1)$ *vectors, distributed i.i.d.*$N(0, I_n)$ *and let* ε_{2i} *be of dimension* $q \leq n$. *Further let* S *be an* $(q \times q)$ *matrix and* $L_{q\times n} = [0, I_q]$. *Define* $S^* = L'SL$. *Then:*

$$E\left[D\right] = E\left[\varepsilon_i \varepsilon_{2j}' S \varepsilon_{2k} \varepsilon_l'\right] = \begin{cases} S^* + S^{*\prime} + I_n tr\left\{S\right\} & \text{if } i = j = k = l \\ I_n tr\left\{S\right\} & \text{if } i = l \neq j = k \\ S^{*\prime} & \text{if } i = k \neq j = l \\ S^* & \text{if } i = j \neq k = l \\ 0 & \text{otherwise} \end{cases}$$

Proof. First let $\varepsilon_i' = \begin{bmatrix} \epsilon_i^1 & \epsilon_i^2 & \cdots & \epsilon_i^n \end{bmatrix}$, $\varepsilon_{2i}' = \begin{bmatrix} \epsilon_i^{n-q+1} & \cdots & \epsilon_i^n \end{bmatrix}$ and denote the element in row a and column b of matrix S as s^{ab}.

- If $i = j \neq k = l$, then $E\left[\varepsilon_i \varepsilon_{2j}' S \varepsilon_{2k} \varepsilon_l'\right] = E\left[\varepsilon_i \varepsilon_{2i}'\right] M E\left[\varepsilon_{2k}\varepsilon_k'\right] = L'SL = S^*$
- If $i = l \neq j = k$,then $E\left[\varepsilon_{2i}\varepsilon_j' M \varepsilon_k \varepsilon_{2l}'\right] = E\left[\varepsilon_{2i}\varepsilon_{2i}'\right] tr\left\{S\right\} = I_n tr\left\{S\right\}$
- If $i = k \neq j = l$,then $E\left[\varepsilon_{2i}\varepsilon_j' M \varepsilon_k \varepsilon_{2l}'\right] = E\left[\varepsilon_{2i}\varepsilon_i'\right] M' E\left[\varepsilon_j \varepsilon_{2j}'\right] = LS'L' = S^{*\prime}$

- For $i=j=k=l$, et $\delta_2 = 1$ iff $\alpha, \beta \geq n-1+1$ and 0 otherwise then
$E\left[\varepsilon_i \varepsilon_{2j}' S \varepsilon_{2k} \varepsilon_2'\right]$
$= \delta_{ab}\delta_2 E\left[\left(\epsilon_i^a\right)^2 \left(\epsilon_i^b\right)^2 \sum_{\substack{b=1 \\ b \neq a}}^q s^{bb} + \left(\epsilon_i^a\right)^4 s^{aa}\right] + \left(1-\delta_{ab}\right)\delta_2 \left[\left(\epsilon_i^a\right)^2 \left(s^{ab}+s^{ba}\right)\left(\epsilon_i^b\right)^2\right]$
$+\delta_{ab}\left(1-\delta_2\right) E\left[\left(\epsilon_i^a\right)^2 \left(\epsilon_i^b\right)^2 \sum_{b=1}^q s^{bb}\right] + \left(1-\delta_{ab}\right)\left(1-\delta_2\right) 0$
such that we find $E\left[\varepsilon_{2i}\varepsilon_i' S \varepsilon_i \varepsilon_{2l}'\right] = S^* + S^{*\prime} + I_n tr\left\{S\right\}$

- If we have a $\varepsilon-$vector, whose index does not coincide with the index of another $\varepsilon-$vector, then by independence the expectation of the whole expression becomes zero.

■

Lemma 16 *Let $\varepsilon_i, i = 1, ..., T$ be $(n \times 1)$ vectors, distributed i.i.d. $N\left(0, I_n\right)$ and M and N $(n \times n)$ matrices.*
Then:

$$E\left[\varepsilon_i' M \varepsilon_j \varepsilon_k' N \varepsilon_l\right] = \begin{cases} tr\left\{MN\right\} + tr\left\{MN'\right\} + tr\left\{M\right\} tr\left\{N\right\} & \textit{if } i=j=k=l \\ tr\left\{M\right\} tr\left\{N\right\} & \textit{if } i=j\neq k=l \\ tr\left\{MN'\right\} & \textit{if } i=k\neq j=l \\ tr\left\{MN\right\} & \textit{if } i=l\neq j=k \\ 0 & \textit{otherwise} \end{cases} \tag{38}$$

Proof. We proceed as in the last two lemma's and refer to them for notation:

- If $i=j\neq k=l$, then $E\left[\varepsilon_i' M \varepsilon_j \varepsilon_k' N \varepsilon_l\right] = E\left[\varepsilon_i' M \varepsilon_j\right] E\left[\varepsilon_k' N \varepsilon_l\right] = tr\left\{M\right\} tr\left\{N\right\}$

- If $i=l\neq j=k$, then $E\left[\varepsilon_i' M \varepsilon_j \varepsilon_k' N \varepsilon_l\right] = tr\left\{MN\right\}$

- If $i=k\neq j=l$, then $E\left[\varepsilon_i' M \varepsilon_j \varepsilon_k' N \varepsilon_l\right] = E\left[\varepsilon_i' M \varepsilon_j \varepsilon_l' N' \varepsilon_k\right] = tr\left\{MN'\right\}$

- If $i=j=k=l$, then $E\left[\varepsilon_i' M \varepsilon_j \varepsilon_k' N \varepsilon_l\right] = E\left[\varepsilon_i' M \varepsilon_i \varepsilon_i' N \varepsilon_i\right]$
$= E\left[\sum_{a,b,c,d=1}^n \epsilon_i^a m^{ab} \epsilon_i^b \epsilon_i^c n^{cd} \epsilon_i^d\right]$
$= E\left[\sum_{a=1}^n \left(\epsilon_i^a\right)^4 m^{aa} n^{aa}\right] + E\left[\sum_{\substack{a,c=1 \\ a\neq c}}^n \left(\epsilon_i^a\right)^2 \left(\epsilon_i^c\right)^2 m^{aa} n^{cc}\right]$
$+E\left[\sum_{\substack{a,b=1 \\ a\neq b}}^n \left(\epsilon_i^a\right)^2 \left(\epsilon_i^b\right)^2 \left(m^{ab} n^{ab} + m^{ab} n^{ba}\right)\right]$
$= tr\left\{MN\right\} + tr\left\{MN'\right\} + tr\left\{M\right\} tr\left\{N\right\}$

- If we have a $\varepsilon-$vector, whose index does not coincide with the index of another $\varepsilon-$vector, then by independence the expectation of the whole expression becomes zero. Throughout we have used the fact that the first and third moments of the normal distribution is zero.

■

A.2 Proof of Theorem 1

We first consider the model of theorem 1, which concerns a simple hypothesis $\mathcal{H}_0 : A = A_0$

$$Y_t = AX_t + \eta_{2t} \tag{39}$$

where

$$\begin{aligned} X_t &= Q(L)\eta_{t-1} \\ \eta_t &= \left[\begin{array}{cc} \eta_{1t}' & \eta_{2t}' \end{array}\right]' \sim MIIDN(0, \Omega) \end{aligned}$$

where η_t is of dimension n, whereas η_{2t} is of dimension q. Furthermore under $\mathcal{H}_0$, $H = Y - XA$, where with capitals we denote the stacked vectors. For instance $Y = [y_1, \ldots, y_T]'$,$U = [\varepsilon_{21}, \ldots, \varepsilon_{2T}]'$, $H = [\eta_{21}, \ldots, \eta_{2T}]'$. Also $\varepsilon_{2t} = \Omega_{22}^{-\frac{1}{2}}\eta_{2t}$ and $\varepsilon_t = \Omega^{-\frac{1}{2}}\eta_t$.

It is well-known that the ordinary least squares estimator and the maximum likelihood estimator coincide in this model, such that the maximum likelihood estimator can be written as: $\hat{A} = A + (X'X)^{-1}(X'H)$. We substitute this in the likelihood ratio test for $\mathcal{H}_0 : A = A_0$ and expand it, keeping only first order terms:

$$\begin{aligned} -2\ln LR\,(A = A_0) &= -T\log\left|(Y - X\hat{A}')'(Y - X\hat{A}')\right| |(H'H)|^{-1} \\ &= -T\log\left|I_q - \left(\Omega_{22}^{\frac{1}{2}}U'U\Omega_{22}^{\frac{1}{2}}\right)^{-1}\left(\Omega_{22}^{\frac{1}{2}}U'X\right)(X'X)^{-1}\left(X'U\Omega_{22}^{\frac{1}{2}}\right)\right| \\ &= -T\log\left|I_q - \Omega_{22}^{-\frac{1}{2}}(U'U)^{-1}(U'X)(X'X)^{-1}(X'U)\,\Omega_{22}^{\frac{1}{2}}\right| \\ &= -T\log\left|\Omega_{22}^{-\frac{1}{2}}\right|\left|I_q - \Omega_{22}^{-\frac{1}{2}}(U'U)^{-1}(U'X)(X'X)^{-1}(X'U)\,\Omega_{22}^{\frac{1}{2}}\right|\left|\Omega_{22}^{\frac{1}{2}}\right| \\ &= -T\log\left|I_q - (U'U)^{-1}(U'X)(X'X)^{-1}(X'U)\right| \\ &= -T\,|I_q - K| \\ &\stackrel{1}{=} tr(K) + \frac{1}{2T}tr(K^2) \end{aligned}$$

where we have defined $K \equiv T\,(U'U)^{-1}(U'X)(X'X)^{-1}(X'U)\,.$
The probability limits of the two matrices, whose inverses enter K, are:

$$\begin{aligned} \left(\frac{1}{T}U'U\right) &\xrightarrow{P} I_q \\ \left(\frac{1}{T}X'X\right) &\xrightarrow{P} \Phi = \sum_{\eta=0}^{\infty} C_\eta C_\eta' = Var(X_t) \end{aligned}$$

and their first order expansions are:

$$\left(\frac{1}{T}U'U\right)^{-1} = \left(I_q - \left(I_q - \frac{1}{T}U'U\right)\right)^{-1}$$
$$\stackrel{1}{=} I_q + \left(I_q - \frac{1}{T}U'U\right) + \left(I_q - \frac{1}{T}U'U\right)^2 \tag{40}$$
$$\left(\frac{1}{T}X'X\right)^{-1} = \left(\Phi - \left(\Phi - \frac{1}{T}X'X\right)\right)^{-1}$$
$$\stackrel{1}{=} \Phi^{-1} + \Phi^{-1}\left(\Phi - \frac{1}{T}X'X\right)\Phi^{-1} + \Phi^{-1}\left(\Phi - \frac{1}{T}X'X\right)\Phi^{-1}\left(\Phi - \frac{1}{T}X'X\right)\Phi^{-1} \tag{41}$$

Using (40) and (41) we can write the first order expansion of the expected value of K as:

$$E\left[tr(K)\right] \stackrel{1}{=} tr\left\{E\left(\frac{1}{\sqrt{T}}U'X\right)\Phi^{-1}\left(\frac{1}{\sqrt{T}}X'U\right)\right\}$$
$$+\, tr\left\{E\left(I_q - \frac{1}{T}U'U\right)\left(\frac{1}{\sqrt{T}}U'X\right)\Phi^{-1}\left(\frac{1}{\sqrt{T}}X'U\right)\right\}$$
$$+\, tr\left\{E\left(\frac{1}{\sqrt{T}}U'X\right)\Phi^{-1}\left(\Phi - \frac{1}{T}X'X\right)\Phi^{-1}\left(\frac{1}{\sqrt{T}}X'U\right)\right\}$$
$$+\, tr\left\{E\left(I_q - \frac{1}{T}U'U\right)\left(\frac{1}{\sqrt{T}}U'X\right)\Phi^{-1}\left(\Phi - \frac{1}{T}X'X\right)\Phi^{-1}\left(\frac{1}{\sqrt{T}}X'U\right)\right\}$$
$$+\, tr\left\{E\left(I_q - \frac{1}{T}U'U\right)^2\left(\frac{1}{\sqrt{T}}U'X\right)\Phi^{-1}\left(\frac{1}{\sqrt{T}}X'U\right)\right\}$$
$$+\, tr\left\{E\left(\frac{1}{\sqrt{T}}U'X\right)\Phi^{-1}\left(\Phi - \frac{1}{T}X'X\right)\Phi^{-1}\left(\Phi - \frac{1}{T}X'X\right)\Phi^{-1}\left(\frac{1}{\sqrt{T}}X'U\right)\right\}$$

The names of these terms shall be D_1 to D_6. Together with $\frac{1}{2T}E\left[tr(K^2)\right]$ these terms form the expansion of the expectation of the likelihood ratio test. Their expectations are worked out one by one in the following pages.

A.3 Derivation of D_1

$$tr\left\{E\left(\frac{1}{\sqrt{T}}U'X\right)\Phi^{-1}\left(\frac{1}{\sqrt{T}}X'U\right)\right\}$$
$$= tr\left\{E\left[\frac{1}{T}\sum_{t,s=1}^{T}\sum_{\zeta,\eta=0}^{\infty}\varepsilon_{2t}\varepsilon_{t-1-\zeta}C'_{\zeta}\Phi^{-1}C_{\eta}\varepsilon_{s-1-\eta}\varepsilon'_{2s}\right]\right\}$$

There is only one way in which this terms gives a non-zero expectation: $t = s, \eta = \zeta$. We then get:

$$
\begin{aligned}
& tr\left\{\frac{1}{T}E\left[\sum_{s=1}^{T}\varepsilon_{2s}'\varepsilon_{2s}\right]E\left[\sum_{\eta=0}^{\infty}\varepsilon_{s-1-\eta}'C_\zeta'\Phi^{-1}C_\zeta\varepsilon_{s-1-\eta}\right]\right\} \\
& = q\times tr\left\{\sum_{\eta=0}^{\infty}C_\zeta'\Phi^{-1}C_\zeta\right\} \\
& = q\times tr\left\{I_n\right\} \\
& = qn
\end{aligned}
$$

$$
\boxed{D_1 = qn}
$$

A.4 Derivation of D_2

$$
\begin{aligned}
& tr\left\{E\left(I_q-\frac{1}{T}U'U\right)\left(\frac{1}{\sqrt{T}}U'X\right)\Phi^{-1}\left(\frac{1}{\sqrt{T}}X'U\right)\right\} \\
& = -tr\left\{\frac{1}{T^2}E\sum_{t,s,r=1}^{T}\sum_{\zeta,\eta=0}^{\infty}\left(\varepsilon_{2r}\varepsilon_{2r}'-I_q\right)\varepsilon_{2t}\varepsilon_{t-1-\zeta}'C_\zeta'\Phi^{-1}C_\eta\varepsilon_{s-1-\eta}\varepsilon_{2s}'\right\}
\end{aligned}
$$

There are two ways in which this combination gives has an expectation of at least $O(\frac{1}{T})$: Either $t=s=r$ and $\eta=\zeta$ or $t=s$ and $s-1-\eta=t-1-\zeta=r$.

A.4.1 The first combination

$t=s=r$ and $\eta=\zeta$

I find

$$
\begin{aligned}
& -tr\left\{\frac{1}{T^2}E\left[\sum_{s=1}^{T}\varepsilon_{2s}\varepsilon_{2s}'\varepsilon_{2s}\varepsilon_{2s}'-\sum_{s=1}^{T}\varepsilon_s\varepsilon_s'\right]E\left[\sum_{\eta=0}^{\infty}\varepsilon_{s-1-\eta}'C_\eta'\Phi^{-1}C_\eta\varepsilon_{s-1-\eta}\right]\right\} \\
& = -tr\left\{\frac{1}{T^2}\left(T\left(q+1\right)\times I_q\right)\times tr\left\{I_n\right\}\right\} \\
& = -\frac{q^2n+nq}{T}
\end{aligned}
$$

where we applied lemma 14 in the second line.

$$
\boxed{D_{21} = -\frac{q^2n+nq}{T}}
$$

A.4.2 The second combination

$t = s$ and $s - 1 - \eta = t - 1 - \zeta = r$.

Here the reasoning goes as follows:

$$
\begin{aligned}
& - tr\left\{ \frac{1}{T^2} E \sum_{t,s,r=1}^{T} \sum_{\zeta,\eta=0}^{\infty} \left(\varepsilon_{2r}\varepsilon_{2r}' - I_q\right) \varepsilon_{2t}\varepsilon_{t-1-\zeta}' C_\zeta' \Phi^{-1} C_\eta \varepsilon_{s-1-\eta} \varepsilon_{2s}' \right\} \\
& = -tr\left\{ \frac{1}{T^2} E\left[\sum_{\eta=0}^{\infty} \varepsilon_{2s-1-\eta}\varepsilon_{s-1-\eta}' C_\eta' \Phi^{-1} C_\eta \varepsilon_{s-1-\eta} \varepsilon_{2s-1-\eta}' \right] E\left[\sum_{s=1}^{T} \varepsilon_{2s}\varepsilon_{2s}' \right] \right\} \\
& + tr\left\{ \frac{1}{T^2} E\left[\sum_{\eta=0}^{\infty} \varepsilon_{s-1-\eta}' C_\eta' \Phi^{-1} C_\eta \varepsilon_{s-1-\eta} \right] E\left[\sum_{s=1}^{T} \varepsilon_{2s}\varepsilon_{2s}' \right] \right\} \\
& = -tr\left\{ \frac{1}{T^2} \left((n+2) \times I_q\right) \times \left(T \times I_q\right) \right\} + \frac{nq}{T} \\
& = -\frac{2q}{T}
\end{aligned}
$$

where we have applied lemma 15 in the third passage. We thus conclude that

$$\boxed{D_{22} = -\frac{2q}{T}}$$

A.5 Derivation of D_3

$$
\begin{aligned}
& tr\left\{ E\left(\frac{1}{\sqrt{T}} U'X \right) \Phi^{-1} \left(\Phi - \frac{1}{T} X'X \right) \Phi^{-1} \left(\frac{1}{\sqrt{T}} X'U \right) \right\} \\
& = -tr\left\{ \frac{1}{T^2} E \sum_{t,s,v=1}^{T} \sum_{\zeta,\eta,\kappa,\lambda=0}^{\infty} \varepsilon_{2t}\varepsilon_{t-1-\zeta}' C_\zeta' \Phi^{-1} C_\kappa \left(\varepsilon_{v-1-\kappa}\varepsilon_{v-1-\lambda}' - \delta_{\kappa\lambda} I_n \right) C_\lambda' \Phi^{-1} C_\eta \varepsilon_{s-1-\eta} \varepsilon_{2s}' \right\}
\end{aligned}
$$

There are five ways in which this expression gives an expectation of at least $O(\frac{1}{T})$:

1. $s = t = v - 1 - \lambda = v - 1 - \kappa$ and $t - 1 - \zeta = s - 1 - \eta$ and $\lambda = \kappa$
2. $s = t$ and $t - 1 - \zeta = s - 1 - \eta = v - 1 - \lambda = v - 1 - \kappa$
3. $v - 1 - \kappa = s - 1 - \eta \neq v - 1 - \lambda = t - 1 - \zeta$ and $s = t$ (also change κ and λ to get two combinations in total)
4. $s = v - 1 - \kappa$ and $t = s - 1 - \eta$ and $v - 1 - \lambda = t - 1 - \zeta$ (also change both κ and λ and ζ and η for four combinations)
5. $t = v - 1 - \kappa$ and $s = v - 1 - \lambda$ and $t - 1 - \zeta = s - 1 - \eta$ (also change κ and λ, deriving two expressions)

These five combinations, some of them consisting of subcombinations, shall now be dealt with one by one:

A.5.1 Derivation of D_{31}

$s = t = v-1-\lambda = v-1-\kappa$ and $\zeta = \eta$ and $\lambda = \kappa$

$$- tr\left\{\frac{1}{T^2}E\sum_{t,s,v=1}^{T}\sum_{\zeta,\eta,\kappa,\lambda=0}^{\infty}\varepsilon_{2t}\varepsilon'_{t-1-\zeta}C'_{\zeta}\Phi^{-1}C_{\kappa}\left(\varepsilon_{v-1-\kappa}\varepsilon'_{v-1-\lambda}-\delta_{\kappa\lambda}I_n\right)C'_{\lambda}\Phi^{-1}C_{\eta}\varepsilon_{s-1-\eta}\varepsilon'_{2s}\right\}$$

$$= -tr\left\{\frac{1}{T^2}E\left[\sum_{s=1}^{T}\varepsilon_s\varepsilon'_{2s}\varepsilon_{2s}\varepsilon'_s - \sum_{s=1\kappa\lambda}^{T}I_n\varepsilon'_s\varepsilon_s\right]E\left[\sum_{\kappa=0}^{\infty}\sum_{\eta=0}^{\infty}C'_{\kappa}\Phi^{-1}C_{\eta}\varepsilon_{s-1-\eta}\varepsilon'_{s-1-\eta}C'_{\eta}\Phi^{-1}C_{\kappa}\right]\right\}$$

$$= -tr\left\{\frac{1}{T^2}\left(2T\times I_q^*\right)\times\sum_{\kappa=0}^{\infty}C'_{\kappa}\Phi^{-1}C_{\kappa}\right\}$$

$$= -\frac{2}{T}tr\left\{\left[\sum\nolimits_{\kappa=0}^{\infty}C'_{\kappa}\Phi^{-1}C_{\kappa}\right]_{22}\right\}$$

where we defined the $(n\times n)$ matrix I_q^* as:

$$I_q^* = \begin{bmatrix} 0 & 0 \\ 0 & I_q \end{bmatrix}$$

$$\boxed{D_{31} = -\tfrac{2}{T}tr\left\{\left[\sum\nolimits_{\kappa=0}^{\infty}C'_{\kappa}\Phi^{-1}C_{\kappa}\right]_{22}\right\}}$$

A.5.2 Derivation of D_{32}

$s = t$ and $t-1-\zeta = s-1-\eta = v-1-\lambda = v-1-\kappa$

$$- tr\left\{\frac{1}{T^2}E\sum_{t,s,v=1}^{T}\sum_{\zeta,\eta,\kappa,\lambda=0}^{\infty}\varepsilon_{2t}\varepsilon'_{t-1-\zeta}C'_{\zeta}\Phi^{-1}C_{\kappa}\left(\varepsilon_{v-1-\kappa}\varepsilon'_{v-1-\lambda}-\delta_{\kappa\lambda}I_n\right)C'_{\lambda}\Phi^{-1}C_{\eta}\varepsilon_{s-1-\eta}\varepsilon'_{2s}\right\}$$

$$= -\frac{1}{T^2}E\left[\sum_{s=1}^{T}\varepsilon'_{2s}\varepsilon_{2s}\right]E\left[\sum_{\kappa=0}^{\infty}\sum_{\eta=0}^{\infty}\varepsilon'_uC'_{\eta}\Phi^{-1}C_{\kappa}\left(\varepsilon_u\varepsilon'_u - I_n\right)C'_{\kappa}\Phi^{-1}C_{\eta}\varepsilon_u\right]$$

$$= -\frac{q}{T}E\left[\sum_{\kappa=0}^{\infty}\sum_{\eta=0}^{\infty}\varepsilon'_uC'_{\eta}\Phi^{-1}C_{\kappa}\varepsilon_u\varepsilon'_uC'_{\kappa}\Phi^{-1}C_{\eta}\varepsilon_u\right] + \frac{q}{T}E\left[\sum_{\kappa=0}^{\infty}\sum_{\eta=0}^{\infty}\varepsilon'_uC'_{\eta}\Phi^{-1}C_{\kappa}C'_{\kappa}\Phi^{-1}C_{\eta}\varepsilon_u\right]$$

$$= -\frac{q}{T}\sum_{\kappa=0}^{\infty}\sum_{\eta=0}^{\infty}tr\left\{\left(C'_{\eta}\Phi^{-1}C_{\kappa}\right)^2\right\} - \frac{q}{T}\sum_{\kappa=0}^{\infty}\sum_{\eta=0}^{\infty}tr\left\{\left(C'_{\eta}\Phi^{-1}C_{\kappa}C'_{\kappa}\Phi^{-1}C_{\eta}\right)\right\}$$

$$\frac{q}{T}\sum_{\kappa=0}^{\infty}\sum_{\eta=0}^{\infty}tr^2\left\{\left(C'_{\kappa}\Phi^{-1}C_{\eta}\right)\right\} + \frac{q}{T}\sum_{\kappa=0}^{\infty}\sum_{\eta=0}^{\infty}tr\left\{\left(C'_{\eta}\Phi^{-1}C_{\kappa}C'_{\kappa}\Phi^{-1}C_{\eta}\right)\right\}$$

where we have applied lemma 16 in the third passage, such that we conclude that the total contribution of D32 is equal to:

$$\boxed{D_{32} = -\frac{q}{T}\sum_{\kappa=0}^{\infty}\sum_{\eta=0}^{\infty} tr\left\{\left(C'_\eta\Phi^{-1}C_\kappa\right)^2\right\} - \frac{q}{T}\sum_{\kappa=0}^{\infty}\sum_{\eta=0}^{\infty} tr^2\left\{\left(C'_\kappa\Phi^{-1}C_\eta\right)\right\}}$$

A.5.3 Derivation of D_{33}

First combination This is the way to derive the first combination of D33
$v-1-\lambda = s-1-\eta \neq v-1-\kappa = t-1-\zeta$ and $s=t$, $\kappa\neq\lambda$
which means that:

$$\boxed{\zeta+\lambda=\eta+\kappa\,,\,\kappa\neq\lambda}$$

$$- \, tr\left\{\frac{1}{T^2}E\sum_{t,s,v=1}^{T}\sum_{\zeta,\eta,\kappa,\lambda=0}^{\infty}\varepsilon_{2t}\varepsilon'_{t-1-\zeta}C'_\zeta\Phi^{-1}C_\kappa\left(\varepsilon_{v-1-\kappa}\varepsilon'_{v-1-\lambda}-\delta_{\kappa\lambda}I_n\right)C'_\lambda\Phi^{-1}C_\eta\varepsilon_{s-1-\eta}\varepsilon'_{2s}\right\}$$
$$= -\frac{1}{T}\sum_{s}^{T}\sum_{\substack{\kappa+\eta=\lambda+\zeta\\ \kappa\neq\lambda}} tr\left\{E\left[\varepsilon_{2s}\varepsilon'_w C'_\zeta\Phi^{-1}C_\kappa\varepsilon_w\varepsilon'_u C'_\lambda\Phi^{-1}C_\eta\varepsilon_u\varepsilon'_{2s}\right]\right\}$$
$$= -\frac{q}{T}\sum_{\substack{\kappa+\eta=\lambda+\zeta\\ \kappa\neq\lambda}} tr\left\{C'_\zeta\Phi^{-1}C_\kappa\right\} tr\left\{C'_\lambda\Phi^{-1}C_\eta\right\}$$

So the total contribution of the first part of the D33 term is:

$$-\frac{q}{T}\sum_{\substack{\kappa+\eta=\lambda+\zeta\\ \kappa\neq\lambda}} tr\left\{C'_\zeta\Phi^{-1}C_\kappa\right\} tr\left\{C'_\lambda\Phi^{-1}C_\eta\right\}$$

Second combination $v-1-\kappa = s-1-\eta \neq v-1-\lambda = t-1-\zeta$ and $s=t$, $\kappa\neq\lambda$
Stated alternatively:

$$\boxed{\zeta+\kappa=\eta+\lambda\,,\,\kappa\neq\lambda}$$

$$- \, tr\left\{\frac{1}{T^2}E\left[\sum_{t,s,v=1}^{T}\sum_{\zeta,\eta,\kappa,\lambda=0}^{\infty}\varepsilon_{2t}\varepsilon'_{t-1-\zeta}C'_\zeta\Phi^{-1}C_\kappa\left(\varepsilon_{v-1-\kappa}\varepsilon'_{v-1-\lambda}-\delta_{\kappa\lambda}I_n\right)C'_\lambda\Phi^{-1}C_\eta\varepsilon_{s-1-\eta}\varepsilon'_{2s}\right]\right\}$$
$$= -\frac{q}{T}tr\left\{E\left[\sum_{\substack{\lambda+\eta=\kappa+\zeta\\ \kappa\neq\lambda}}\varepsilon'_{t-1-\zeta}C'_\zeta\Phi^{-1}C_\kappa\varepsilon_{v-1-\kappa}\varepsilon'_{v-1-\lambda}C'_\lambda\Phi^{-1}C_\eta\varepsilon_{s-1-\eta}\right]\right\} \tag{42}$$
$$= -\frac{q}{T}\sum_{\substack{\lambda+\eta=\kappa+\zeta\\ \kappa\neq\lambda}} tr\left\{C'_\zeta\Phi^{-1}C_\kappa C'_\eta\Phi^{-1}C_\lambda\right\} \tag{43}$$

Total
The total summation of D_{33} is equal to:

$$\boxed{\boxed{\begin{array}{c} D_{33} = -\frac{q}{T}\sum_{\substack{\kappa+\eta=\lambda+\zeta \\ \kappa\neq\lambda}} tr\left\{C'_\zeta\Phi^{-1}C_\kappa\right\} tr\left\{C'_\lambda\Phi^{-1}C_\eta\right\} \\ -\frac{q}{T}\sum_{\substack{\lambda+\eta=\kappa+\zeta \\ \kappa\neq\lambda}} tr\left\{C'_\zeta\Phi^{-1}C_\kappa C'_\eta\Phi^{-1}C_\lambda\right\} \end{array}}}$$

A.5.4 Derivation of D_{34}

First combination $s = v - 1 - \kappa$ and $t = s - 1 - \eta$ and $v - 1 - \lambda = t - 1 - \zeta$

Note that $\kappa \neq \lambda$ and

$$\boxed{\lambda = \kappa + \zeta + \eta + 2}$$

$$- tr\left\{\frac{1}{T^2}E\left[\sum_{t,s,v=1}^{T}\sum_{\zeta,\eta,\kappa,\lambda=0}^{\infty}\varepsilon_{2t}\varepsilon'_{t-1-\zeta}C'_\zeta\Phi^{-1}C_\kappa\left(\varepsilon_{v-1-\kappa}\varepsilon'_{v-1-\lambda}-\delta_{\kappa\lambda}I_n\right)C'_\lambda\Phi^{-1}C_\eta\varepsilon_{s-1-\eta}\varepsilon'_{2s}\right]\right\}$$

$$= -tr\left\{\frac{1}{T^2}E\left[\sum_{s=1}^{T}\sum_{\zeta,\eta,\kappa=0}^{\infty}\varepsilon_{2t}\varepsilon'_{t-1-\zeta}C'_\zeta\Phi^{-1}C_\kappa\varepsilon_{v-1-\kappa}\varepsilon'_{v-1-\lambda}C'_\lambda\Phi^{-1}C_\eta\varepsilon_{s-1-\eta}\varepsilon'_{2s}\right]\right\}$$

$$= -tr\left\{\frac{1}{T^2}E\left[\sum_{s=1}^{T}\sum_{\zeta,\eta,\kappa=0}^{\infty}\varepsilon_{2s}\varepsilon'_{v-1-\kappa}C'_\kappa\Phi^{-1}C_\zeta\varepsilon_{t-1-\zeta}\varepsilon'_{v-1-\lambda}C'_{\kappa+\zeta+\eta+2}\Phi^{-1}C_\eta\varepsilon_{s-1-\eta}\varepsilon'_{2t}\right]\right\}$$

$$= -\frac{1}{T}tr\left\{\left[\sum_{\zeta,\eta,\kappa=0}^{\infty}C'_\kappa\Phi^{-1}C_\zeta C'_{\kappa+\zeta+\eta+2}\Phi^{-1}C_\eta\right]_{22}\right\}$$

The second combination $s = v - 1 - \lambda$ and $t = s - 1 - \eta$ and $v - 1 - \kappa = t - 1 - \zeta$

Note that $\kappa \neq \lambda$ and

$$\boxed{\kappa = \lambda + \zeta + \eta + 2}$$

$$- tr\left\{\frac{1}{T^2}E\left[\sum_{t,s,v=1}^{T}\sum_{\zeta,\eta,\kappa,\lambda=0}^{\infty}\varepsilon_{2t}\varepsilon'_{t-1-\zeta}C'_\zeta\Phi^{-1}C_\kappa\left(\varepsilon_{v-1-\kappa}\varepsilon'_{v-1-\lambda}-\delta_{\kappa\lambda}I_n\right)C'_\lambda\Phi^{-1}C_\eta\varepsilon_{s-1-\eta}\varepsilon'_{2s}\right]\right\}$$

$$= -tr\left\{\frac{1}{T^2}\sum_{s=1}^{T}\sum_{\zeta,\eta,\lambda=0}^{\infty}E\left[\varepsilon'_{t-1-\zeta}C'_\zeta\Phi^{-1}C_{\lambda+\zeta+\eta+2}\varepsilon_{v-1-\kappa}\right]E\left[\varepsilon_{2t}\varepsilon'_{s-1-\eta}\right]C'_\eta\Phi^{-1}C_\lambda E\left[\varepsilon_{v-1-\lambda}\varepsilon'_{2s}\right]\right\}$$

$$= -\frac{1}{T}\sum_{\zeta,\eta,\lambda=0}^{\infty}tr\left\{C'_\zeta\Phi^{-1}C_{\lambda+\zeta+\eta+2}\right\}tr\left\{\left[C'_\eta\Phi^{-1}C_\lambda\right]_{22}\right\}$$

The third combination $t = v - 1 - \kappa$ and $s = t - 1 - \zeta$ and $v - 1 - \lambda = s - 1 - \eta$

Note that $\kappa \neq \lambda$ and

$$\boxed{\lambda = \kappa + \zeta + \eta + 2}$$

$$
\begin{aligned}
&- tr\left\{\frac{1}{T^2}E\left[\sum_{t,s,v=1}^{T}\sum_{\zeta,\eta,\kappa,\lambda=0}^{\infty}\varepsilon_{2t}\varepsilon'_{t-1-\zeta}C'_{\zeta}\Phi^{-1}C_{\kappa}\left(\varepsilon_{v-1-\kappa}\varepsilon'_{v-1-\lambda}-\delta_{\kappa\lambda}I_n\right)C'_{\lambda}\Phi^{-1}C_{\eta}\varepsilon_{s-1-\eta}\varepsilon'_{2s}\right]\right\}\\
&= -tr\left\{\frac{1}{T^2}\sum_{t=1}^{T}\sum_{\zeta,\eta,\kappa=0}^{\infty}E\left[\varepsilon_{2t}\varepsilon'_{v-1-\kappa}\right]C'_{\kappa}\Phi^{-1}C_{\zeta}E\left[\varepsilon_{t-1-\zeta}\varepsilon'_{2s}\right]E\left[\varepsilon'_{v-1-\lambda}C'_{\kappa+\zeta+\eta+2}\Phi^{-1}C_{\eta}\varepsilon_{s-1-\eta}\right]\right\}\\
&= -\frac{1}{T}\sum_{\zeta,\eta,\kappa=0}^{\infty}tr\left\{\left[C'_{\kappa}\Phi^{-1}C_{\zeta}\right]_{22}\right\}tr\left\{C'_{\kappa+\zeta+\eta+2}\Phi^{-1}C_{\eta}\right\}
\end{aligned}
$$

The fourth combination $t = v - 1 - \lambda$ and $s = t - 1 - \zeta$ and $v - 1 - \kappa = s - 1 - \eta$
Note that $\kappa \neq \lambda$ and $\kappa = \lambda + \zeta + \eta + 2$

$$
\begin{aligned}
&- tr\left\{\frac{1}{T^2}E\left[\sum_{t,s,v=1}^{T}\sum_{\zeta,\eta,\kappa,\lambda=0}^{\infty}\varepsilon_{2t}\varepsilon'_{t-1-\zeta}C'_{\zeta}\Phi^{-1}C_{\kappa}\left(\varepsilon_{v-1-\kappa}\varepsilon'_{v-1-\lambda}-\delta_{\kappa\lambda}I_n\right)C'_{\lambda}\Phi^{-1}C_{\eta}\varepsilon_{s-1-\eta}\varepsilon'_{2s}\right]\right\}\\
&= -tr\left\{\frac{1}{T^2}\sum_{t=1}^{T}\sum_{\zeta,\eta,\lambda=0}^{\infty}E\left[\varepsilon_{2t}\varepsilon'_{v-1-\lambda}\right]C'_{\lambda}\Phi^{-1}C_{\eta}E\left[\varepsilon_{s-1-\eta}\varepsilon'_{v-1-\kappa}\right]C'_{\lambda+\zeta+\eta+2}\Phi^{-1}C_{\zeta}E\left[\varepsilon_{t-1-\zeta}\varepsilon'_{2s}\right]\right\}\\
&= -\frac{1}{T}\sum_{\zeta,\eta,\lambda=0}^{\infty}tr\left\{\left[C'_{\lambda}\Phi^{-1}C_{\eta}C'_{\lambda+\zeta+\eta+2}\Phi^{-1}C_{\zeta}\right]_{22}\right\}
\end{aligned}
$$

Total The total contribution of D34 is therefore:

$$
\boxed{\begin{aligned}D_{34} = &-\tfrac{2}{T}\textstyle\sum_{\zeta,\eta,\kappa=0}^{\infty}tr\left\{\left[C'_{\kappa}\Phi^{-1}C_{\zeta}\right]_{22}\right\}tr\left\{C'_{\kappa+\zeta+\eta+2}\Phi^{-1}C_{\eta}\right\}\\ &-\tfrac{2}{T}\textstyle\sum_{\zeta,\eta,\lambda=0}^{\infty}tr\left\{\left[C'_{\lambda}\Phi^{-1}C_{\eta}C'_{\lambda+\zeta+\eta+2}\Phi^{-1}C_{\zeta}\right]_{22}\right\}\end{aligned}}
$$

A.5.5 Derivation of D_{35}

First combination $t = v - 1 - \kappa$ and $s = v - 1 - \lambda$ and $t - 1 - \zeta = s - 1 - \eta$ and $\kappa \neq \lambda$

$$
\boxed{\kappa + \zeta = \eta + \lambda}
$$

$$-\,tr\left\{\frac{1}{T^2}E\left[\sum_{t,s,v=1}^{T}\sum_{\zeta,\eta,\kappa,\lambda=0}^{\infty}\varepsilon_{2t}\varepsilon'_{t-1-\zeta}C'_\zeta\Phi^{-1}C_\kappa\left(\varepsilon_{v-1-\kappa}\varepsilon'_{v-1-\lambda}-\delta_{\kappa\lambda}I_n\right)C'_\lambda\Phi^{-1}C_\eta\varepsilon_{s-1-\eta}\varepsilon'_{2s}\right]\right\}$$

$$=-tr\left\{\frac{1}{T^2}\sum_{t=1}^{T}\sum_{\substack{\lambda+\eta=\kappa+\zeta\\ \kappa\neq\lambda}}E\left[\varepsilon_{2t}\varepsilon'_{v-1-\kappa}\right]C'_\kappa\Phi^{-1}C_\zeta E\left[\varepsilon_{t-1-\zeta}\varepsilon'_{s-1-\eta}\right]C'_\eta\Phi^{-1}C_\lambda E\left[\varepsilon_{v-1-\lambda}\varepsilon'_{2s}\right]\right\}$$

$$-\frac{2}{T}\sum_{\kappa=0}^{\infty}\sum_{\lambda=\kappa+1}^{\infty}\sum_{\eta=0}^{\infty}tr\left\{\left[C'_\kappa\Phi^{-1}C_{\lambda+\eta-\kappa}C'_\eta\Phi^{-1}C_\lambda\right]_{22}\right\}$$

$$=-\frac{2}{T}\sum_{\kappa,\eta,\alpha=0}^{\infty}tr\left\{\left[C'_\kappa\Phi^{-1}C_{\alpha+\eta-1}C'_\eta\Phi^{-1}C_{\alpha+\kappa+1}\right]_{22}\right\}$$

Second combination $t=v-1-\lambda$ and $s=v-1-\kappa$ and $t-1-\zeta=s-1-\eta$ and $\kappa\neq\lambda$

$$\boxed{\kappa+\eta=\zeta+\lambda}$$

$$-\,tr\left\{\frac{1}{T^2}E\left[\sum_{t,s,v=1}^{T}\sum_{\zeta,\eta,\kappa,\lambda=0}^{\infty}\varepsilon_{2t}\varepsilon'_{t-1-\zeta}C'_\zeta\Phi^{-1}C_\kappa\left(\varepsilon_{v-1-\kappa}\varepsilon'_{v-1-\lambda}-\delta_{\kappa\lambda}I_n\right)C'_\lambda\Phi^{-1}C_\eta\varepsilon_{s-1-\eta}\varepsilon'_{2s}\right]\right\}$$

$$=-tr\left\{\frac{1}{T^2}\sum_{t=1}^{T}\sum_{\substack{\kappa+\eta=\zeta+\lambda\\ \kappa\neq\lambda}}E\left[\varepsilon_{2t}\varepsilon'_{v-1-\lambda}\right]C'_\lambda\Phi^{-1}C_\eta E\left[\varepsilon_{s-1-\eta}\varepsilon'_{t-1-\zeta}\right]C'_\zeta\Phi^{-1}C_\kappa E\left[\varepsilon_{v-1-\kappa}\varepsilon'_{2s}\right]\right\}$$

$$=-\frac{2}{T}\sum_{\kappa=0}^{\infty}\sum_{\lambda=\kappa+1}^{\infty}\sum_{\zeta=0}^{\infty}tr\left\{\left[C'_\lambda\Phi^{-1}C_{\zeta+\lambda-\kappa}C'_\zeta\Phi^{-1}C_\kappa\right]_{22}\right\}$$

$$=-\frac{2}{T}\sum_{\kappa,\alpha,\zeta=0}^{\infty}tr\left\{\left[C'_{\alpha+\kappa+1}\Phi^{-1}C_{\zeta+\alpha+1}C'_\zeta\Phi^{-1}C_\kappa\right]_{22}\right\}$$

The total contribution of D35 is therefore:

$$\boxed{\begin{aligned}D_{35}=&-\tfrac{2}{T}\textstyle\sum_{\kappa,\eta,\alpha=0}^{\infty}tr\left\{\left[C'_\kappa\Phi^{-1}C_{\alpha+\eta-1}C'_\eta\Phi^{-1}C_{\alpha+\kappa+1}\right]_{22}\right\}\\ &-\tfrac{2}{T}\textstyle\sum_{\kappa,\alpha,\zeta=0}^{\infty}tr\left\{\left[C'_{\alpha+\kappa+1}\Phi^{-1}C_{\zeta+\alpha+1}C'_\zeta\Phi^{-1}C_\kappa\right]_{22}\right\}\end{aligned}}$$

A.6 Derivation of D_4

$$tr\left\{E\left(I_1-\frac{1}{T}U'U\right)\left(\frac{1}{\sqrt{T}}U'X\right)\Phi^{-1}\left(\Phi-\frac{1}{T}X'X\right)\Phi^{-1}\left(\frac{1}{\sqrt{T}}X'U\right)\right\}$$
$$=tr\left\{\frac{1}{T^3}E\sum_{t,s,r,\nu=1}^{T}\sum_{\zeta,\eta,\kappa,\lambda=0}^{\infty}\left(\varepsilon_{2r}\varepsilon_{2r}'-I_q\right)\varepsilon_{2t}\varepsilon_{t-1-\zeta}'C_\zeta'\Phi^{-1}C_\kappa\left(\varepsilon_{v-1-\kappa}\varepsilon_{v-1-\lambda}'-\delta_{\kappa\lambda}I_n\right)C_\lambda'\Phi^{-1}C_\eta\varepsilon_{s-1-\eta}\varepsilon_{2s}'\right\}$$

I only find one combination in this case:

$t-1-\zeta=s-1-\eta$ and $v-1-\lambda=v-1-\kappa=r$ which implies that $\lambda=\kappa$.

Take the four separate terms one by one, starting with the case in which we take both identity matrices:

$$tr\left\{\frac{1}{T^3}E\sum_{t,\nu=1}^{T}\sum_{\zeta,\lambda=0}^{\infty}\varepsilon_{2t}\varepsilon_{t-1-\zeta}'C_\zeta'\Phi^{-1}C_\lambda C_\lambda'\Phi^{-1}C_\zeta\varepsilon_{s-1-\zeta}\varepsilon_{2t}'\right\}$$
$$=\frac{1}{T}qn$$

Then

$$-tr\left\{\frac{1}{T^3}E\sum_{t,\nu=1}^{T}\sum_{\zeta,\lambda=0}^{\infty}\varepsilon_{2r}\varepsilon_{2r}'\varepsilon_{2t}\varepsilon_{t-1-\zeta}'C_\zeta'\Phi^{-1}C_\lambda C_\lambda'\Phi^{-1}C_\zeta\varepsilon_{s-1-\zeta}\varepsilon_{2t}'\right\}$$
$$=-\frac{1}{T}qn$$

$$-tr\left\{\frac{1}{T^3}E\sum_{t,r=1}^{T}\sum_{\zeta,\lambda=0}^{\infty}\varepsilon_{2t}\varepsilon_{t-1-\zeta}'C_\zeta'\Phi^{-1}C_\lambda\varepsilon_r\varepsilon_r'C_\lambda'\Phi^{-1}C_\zeta\varepsilon_{s-1-\zeta}\varepsilon_{2t}'\right\}$$
$$=-\frac{1}{T}qn$$

and the most complicated one:

$$tr\left\{\frac{1}{T^3}E\sum_{t,r=1}^{T}\sum_{\zeta,\lambda=0}^{\infty}\varepsilon_{2r}\varepsilon_{2r}'\varepsilon_{2t}\varepsilon_{t-1-\zeta}'C_\zeta'\Phi^{-1}C_{v-1-\lambda}\varepsilon_r\varepsilon_r'C_{v-1-\lambda}'\Phi^{-1}C_\eta\varepsilon_{s-1-\eta}\varepsilon_{2t}'\right\}$$
$$=tr\left\{\frac{1}{T^3}E\sum_{t,r=1}^{T}\sum_{\zeta,\lambda=0}^{\infty}\left[\varepsilon_{2t}\varepsilon_{2t}'\right]\varepsilon_{2r}\varepsilon_r'C_{v-1-\lambda}'\Phi^{-1}C_\zeta\varepsilon_{t-1-\zeta}\varepsilon_{s-1-\zeta}'C_\eta'\Phi^{-1}C_{v-1-\lambda}\varepsilon_r\varepsilon_{2r}'\right\}$$
$$=\frac{2}{T}tr\left\{\sum_{\zeta=0}^{\infty}\left[C_\zeta'\Phi^{-1}C_\zeta\right]_{22}\right\}+\frac{1}{T}qn$$

The total expression then becomes equal to:

$$\boxed{D_4=\frac{2}{T}tr\left\{\sum_{\zeta=0}^{\infty}\left[C_\zeta'\Phi^{-1}C_\zeta\right]_{22}\right\}}$$

A.7 Derivation of D_5

$$tr\left\{E\left(I_q-\frac{1}{T}U'U\right)^2\left(\frac{1}{\sqrt{T}}U'X\right)\Phi^{-1}\left(\frac{1}{\sqrt{T}}X'U\right)\right\}$$
$$=tr\left\{\frac{1}{T^3}E\sum_{t,s,y,r=1}^{T}\sum_{\zeta,\eta=0}^{\infty}\left(\varepsilon_{2y}\varepsilon_{2y}'-I_q\right)\left(\varepsilon_{2r}\varepsilon_{2r}'-I_q\right)\varepsilon_{2t}\varepsilon_{t-1-\zeta}'C_\zeta'\Phi^{-1}C_\eta'\varepsilon_{s-1-\eta}\varepsilon_{2s}'\right\}$$

Here I find only one combination:
$y=r, s-1-\eta=t-1-\zeta, t=s$

$$tr\left\{\frac{1}{T^3}E\sum_{t,r=1}^{T}\sum_{\zeta=0}^{\infty}\left(\varepsilon_{2r}\varepsilon_{2r}'-I_q\right)\left(\varepsilon_{2r}\varepsilon_{2r}'-I_q\right)\varepsilon_{2t}\varepsilon_{t-1-\zeta}'C_\zeta'\Phi^{-1}C'\varepsilon_{t-1-\zeta}\varepsilon_{2t}'\right\}$$
$$=\frac{1}{T}tr\left\{I_n\right\}tr\left\{E\left[\left(\varepsilon_{2r}\varepsilon_{2r}'-I_q\right)\left(\varepsilon_{2r}\varepsilon_{2r}'-I_q\right)\right]\right\}$$
$$=\frac{1}{T}\left(nq+nq^2\right)$$

Therefore:

$$\boxed{D_5=\frac{1}{T}\left(nq+nq^2\right)}$$

A.8 Derivation of D_6

$$tr\left\{E\left(\frac{1}{\sqrt{T}}U'X\right)\Phi^{-1}\left(\Phi-\frac{1}{T}X'X\right)\Phi^{-1}\left(\Phi-\frac{1}{T}X'X\right)\Phi^{-1}\left(\frac{1}{\sqrt{T}}X'U\right)\right\}$$
$$=\frac{1}{T^3}Etr\left\{\sum_{t,s,v,w=1}^{T}\sum_{\zeta,\eta,\kappa,\lambda,\alpha,\beta=0}^{\infty}\varepsilon_{2t}\varepsilon_{t-1-\zeta}'C_\zeta'\Phi^{-1}C_{v-1-\kappa}\left(\varepsilon_{v-1-\kappa}\varepsilon_{v-1-\lambda}'-\delta_{\kappa\lambda}I_n\right)C_{v-1-\lambda}'\Phi^{-1}\right.$$
$$\left.\times C_\alpha\left(\varepsilon_{w-1-\alpha}\varepsilon_{w-1-\beta}'-\delta_{\alpha\beta}I_n\right)C_\beta'\Phi^{-1}C_\eta\varepsilon_{s-1-\eta}\varepsilon_s'\right\}$$

The combinations, which give non-zero expectations of order $\frac{1}{T}$can be logically subdivided in three groups:

1. The $\left(\varepsilon_{v-1-\kappa}\varepsilon_{v-1-\lambda}'-\delta_{\kappa\lambda}I_n\right)$ and $\left(\varepsilon_{w-1-\alpha}\varepsilon_{w-1-\beta}'-\delta_{\alpha\beta}I_n\right)$ all coincide. At the same time $s=t$ and $t-1-\zeta=s-1-\eta$.

2. One of $\varepsilon_{v-1-\kappa}$ and $\varepsilon_{v-1-\lambda}'$ coincides with one of $\varepsilon_{w-1-\alpha}/\varepsilon_{w-1-\beta}'$. The two remaining ones then also coincide. Obviously we have two different combinations and $s=t$ and $t-1-\zeta=s-1-\eta$.

3. ε_s' coincides with one of $\varepsilon_{v-1-\kappa}$ and $\varepsilon_{v-1-\lambda}'$. $\varepsilon_{s-1-\eta}$ then coincides with the other. Similarly ε_t and $\varepsilon_{t-1-\zeta}'$ each coincide with one of $\varepsilon_{w-1-\alpha}$ /$\varepsilon_{w-1-\beta}'$. Note that there are eight of such combinations, which are listed one by one below in the derivation of C63

Each of these possibilities shall now be dealt with in turn.

A.8.1 Derivation of D_{61}

$$\frac{1}{T^3}Etr\left\{\sum_{t,s,v,w=1}^{T}\sum_{\zeta,\eta,\kappa,\lambda,\alpha,\beta=0}^{\infty}\varepsilon_{2t}\varepsilon_{t-1-\zeta}'C_\zeta'\Phi^{-1}C_\kappa\left(\varepsilon_{v-1-\kappa}\varepsilon_{v-1-\lambda}'-\delta_{\kappa\lambda}I_n\right)C_\lambda'\Phi^{-1}\right.$$
$$\left.\times C_\alpha\left(\varepsilon_{w-1-\alpha}\varepsilon_{w-1-\beta}'-\delta_{\alpha\beta}I_n\right)C_\beta'\Phi^{-1}C_\eta\varepsilon_{s-1-\eta}\varepsilon_{2s}'\right\}$$

For the first combination, we have $t-1-\zeta=s-1-\eta, s=t, v-1-\kappa=v-1-\lambda=w-1-\alpha=w-1-\beta=y$

which can be rephrased as:

$$\boxed{s=t,\zeta=\eta,\alpha=\beta,\kappa=\lambda}$$

w can also vary.

For simplicity we shall just use the $\sum$ for now.

$$\sum tr\left\{\varepsilon_{2t}\varepsilon_{t-1-\zeta}'C_\zeta'\Phi^{-1}C_\kappa\left(\varepsilon_y\varepsilon_y'-I_n\right)C_\kappa'\Phi^{-1}\right.$$
$$\left.\times C_\alpha\left(\varepsilon_y\varepsilon_y'-I_n\right)C_\alpha'\Phi^{-1}C_\zeta\varepsilon_{t-1-\zeta}\varepsilon_{2t}'\right\}$$
$$=\frac{q}{T}tr\left\{\sum_{\alpha,\kappa}C_\alpha'\Phi^{-1}C_\kappa\left(\varepsilon_y\varepsilon_y'-I_n\right)C_\kappa'\Phi^{-1}C_\alpha\left(\varepsilon_y\varepsilon_y'-I_n\right)\right\}$$
$$=\frac{q}{T}\sum_{\alpha,\kappa}tr^2\left\{C_\alpha'\Phi^{-1}C_\kappa\right\}+\frac{q}{T}tr\sum_{\alpha,\kappa}\left\{\left(C_\alpha'\Phi^{-1}C_\kappa\right)^2\right\}$$

So we have that the total is equal to:

$$\boxed{D_{61}=\frac{q}{T}\sum_{\alpha,\kappa}tr^2\left\{C_\alpha'\Phi^{-1}C_\kappa\right\}+\frac{q}{T}tr\sum_{\alpha,\kappa}\left\{\left(C_\alpha'\Phi^{-1}C_\kappa\right)^2\right\}}$$

A.8.2 Derivation of D_{62}

$$\frac{1}{T^3}Etr\left\{\sum_{t,s,v,w=1}^{T}\sum_{\zeta,\eta,\kappa,\lambda,\alpha,\beta=0}^{\infty}\varepsilon_{2t}\varepsilon_{t-1-\zeta}'C_\zeta'\Phi^{-1}C_\kappa\left(\varepsilon_{v-1-\kappa}\varepsilon_{v-1-\lambda}'-\delta_{\kappa\lambda}I_n\right)C_\lambda'\Phi^{-1}\right.$$
$$\left.\times C_\alpha\left(\varepsilon_{w-1-\alpha}\varepsilon_{w-1-\beta}'-\delta_{\alpha\beta}I_n\right)C_\beta'\Phi^{-1}C_\eta\varepsilon_{s-1-\eta}\varepsilon_{2s}'\right\}$$

There are two combinations:

1. $t = s, \eta = \zeta,$
   $v - 1 - \kappa = w - 1 - \alpha$
   $v - 1 - \lambda = w - 1 - \beta$
   $\kappa \neq \lambda$

2. $t = s, \eta = \zeta,$
   $v - 1 - \lambda = w - 1 - \alpha$
   $v - 1 - \kappa = w - 1 - \beta$
   $\kappa \neq \lambda$

First combination This combination implies that:

$$\boxed{\beta + \kappa = \alpha + \lambda, \kappa \neq \lambda}$$

$$\frac{1}{T^3} Etr \left\{ \sum_{t,v,w=1}^{T} \sum_{\zeta=0}^{\infty} \sum_{\substack{\beta+\kappa=\alpha+\lambda \\ \kappa\neq\lambda}} \varepsilon_{2t}\varepsilon'_{t-1-\zeta} C'_\zeta \Phi^{-1} C_\kappa \varepsilon_{v-1-\kappa}\varepsilon'_{v-1-\lambda} C'_\lambda \Phi^{-1} \right.$$
$$\left. \times C_\alpha \varepsilon_{w-1-\alpha}\varepsilon'_{w-1-\beta} C'_{\alpha+\lambda-\kappa} \Phi^{-1} C_\zeta \varepsilon_{t-1-\zeta}\varepsilon'_{2t} \right\}$$
$$= \frac{q}{T} Etr \left\{ \sum_{\substack{\beta+\kappa=\alpha+\lambda \\ \kappa\neq\lambda}} \Phi^{-1} C_\kappa \varepsilon_{v-1-\kappa}\varepsilon'_{v-1-\lambda} C'_\lambda \Phi^{-1} C_\alpha \varepsilon_{w-1-\alpha}\varepsilon'_{w-1-\beta} C'_\beta \right\}$$
$$= \frac{q}{T} tr \left\{ \sum_{\substack{\beta+\kappa=\alpha+\lambda \\ \kappa\neq\lambda}} C'_\beta \Phi^{-1} C_\kappa C'_\alpha \Phi^{-1} C_\lambda \right\}$$

Note that this expression is exactly the opposite of expression D33. So we conclude that the expectation of this combination is equal to:

$$\frac{q}{T} tr \left\{ \sum_{\substack{\beta+\kappa=\alpha+\lambda \\ \kappa\neq\lambda}} C'_\beta \Phi^{-1} C_\kappa C'_\alpha \Phi^{-1} C_\lambda \right\}$$

Second combination Combining the conditions, we obtain:

$$\boxed{\beta + \lambda = \alpha + \kappa, \kappa \neq \lambda}$$

$$tr\left\{E\left(\frac{1}{\sqrt{T}}U'X\right)\Phi^{-1}\left(\Phi-\frac{1}{T}X'X\right)\Phi^{-1}\left(\Phi-\frac{1}{T}X'X\right)\Phi^{-1}\left(\frac{1}{\sqrt{T}}X'U\right)\right\}$$
$$=\frac{1}{T^3}Etr\left\{\sum_{t,v,w=1}^{T}\sum_{\zeta,\kappa,\lambda,\alpha,\beta=0}^{\infty}\varepsilon_{2t}\varepsilon'_{t-1-\zeta}C'_\zeta\Phi^{-1}C_\kappa\varepsilon_{v-1-\kappa}\varepsilon'_{v-1-\lambda}C'_\lambda\Phi^{-1}\right.$$
$$\left.\times C_\alpha\varepsilon_{w-1-\alpha}\varepsilon'_{w-1-\beta}C'_{\alpha-\lambda+\kappa}\Phi^{-1}C_\zeta\varepsilon_{t-1-\zeta}\varepsilon'_{2t}\right\}$$
$$=\frac{q}{T}\sum_{\substack{\beta+\lambda=\alpha+\kappa\\ \kappa\neq\lambda}}tr\left\{C'_\beta\Phi^{-1}C_\kappa\right\}tr\left\{C'_\lambda\Phi^{-1}C_\alpha\right\}$$

Total The total expectation of this term is therefore equal to:

$$\boxed{\begin{aligned}D_{62}=&\tfrac{q}{T}tr\textstyle\sum_{\substack{\kappa,\lambda,\alpha=0\\ \kappa\neq\lambda}}^{\infty}\left\{C'_{\alpha+\lambda-\kappa}\Phi^{-1}C_\kappa C'_\alpha\Phi^{-1}C_\lambda\right\}\\ &+\tfrac{q}{T}\textstyle\sum_{\substack{\beta+\lambda=\alpha+\kappa\\ \kappa\neq\lambda}}tr\left\{C'_\beta\Phi^{-1}C_\kappa\right\}tr\left\{C'_\lambda\Phi^{-1}C_\alpha\right\}\end{aligned}}$$

A.8.3 Derivation of D_{63}

$$tr\left\{E\left(\frac{1}{\sqrt{T}}U'X\right)\Phi^{-1}\left(\Phi-\frac{1}{T}X'X\right)\Phi^{-1}\left(\Phi-\frac{1}{T}X'X\right)\Phi^{-1}\left(\frac{1}{\sqrt{T}}X'U\right)\right\}$$
$$=\frac{1}{T^3}Etr\left\{\sum_{t,s,v,w=1}^{T}\sum_{\zeta,\eta,\kappa,\lambda,\alpha,\beta=0}^{\infty}\varepsilon_{2t}\varepsilon'_{t-1-\zeta}C'_\zeta\Phi^{-1}C_\kappa\left(\varepsilon_{v-1-\kappa}\varepsilon'_{v-1-\lambda}-\delta_{\kappa\lambda}I_n\right)C'_\lambda\Phi^{-1}\right.$$
$$\left.\times C_\alpha\left(\varepsilon_{w-1-\alpha}\varepsilon'_{w-1-\beta}-\delta_{\alpha\beta}I_n\right)C'_\beta\Phi^{-1}C_\eta\varepsilon_{s-1-\eta}\varepsilon'_{2s}\right\}$$

There are eight possible constellations, which give rise to first order terms:

1.	$s=w-1-\beta$	$s-1-\eta=w-1-\alpha$	$t=v-1-\kappa$	$t-1-\zeta=v-1-\lambda$
2.	$s=w-1-\alpha$	$s-1-\eta=w-1-\beta$	$t=v-1-\kappa$	$t-1-\zeta=v-1-\lambda$
3	$s=w-1-\beta$	$s-1-\eta=w-1-\alpha$	$t=v-1-\lambda$	$t-1-\zeta=v-1-\kappa$
4	$s=w-1-\alpha$	$s-1-\eta=w-1-\beta$	$t=v-1-\lambda$	$t-1-\zeta=v-1-\kappa$
5	$s=v-1-\kappa$	$s-1-\eta=v-1-\lambda$	$t=w-1-\beta$	$t-1-\zeta=w-1-\alpha$
6	$s=v-1-\lambda$	$s-1-\eta=v-1-\kappa$	$t=w-1-\beta$	$t-1-\zeta=w-1-\alpha$
7	$s=v-1-\kappa$	$s-1-\eta=v-1-\lambda$	$t=w-1-\alpha$	$t-1-\zeta=w-1-\beta$
8	$s=v-1-\lambda$	$s-1-\eta=v-1-\kappa$	$t=w-1-\alpha$	$t-1-\zeta=w-1-\beta$

In all of these eight constellations we have that $\kappa\neq\lambda,\alpha\neq\beta$. We shall now take them one by one:

First combination This combination implies that:

$$\boxed{\alpha=\beta+\eta+1\text{ and }\lambda=\kappa+\zeta+1}$$

$$
\begin{aligned}
&\frac{1}{T^3}Etr\left\{\sum\varepsilon_{2t}\varepsilon'_{t-1-\zeta}C'_\zeta\Phi^{-1}C_\kappa\varepsilon_{v-1-\kappa}\varepsilon'_{v-1-\lambda}C'_\lambda\Phi^{-1}\right.\\
&\left.\times C_\alpha\varepsilon_{w-1-\alpha}\varepsilon'_{w-1-\beta}C'_\beta\Phi^{-1}C_\eta\varepsilon_{s-1-\eta}\varepsilon'_{2s}\right\}\\
&=\frac{1}{T^3}Etr\sum\left\{\varepsilon_{2t}\varepsilon'_{v-1-\kappa}C'_\kappa\Phi^{-1}C_\zeta\varepsilon_{t-1-\zeta}\varepsilon'_{v-1-\lambda}C'_\lambda\Phi^{-1}C_\alpha\varepsilon_{w-1-\alpha}\right.\\
&\left.\times\varepsilon'_{s-1-\eta}C'_\eta\Phi^{-1}C_\beta\varepsilon_{w-1-\beta}\varepsilon'_{2s}\right\}\\
&=\frac{1}{T}tr\left\{\sum_{\beta,\eta,\kappa,\zeta}\left[C'_\kappa\Phi^{-1}C_\zeta C'_{\kappa+\zeta+1}\Phi^{-1}C_{\beta+\eta+1}C'_\eta\Phi^{-1}C_\beta\right]_{22}\right\}
\end{aligned}
$$

Second combination

$$\boxed{\beta=\alpha+\eta+1 \text{ and } \lambda=\kappa+\zeta+1}$$

$$
\begin{aligned}
&\frac{1}{T^3}Etr\left\{\sum\varepsilon_{2t}\varepsilon'_{t-1-\zeta}C'_\zeta\Phi^{-1}C_\kappa\varepsilon_{v-1-\kappa}\varepsilon'_{v-1-\lambda}C'_\lambda\Phi^{-1}\right.\\
&\left.\times C_\alpha\varepsilon_{w-1-\alpha}\varepsilon'_{w-1-\beta}C'_\beta\Phi^{-1}C_\eta\varepsilon_{s-1-\eta}\varepsilon'_{2s}\right\}\\
&=\frac{1}{T^3}E\sum tr\left\{\varepsilon_{2t}\varepsilon'_{v-1-\kappa}C'_\kappa\Phi^{-1}C_\zeta\varepsilon_{t-1-\zeta}\varepsilon'_{v-1-\lambda}C'_\lambda\Phi^{-1}C_\alpha\varepsilon_{w-1-\alpha}\varepsilon'_{2s}\right\}\times\\
&tr\left\{\varepsilon'_{w-1-\beta}C'_\beta\Phi^{-1}C_\eta\varepsilon_{s-1-\eta}\right\}\\
&=\frac{1}{T}\sum_{\alpha,\eta,\kappa,\zeta=0}^{\infty}tr\left\{\left[C'_\kappa\Phi^{-1}C_\zeta C'_{\kappa+\zeta+1}\Phi^{-1}C_\alpha\right]_{22}\right\}tr\left\{C'_{\alpha+\eta+1}\Phi^{-1}C_\eta\right\}
\end{aligned}
$$

Third combination

$$\boxed{\alpha=\beta+\eta+1 \text{ and } \kappa=\lambda+\zeta+1}$$

$$
\begin{aligned}
&\frac{1}{T^3}Etr\left\{\sum\varepsilon_{2t}\varepsilon'_{t-1-\zeta}C'_\zeta\Phi^{-1}C_\kappa\varepsilon_{v-1-\kappa}\varepsilon'_{v-1-\lambda}C'_\lambda\Phi^{-1}\right.\\
&\left.\times C_\alpha\varepsilon_{w-1-\alpha}\varepsilon'_{w-1-\beta}C'_\beta\Phi^{-1}C_\eta\varepsilon_{s-1-\eta}\varepsilon'_{2s}\right\}\\
&=\frac{1}{T^3}E\sum tr\left\{\varepsilon_{2t}\varepsilon'_{v-1-\lambda}C'_\lambda\Phi^{-1}C_\alpha\varepsilon_{w-1-\alpha}\varepsilon'_{s-1-\eta}C'_\eta\Phi^{-1}C_\beta\varepsilon'_{w-1-\beta}\varepsilon'_{2s}\right\}\times\\
&tr\left\{\varepsilon'_{t-1-\zeta}C'_\zeta\Phi^{-1}C_\kappa\varepsilon_{v-1-\kappa}\right\}\\
&=\frac{1}{T}\sum_{\beta,\eta,\lambda,\zeta=0}^{\infty}tr\left\{\left[C'_\lambda\Phi^{-1}C_{\beta+\eta+1}C'_\eta\Phi^{-1}C_\beta\right]_{22}\right\}tr\left\{C'_\zeta\Phi^{-1}C_{\lambda+\zeta+1}\right\}
\end{aligned}
$$

which incidentally is equal to the second combination

Fourth combination

$$\boxed{\beta = \alpha + \eta + 1 \text{ and } \kappa = \lambda + \zeta + 1}$$

$$\frac{1}{T^3} Etr\left\{\sum \varepsilon_{2t}\varepsilon'_{t-1-\zeta}C'_\zeta\Phi^{-1}C_\kappa\varepsilon_{v-1-\kappa}\varepsilon'_{v-1-\lambda}C'_\lambda\Phi^{-1}\right.$$
$$\left.\times C_\alpha\varepsilon_{w-1-\alpha}\varepsilon'_{w-1-\beta}C'_\beta\Phi^{-1}C_\eta\varepsilon_{s-1-\eta}\varepsilon'_{2s}\right\}$$
$$= \frac{1}{T^3} E\sum tr\left\{\varepsilon_{2t}\varepsilon'_{v-1-\lambda}C'_\lambda\Phi^{-1}C_\alpha\varepsilon_{w-1-\alpha}\varepsilon'_{2s}\right\}$$
$$\times\, tr\left\{\varepsilon'_{t-1-\zeta}C'_\zeta\Phi^{-1}C_\kappa\varepsilon_{v-1-\kappa}\right\} tr\left\{\varepsilon'_{w-1-\beta}C'_\beta\Phi^{-1}C_\eta\varepsilon_{s-1-\eta}\right\}$$
$$= \frac{1}{T}\sum_{\alpha,\eta,\lambda,\zeta=0}^{\infty} tr\left\{\left[C'_\lambda\Phi^{-1}C_\alpha\right]_{22}\right\} tr\left\{C'_\zeta\Phi^{-1}C_{\lambda+\zeta+1}\right\} tr\left\{C'_{\alpha+\eta+1}\Phi^{-1}C_\eta\right\}$$

Fifth combination

$$\boxed{\lambda = \kappa + \eta + 1 \text{ and } \alpha = \beta + \zeta + 1}$$

$$\frac{1}{T^3} Etr\left\{\sum \varepsilon_{2t}\varepsilon'_{t-1-\zeta}C'_\zeta\Phi^{-1}C_\kappa\varepsilon_{v-1-\kappa}\varepsilon'_{v-1-\lambda}C'_\lambda\Phi^{-1}\right.$$
$$\left.\times C_\alpha\varepsilon_{w-1-\alpha}\varepsilon'_{w-1-\beta}C'_\beta\Phi^{-1}C_\eta\varepsilon_{s-1-\eta}\varepsilon'_{2s}\right\}$$
$$= \frac{1}{T^3} Etr\left\{\sum \varepsilon_{2t}\varepsilon'_{w-1-\beta}C'_\beta\Phi^{-1}C_\eta\varepsilon_{s-1-\eta}\varepsilon'_{v-1-\lambda}C'_\lambda\Phi^{-1}C_\alpha\varepsilon_{w-1-\alpha}\right.$$
$$\left.\times\varepsilon'_{t-1-\zeta}C'_\zeta\Phi^{-1}C_\kappa\varepsilon_{v-1-\kappa}\varepsilon'_{2s}\right\}$$
$$= \sum_{\beta,\eta,\kappa,\zeta=0}^{\infty} tr\left\{\left[C'_\beta\Phi^{-1}C_\eta C'_{\kappa+\eta+1}\Phi^{-1}C_{\beta+\zeta+1}C'_\zeta\Phi^{-1}C_\kappa\right]_{22}\right\}$$

Sixth combination

$$\boxed{\kappa = \lambda + \eta + 1 \text{ and } \alpha = \beta + \zeta + 1}$$

$$\frac{1}{T^3} Etr\left\{\sum \varepsilon_{2t}\varepsilon'_{t-1-\zeta}C'_\zeta\Phi^{-1}C_\kappa\varepsilon_{v-1-\kappa}\varepsilon'_{v-1-\lambda}C'_\lambda\Phi^{-1}\right.$$
$$\left.\times C_\alpha\varepsilon_{w-1-\alpha}\varepsilon'_{w-1-\beta}C'_\beta\Phi^{-1}C_\eta\varepsilon_{s-1-\eta}\varepsilon'_{2s}\right\}$$
$$= \frac{1}{T^3} E\sum tr\left\{\varepsilon_{2t}\varepsilon'_{w-1-\beta}C'_\beta\Phi^{-1}C_\eta\varepsilon_{s-1-\eta}\varepsilon'_{v-1-\kappa}C'_\kappa\Phi^{-1}C_\zeta\varepsilon_{t-1-\zeta}\varepsilon'_{w-1-\alpha}C'_\alpha\Phi^{-1}C_\lambda\varepsilon_{v-1-\lambda}\varepsilon'_{2s}\right\}$$
$$tr\left\{\varepsilon'_{w-1-\beta}C'_\beta\Phi^{-1}C_\eta\varepsilon_{s-1-\eta}\right\}$$
$$= \sum_{\beta,\eta,\lambda,\zeta=0}^{\infty} tr\left\{\left[C'_\beta\Phi^{-1}C_\eta C'_{\lambda+\eta+1}\Phi^{-1}C_\zeta C'_{\beta+\zeta+1}\Phi^{-1}C_\lambda\right]_{22}\right\} tr\left\{C'_\beta\Phi^{-1}C_\eta\right\}$$

Seventh combination

$$\boxed{\lambda = \kappa + \eta + 1 \text{ and } \beta = \alpha + \zeta + 1}$$

$$\frac{1}{T^3}Etr\left\{\sum \varepsilon_{2t}\varepsilon'_{t-1-\zeta}C'_\zeta\Phi^{-1}C_\kappa\varepsilon_{v-1-\kappa}\varepsilon'_{v-1-\lambda}C'_\lambda\Phi^{-1}\right.$$
$$\left.\times C_\alpha\varepsilon_{w-1-\alpha}\varepsilon'_{w-1-\beta}C'_\beta\Phi^{-1}C_\eta\varepsilon_{s-1-\eta}\varepsilon'_{2s}\right\}$$
$$=\frac{1}{T^3}E\sum tr\left\{\varepsilon_{2t}\varepsilon'_{w-1-\alpha}C'_\alpha\Phi^{-1}C_\lambda\varepsilon_{v-1-\lambda}\varepsilon'_{s-1-\eta}C'_\eta\Phi^{-1}C_\beta\varepsilon_{w-1-\beta}\right.$$
$$\left.\times\varepsilon'_{t-1-\zeta}C'_\zeta\Phi^{-1}C_\kappa\varepsilon_{v-1-\kappa}\varepsilon'_{2s}\right\}$$
$$=\frac{1}{T}\sum_{\alpha,\eta,\kappa,\zeta=0}^{\infty}tr\left\{\left[C'_\alpha\Phi^{-1}C_{\kappa+\eta+1}C'_\eta\Phi^{-1}C_{\alpha+\zeta+1}C'_\zeta\Phi^{-1}C_\kappa\right]_{22}\right\}$$

Eighth combination

$$\boxed{\kappa=\lambda+\eta+1 \text{ and } \beta=\alpha+\zeta+1}$$

$$\frac{1}{T^3}Etr\left\{\sum \varepsilon_{2t}\varepsilon'_{t-1-\zeta}C'_\zeta\Phi^{-1}C_\kappa\varepsilon_{v-1-\kappa}\varepsilon'_{v-1-\lambda}C'_\lambda\Phi^{-1}\right.$$
$$\left.\times C_\alpha\varepsilon_{w-1-\alpha}\varepsilon'_{w-1-\beta}C'_\beta\Phi^{-1}C_\eta\varepsilon_{s-1-\eta}\varepsilon'_{2s}\right\}$$
$$=\frac{1}{T^3}E\sum tr\left\{\varepsilon_{2t}\varepsilon'_{w-1-\alpha}C'_\alpha\Phi^{-1}C_\lambda\varepsilon_{v-1-\lambda}\varepsilon'_{2s}\right\}$$
$$tr\left\{\varepsilon_{w-1-\beta}\varepsilon'_{t-1-\zeta}C'_\zeta\Phi^{-1}C_\kappa\varepsilon_{v-1-\kappa}\varepsilon'_{s-1-\eta}C'_\eta\Phi^{-1}C_\beta\right\}$$
$$=\frac{1}{T}\sum_{\alpha,\eta,\lambda,\zeta=0}^{\infty}tr\left\{\left[C'_\alpha\Phi^{-1}C_\lambda\right]_{22}\right\}tr\left\{C'_\zeta\Phi^{-1}C_{\lambda+\eta+1}C'_\eta\Phi^{-1}C_{\alpha+\zeta+1}\right\}$$

which is seen to equal the sixth combination

Total The total of the D_{63} term then becomes:

$$\boxed{\begin{aligned}
&D_{63}=\tfrac{1}{T}tr\left\{\textstyle\sum_{\beta,\eta,\kappa,\zeta}\left[C'_\kappa\Phi^{-1}C_\zeta C'_{\kappa+\zeta+1}\Phi^{-1}C_{\beta+\eta+1}C'_\eta\Phi^{-1}C_\beta\right]_{22}\right\}\\
&+\tfrac{2}{T}\textstyle\sum_{\alpha,\eta,\kappa,\zeta=0}^{\infty}tr\left\{\left[C'_\kappa\Phi^{-1}C_\zeta C'_{\kappa+\zeta+1}\Phi^{-1}C_\alpha\right]_{22}\right\}tr\left\{C'_{\alpha+\eta+1}\Phi^{-1}C_\eta\right\}\\
&+\tfrac{1}{T}\textstyle\sum_{\alpha,\eta,\lambda,\zeta=0}^{\infty}tr\left\{\left[C'_\lambda\Phi^{-1}C_\alpha\right]_{22}\right\}tr\left\{C'_\zeta\Phi^{-1}C_{\lambda+\zeta+1}\right\}tr\left\{C'_{\alpha+\eta+1}\Phi^{-1}C_\eta\right\}\\
&+\tfrac{1}{T}\textstyle\sum_{\beta,\eta,\kappa,\zeta=0}^{\infty}tr\left\{\left[C'_\beta\Phi^{-1}C_\eta C'_{\kappa+\eta+1}\Phi^{-1}C_{\beta+\zeta+1}C'_\zeta\Phi^{-1}C_\kappa\right]_{22}\right\}\\
&+\tfrac{2}{T}\textstyle\sum_{\beta,\eta,\lambda,\zeta=0}^{\infty}tr\left\{\left[C'_\beta\Phi^{-1}C_\eta C'_{\lambda+\eta+1}\Phi^{-1}C_\zeta C'_{\beta+\zeta+1}\Phi^{-1}C_\lambda\right]_{22}\right\}\\
&+\tfrac{1}{T}\textstyle\sum_{\alpha,\eta,\lambda,\zeta=0}^{\infty}tr\left\{\left[C'_\alpha\Phi^{-1}C_\lambda\right]_{22}\right\}tr\left\{C'_\zeta\Phi^{-1}C_{\lambda+\eta+1}C'_\eta\Phi^{-1}C_{\alpha+\zeta+1}\right\}
\end{aligned}}$$

A.9 The second term: $E\left[\frac{1}{2T}tr(K^2)\right]$

This term is already of the order $\frac{1}{T}$, so we just have to take the nullth order expansion of $F = E\left[tr(K^2)\right]$

$$tr\left\{E\left[K^2\right]\right\}$$
$$= tr\left\{E\left[\left(\frac{1}{T}U'U\right)^{-1}\left(\frac{1}{\sqrt{T}}U'X\right)\left(\frac{1}{T}X'X\right)^{-1}\left(\frac{1}{\sqrt{T}}X'U\right)\left(\frac{1}{T}U'U\right)^{-1}\right.\right.$$
$$\left.\left.\times\left(\frac{1}{\sqrt{T}}U'X\right)\left(\frac{1}{T}X'X\right)^{-1}\left(\frac{1}{\sqrt{T}}X'U\right)\right]\right\}$$
$$\overset{0}{=} tr\left\{E\left[\left(\frac{1}{\sqrt{T}}U'X\right)\Phi^{-1}\left(\frac{1}{\sqrt{T}}X'U\right)\left(\frac{1}{\sqrt{T}}U'X\right)\Phi^{-1}\left(\frac{1}{\sqrt{T}}X'U\right)\right]\right\}$$
$$= tr\left\{E\left[\frac{1}{T}\sum_{t,s,m,n=1}^{T}\sum_{\zeta,\eta,\gamma,\chi=0}^{\infty}\varepsilon_{2t}\varepsilon'_{t-1-\zeta}C'_\zeta\Phi^{-1}C_\eta\varepsilon_{s-1-\eta}\varepsilon'_{2s}\varepsilon_{2m}\varepsilon'_{m-1-\gamma}C'_\gamma\Phi^{-1}C_\chi\varepsilon_{n-1-\chi}\varepsilon'_{2n}\right]\right\}$$

There are three possible combinations:

1. $t = s, \zeta = \eta, m = n, \gamma = \chi$
2. $t = m, \zeta = \gamma, s = n, \eta = \chi$
3. $t = n, \zeta = \chi, s = m, \gamma = \eta$

A.9.1 First combination

$$tr\left\{E\left[\frac{1}{T^2}\sum_{t,s,m,n=1}^{T}\sum_{\zeta,\eta,\gamma,\chi=0}^{\infty}\varepsilon_{2t}\varepsilon'_{t-1-\zeta}C'_\zeta\Phi^{-1}C_\eta\varepsilon_{s-1-\eta}\varepsilon'_{2s}\varepsilon_{2m}\varepsilon'_{m-1-\gamma}C'_\gamma\Phi^{-1}C_\chi\varepsilon_{n-1-\chi}\varepsilon'_{2n}\right]\right\}$$
$$= \frac{1}{T^2}E\sum_{t,m=1}^{T}\sum_{\zeta,\gamma=0}^{\infty}tr\left\{\left[\varepsilon_{2t}\varepsilon'_{2s}\varepsilon_{2m}\varepsilon'_{2n}\right]\right\}tr\left\{\varepsilon'_{t-1-\zeta}C'_\zeta\Phi^{-1}C_\zeta\varepsilon_{s-1-\eta}\right\}tr\left\{\varepsilon'_{m-1-\gamma}C'_\gamma\Phi^{-1}C_\gamma\varepsilon_{n-1-\chi}\right\}$$
$$= qn^2$$

A.9.2 Second combination

$$tr\left\{E\left[\frac{1}{T^2}\sum_{t,s,m,n=1}^{T}\sum_{\zeta,\eta,\gamma,\chi=0}^{\infty}\varepsilon_{2t}\varepsilon'_{t-1-\zeta}C'_\zeta\Phi^{-1}C_\eta\varepsilon_{s-1-\eta}\varepsilon'_{2s}\varepsilon_{2m}\varepsilon'_{m-1-\gamma}C'_\gamma\Phi^{-1}C_\chi\varepsilon_{n-1-\chi}\varepsilon'_{2n}\right]\right\}$$
$$= \frac{1}{T^2}E\sum_{t,s=1}^{T}\sum_{\zeta,\eta=0}^{\infty}tr\left\{\varepsilon_{2t}\varepsilon'_{2m}\varepsilon_{2s}\varepsilon'_{2n}\right\}tr\left\{\varepsilon_{n-1-\gamma}\varepsilon'_{t-1-\zeta}C'_\zeta\Phi^{-1}C_\eta\varepsilon_{s-1-\eta}\varepsilon'_{m-1-\chi}C'_\eta\Phi^{-1}C_\zeta\right\}$$
$$= qn$$

A.9.3 Third combination

$$
\begin{aligned}
& tr\left\{E\left[\frac{1}{T^2}\sum_{t,s,m,n=1}^{T}\sum_{\zeta,\eta,\gamma,\chi=0}^{\infty}\varepsilon_{2t}\varepsilon'_{t-1-\zeta}C'_\zeta\Phi^{-1}C_\eta\varepsilon_{s-1-\eta}\varepsilon'_{2s}\varepsilon_{2m}\varepsilon'_{m-1-\gamma}C'_\gamma\Phi^{-1}C_\chi\varepsilon_{n-1-\chi}\varepsilon'_{2n}\right]\right\} \\
&= \frac{1}{T^2}E\sum_{t,s=1}^{T}\sum_{\zeta,\eta=0}^{\infty}tr\left\{\varepsilon'_{2n}\varepsilon_{2t}\varepsilon'_{2s}\varepsilon_{2m}\right\}tr\left\{C_\zeta\varepsilon_{n-1-\chi}\varepsilon'_{t-1-\zeta}C'_\zeta\Phi^{-1}C_\eta\varepsilon_{s-1-\eta}\varepsilon'_{m-1-\gamma}C'_\eta\Phi^{-1}\right\} \\
&= q^2n
\end{aligned}
$$

So the total contribution of this term is:

$$
\boxed{\boxed{\frac{1}{2T}F = \frac{qn}{2T} + \frac{q^2n+qn^2}{2T}}}
$$

A.10 Total

Adding up all the terms, we find:

$$
\begin{aligned}
E\left[LR\right] = qn &+ \frac{1}{2T}\left(-4q+qn+q^2n+qn^2\right) \\
&+ \frac{1}{T}tr\left\{\sum\nolimits_{\beta,\eta,\kappa,\zeta=0}^{\infty}\left[C'_\kappa\Phi^{-1}C_\zeta C'_{\kappa+\zeta+1}\Phi^{-1}C_{\beta+\eta+1}C'_\eta\Phi^{-1}C_\beta\right]_{22}\right\} \\
&+ \frac{2}{T}\sum\nolimits_{\alpha,\eta,\kappa,\zeta=0}^{\infty}tr\left\{\left[C'_\kappa\Phi^{-1}C_\zeta C'_{\kappa+\zeta+1}\Phi^{-1}C_\alpha\right]_{22}\right\}tr\left\{C'_{\alpha+\eta+1}\Phi^{-1}C_\eta\right\} \\
&+ \frac{1}{T}\sum\nolimits_{\alpha,\eta,\lambda,\zeta=0}^{\infty}tr\left\{\left[C'_\lambda\Phi^{-1}C_\alpha\right]_{22}\right\}tr\left\{C'_\zeta\Phi^{-1}C_{\lambda+\zeta+1}\right\}tr\left\{C'_{\alpha+\eta+1}\Phi^{-1}C_\eta\right\} \\
&+ \frac{1}{T}\sum\nolimits_{\beta,\eta,\kappa,\zeta=0}^{\infty}tr\left\{\left[C'_\beta\Phi^{-1}C_\eta C'_{\kappa+\eta+1}\Phi^{-1}C_{\beta+\zeta+1}C'_\zeta\Phi^{-1}C_\kappa\right]_{22}\right\} \\
&+ \frac{2}{T}\sum\nolimits_{\beta,\eta,\lambda,\zeta=0}^{\infty}tr\left\{\left[C'_\beta\Phi^{-1}C_\eta C'_{\lambda+\eta+1}\Phi^{-1}C_\zeta C'_{\beta+\zeta+1}\Phi^{-1}C_\lambda\right]_{22}\right\} \\
&+ \frac{1}{T}\sum\nolimits_{\alpha,\eta,\lambda,\zeta=0}^{\infty}tr\left\{\left[C'_\alpha\Phi^{-1}C_\lambda\right]_{22}\right\}tr\left\{C'_\zeta\Phi^{-1}C_{\lambda+\eta+1}C'_\eta\Phi^{-1}C_{\alpha+\zeta+1}\right\} \\
&- \frac{2}{T}\sum\nolimits_{\zeta,\eta,\kappa=0}^{\infty}tr\left\{\left[C'_\kappa\Phi^{-1}C_\varsigma\right]_{22}\right\}tr\left\{C'_{\kappa+\zeta+\eta+2}\Phi^{-1}C_\eta\right\} \\
&- \frac{2}{T}\sum\nolimits_{\zeta,\eta,\lambda=0}^{\infty}tr\left\{\left[C'_\lambda\Phi^{-1}C_\eta C'_{\lambda+\zeta+\eta+2}\Phi^{-1}C_\varsigma\right]_{22}\right\} \\
&- \frac{2}{T}\sum\nolimits_{\kappa,\eta,\alpha=0}^{\infty}tr\left\{\left[C'_\kappa\Phi^{-1}C_{\alpha+\eta-1}C'_\eta\Phi^{-1}C_{\alpha+\kappa+1}\right]_{22}\right\} \\
&- \frac{2}{T}\sum\nolimits_{\kappa,\zeta,\alpha=0}^{\infty}tr\left\{\left[C'_{\alpha+\kappa+1}\Phi^{-1}C_{\zeta+\alpha+1}C'_\zeta\Phi^{-1}C_\kappa\right]_{22}\right\}
\end{aligned} \tag{44}
$$

substituting $\Gamma_j = E\left[X_tX'_{t-j}\right] = \sum_{\alpha=0}^{\infty}C_{\alpha+j}C'_\alpha$ where possible gives the expression in theorem 1 which is hereby proven.

A.11 Proof of Theorem 2

For theorem 2 we note that the log-likelihood equals:

$$l_T = -\frac{1}{2}Tq\log 2\pi - \frac{T}{2}\log|\Omega_{22}| - \frac{1}{2}tr\left\{\Omega_{22}(Y-XA')'(Y-XA')\right\}$$

Thus for a known variance-covariance matrix $\Omega_{22} = I$, the likelihood ratio statistic equals

$$\begin{aligned}-2\ln LR\left(A=A_0\right) &= tr\left\{(Y-XA_0')'(Y-XA_0')\right\} - tr\left\{(Y-X\hat{A}')'(Y-X\hat{A}')\right\} \\ &= tr\left\{U'U\right\} - tr\left\{(Y-XA_0'+X\left(A_0-\hat{A}\right))'(Y-XA_0'+X\left(A_0-\hat{A}\right))\right\} \\ &= tr\left\{U'U\right\} - tr\left\{(Y-X\left(X'X\right)^{-1}X'U)'(Y-X\left(X'X\right)^{-1}X'U)\right\} \\ &= tr\left\{\left(U'X\right)\left(X'X\right)^{-1}\left(X'U\right)\right\}\end{aligned}$$

where we have used that $\hat{A} = \left(X'X\right)^{-1}\left(X'Y\right) = A_0 + \left(X'X\right)^{-1}\left(X'U\right)$ and defined $U = Y - XA_0$. We thus obtain:

$$-2\ln LR\left(A=A_0\right) = tr\left\{\left(U'X\right)\left(X'X\right)^{-1}\left(X'U\right)\right\}$$

A first order expansion of this expression (using equation (40)) delivers

$$\begin{aligned}&E\left[tr\left\{\left(U'X\right)\left(X'X\right)^{-1}\left(X'U\right)\right\}\right] \\ &\overset{1}{=} tr\left\{E\left(\frac{1}{\sqrt{T}}U'X\right)\Phi^{-1}\left(\frac{1}{\sqrt{T}}X'U\right)\right\} \\ &+ tr\left\{E\left(\frac{1}{\sqrt{T}}U'X\right)\Phi^{-1}\left(\Phi-\frac{1}{T}X'X\right)\Phi^{-1}\left(\frac{1}{\sqrt{T}}X'U\right)\right\} \\ &+ tr\left\{E\left(\frac{1}{\sqrt{T}}U'X\right)\Phi^{-1}\left(\Phi-\frac{1}{T}X'X\right)\Phi^{-1}\left(\Phi-\frac{1}{T}X'X\right)\Phi^{-1}\left(\frac{1}{\sqrt{T}}X'U\right)\right\}\end{aligned}$$

or stated differently

$$E\left[tr\left\{\left(U'X\right)\left(X'X\right)^{-1}\left(X'U\right)\right\}\right] \overset{1}{=} D_1 + D_3 + D_6$$

Adding up the expressions for all these terms, which were calculated in the last paragraph, deliver the result in theorem 2.

A.12 Proof of theorem 10

We take the terms of theorem 1 one by one, substute $C_\beta = SD^\beta F$ and $D = V\Lambda V^{-1}$ and then simplify. In this proof all ten terms turn out to be different.

$$\begin{aligned}t1' &= \sum\nolimits_{\beta,\eta,\kappa,\zeta=0}^{\infty} tr\left\{C_\kappa'\Phi^{-1}C_\zeta C_{\kappa+\zeta+1}'\Phi^{-1}C_{\beta+\eta+1}C_\eta'\Phi^{-1}C_\beta I_{22}\right\} \\ &= \sum\nolimits_{\beta,\kappa=0}^{\infty} tr\left\{F'D^{\kappa\prime}S'\left(S\Psi S'\right)^{-1}S\Psi D^{\kappa+1\prime}S'\left(S\Psi S'\right)^{-1}SD^{\beta+1}\Psi S'\left(S\Psi S'\right)^{-1}SD^\beta FI_{22}\right\} \\ &= \sum\nolimits_{\beta,\kappa=0}^{\infty} tr\left\{FI_{22}F'D^{\kappa\prime}S'\left(S\Psi S'\right)^{-1}S\Psi D^{\kappa+1\prime}S'\left(S\Psi S'\right)^{-1}SD^{\beta+1}\Psi S'\left(S\Psi S'\right)^{-1}SD^\beta\right\} \\ &= \sum\nolimits_{\beta,\kappa=0}^{\infty} tr\left\{\left(V^{-1}FI_{22}F'V^{-1\prime}\right)\Lambda^\kappa\left(V'PV^{-1\prime}\Lambda\right)\Lambda^\kappa\left(V'S'\Phi^{-1}SV\Lambda\right)\Lambda^\beta\left(V^{-1}P'V\right)\Lambda^\beta\right\} \\ &= tr\left\{A_1\left(A_2\oslash A_8\right)A_3\left(A_4\oslash A_8\right)\right\}\end{aligned}$$

$$
\begin{aligned}
t2' &= \sum\nolimits_{\alpha,\eta,\kappa,\zeta=0}^{\infty} tr\left\{C'_{\kappa}\Phi^{-1}C_{\zeta}C'_{\kappa+\zeta+1}\Phi^{-1}C_{\alpha}I_{22}\right\} tr\left\{C'_{\alpha+\eta+1}\Phi^{-1}C_{\eta}\right\} \\
&= \sum\nolimits_{\alpha,\kappa=0}^{\infty} tr\left\{F'D^{\kappa\prime}S'\Phi^{-1}S\Psi D^{\kappa+1\prime}S'\Phi^{-1}SD^{\alpha}FI_{22}\right\} tr\left\{S'\Phi^{-1}S\Psi D^{\alpha+1\prime}\right\} \\
&= \sum\nolimits_{\alpha,\kappa=0}^{\infty} tr\left\{\left(V'PV^{-1\prime}\Lambda\right)\Lambda^{\alpha}\right\} tr\left\{\left(V^{-1}FI_{22}F'V^{-1\prime}\right)\Lambda^{\kappa}\left(V'PV^{-1\prime}\Lambda\right)\Lambda^{\kappa}\left(V'S'\Phi^{-1}SV\right)\Lambda^{\alpha}\right\} \\
&= \sum\nolimits_{\alpha,\kappa=0}^{\infty} tr\left\{\left(V'PV^{-1\prime}\Lambda\right)\Lambda^{\alpha}\otimes\left(V^{-1}FI_{22}F'V^{-1\prime}\right)\Lambda^{\kappa}\left(V'PV^{-1\prime}\Lambda\right)\Lambda^{\kappa}\left(V'S'\Phi^{-1}SV\right)\Lambda^{\alpha}\right\} \\
&= tr\left\{\left(A_2\otimes A_1\right)\left(I\otimes\left(A_2\oslash\left(ll'-\Lambda^{co}\Lambda^{ro}\right)\right)A_5\right)\left(I_{n^2}-\Lambda\otimes\Lambda\right)^{-1}\right\} \\
&= \sum\nolimits_{i=1}^{n}\left(A_2\right)_{ii} tr\left\{A_1\left(A_2\oslash A_8\right)A_5A_{9i}\right\}
\end{aligned}
$$

$$
\begin{aligned}
t3' &= \sum\nolimits_{\alpha,\eta,\lambda,\zeta=0}^{\infty} tr\left\{C'_{\lambda}\Phi^{-1}C_{\alpha}I_{22}\right\} tr\left\{C'_{\zeta}\Phi^{-1}C_{\lambda+\zeta+1}\right\} tr\left\{C'_{\alpha+\eta+1}\Phi^{-1}C_{\eta}\right\} \\
&= \sum\nolimits_{\alpha,\lambda=0}^{\infty} tr\left\{P'D^{\lambda+1}\right\} tr\left\{PD^{\alpha+1\prime}\right\} tr\left\{FI_{22}F'D^{\lambda\prime}S'\Phi^{-1}SD^{\alpha}\right\} \\
&= \sum\nolimits_{\alpha,\lambda=0}^{\infty} tr\left\{\left(\Lambda V^{-1}P'V\right)\Lambda^{\lambda}\right\} tr\left\{\left(V'PV^{-1\prime}\Lambda\right)\Lambda^{\alpha}\right\} tr\left\{\left(V^{-1}FI_{22}F'V^{-1\prime}\right)\Lambda^{\lambda}\left(V'S'\Phi^{-1}SV\right)\Lambda^{\alpha}\right\} \\
&= \sum\nolimits_{\alpha,\lambda=0}^{\infty} tr\left\{A'_2\Lambda^{\lambda}\right\} tr\left\{A_2\Lambda^{\alpha}\right\} tr\left\{A_1\Lambda^{\lambda}A_5\Lambda^{\alpha}\right\} \\
&= tr\left\{\left(A'_2\otimes A_2\otimes A_1\right)\left(I-\Lambda\otimes I\otimes\Lambda\right)^{-1}\left(I\otimes I\otimes A_5\right)\left(I-I\otimes\Lambda\otimes\Lambda\right)^{-1}\right\} \\
&= \sum\nolimits_{i,j=1}^{n}\left(A_2\right)_{ii}\left(A_2\right)_{jj} tr\left\{A_1A_{9i}A_5A_{9j}\right\}
\end{aligned}
$$

$$
\begin{aligned}
t4' &= \sum\nolimits_{\beta,\eta,\kappa,\zeta=0}^{\infty} tr\left\{C'_{\beta}\Phi^{-1}C_{\eta}C'_{\kappa+\eta+1}\Phi^{-1}C_{\beta+\zeta+1}C'_{\zeta}\Phi^{-1}C_{\kappa}I_{22}\right\} \\
&= \sum\nolimits_{\beta,\eta,\kappa,\zeta=0}^{\infty} tr\left\{F'D^{\beta\prime}PD^{\kappa+1\prime}S'\Phi^{-1}SD^{\beta+1}P'D^{\kappa}FI_{22}\right\} \\
&= \sum\nolimits_{\beta,\kappa=0}^{\infty} tr\left\{\left(V^{-1}FI_{22}F'V^{-1\prime}\right)\Lambda^{\beta}\left(V'PV^{-1\prime}\Lambda\right)\Lambda^{\kappa}\left(V'S'\Phi^{-1}SV\Lambda\right)\Lambda^{\beta}\left(V^{-1}P'V\right)\Lambda^{\kappa}\right\} \\
&= \sum\nolimits_{\beta,\kappa=0}^{\infty} tr\left\{A_1\Lambda^{\beta}A_2\Lambda^{\kappa}A_3\Lambda^{\beta}A_4\Lambda^{\kappa}\right\} \\
&= \sum\nolimits_{i,j,k,m=1}^{n}\frac{\left(A_1\right)_{ij}\left(A_2\right)_{jk}\left(A_3\right)_{km}\left(A_4\right)_{mi}}{(1-\upsilon_j\upsilon_m)(1-\upsilon_i\upsilon_k)}
\end{aligned}
$$

$$
\begin{aligned}
t5' &= \sum\nolimits_{\beta,\eta,\lambda,\zeta=0}^{\infty} tr\left\{C'_{\beta}\Phi^{-1}C_{\eta}C'_{\lambda+\eta+1}\Phi^{-1}C_{\zeta}C'_{\beta+\zeta+1}\Phi^{-1}C_{\lambda}I_{22}\right\} \\
&= \sum\nolimits_{\beta,\lambda=0}^{\infty} tr\left\{F'D^{\beta\prime}PD^{\lambda+1\prime}PD^{\beta+1\prime}S'\Phi^{-1}SD^{\lambda}FI_{22}\right\} \\
&= \sum\nolimits_{\beta,\lambda=0}^{\infty} tr\left\{\left(V^{-1}FI_{22}F'V^{-1\prime}\right)\Lambda^{\kappa}\left(V'PV^{-1\prime}\Lambda\right)\Lambda^{\lambda}\left(V'PV^{-1\prime}\Lambda\right)\Lambda^{\kappa}\left(V'S'\Phi^{-1}SV\right)\Lambda^{\lambda}\right\} \\
&= \sum\nolimits_{\beta,\lambda=0}^{\infty} tr\left\{A_1\Lambda^{\kappa}A_2\Lambda^{\lambda}A_2\Lambda^{\kappa}A_5\Lambda^{\lambda}\right\} \\
&= \sum\nolimits_{i,j,k,m=1}^{n}\frac{\left(A_1\right)_{ij}\left(A_2\right)_{jk}\left(A_2\right)_{km}\left(A_5\right)_{mi}}{(1-\upsilon_j\upsilon_m)(1-\upsilon_i\upsilon_k)}
\end{aligned}
$$

$$
\begin{aligned}
t6' &= \sum\nolimits_{\alpha,\eta,\lambda,\zeta=0}^{\infty} tr\left\{C'_{\alpha}\Phi^{-1}C_{\lambda}I_{22}\right\} tr\left\{C'_{\zeta}\Phi^{-1}C_{\lambda+\eta+1}C'_{\eta}\Phi^{-1}C_{\alpha+\zeta+1}\right\} \\
&= \sum\nolimits_{\alpha,\lambda=0}^{\infty} tr\left\{D^{\alpha+1}P'D^{\lambda+1}P'\right\} tr\left\{F'D^{\alpha\prime}S'\Phi^{-1}SD^{\lambda}FI_{22}\right\} \\
&= \sum\nolimits_{\alpha,\lambda=0}^{\infty} tr\left\{\left(\Lambda V^{-1}P'V\right)\Lambda^{\alpha}\left(\Lambda V^{-1}P'V\right)\Lambda^{\lambda}\right\} tr\left\{\left(V^{-1}FI_{22}F'V^{-1\prime}\right)\Lambda^{\alpha}\left(V'S'\Phi^{-1}SV\right)\Lambda^{\lambda}\right\} \\
&= \sum\nolimits_{\alpha,\lambda=0}^{\infty} tr\left\{A'_2\Lambda^{\alpha}A'_2\Lambda^{\lambda}\right\} tr\left\{A_1\Lambda^{\alpha}A_5\Lambda^{\lambda}\right\} \\
&= tr\left\{\left(A'_2\otimes A_1\right)\left(I-\Lambda\otimes\Lambda\right)^{-1}\left(A'_2\otimes A_5\right)\left(I-\Lambda\otimes\Lambda\right)^{-1}\right\} \\
&= \sum\nolimits_{i,j=1}^{n} \left(A_2\right)_{ji}\left(A_2\right)_{ij} tr\left\{A_1A_{9i}A_5A_{9j}\right\}
\end{aligned}
$$

$$
\begin{aligned}
t7' &= \sum\nolimits_{\zeta,\eta,\kappa=0}^{\infty} tr\left\{C'_{\kappa}\Phi^{-1}C_{\zeta}I_{22}\right\} tr\left\{C'_{\kappa+\zeta+\eta+2}\Phi^{-1}C_{\eta}\right\} \\
&= \sum\nolimits_{\zeta,\kappa=0}^{\infty} tr\left\{D^{\kappa+\zeta+2}P\right\} tr\left\{F'D^{\kappa\prime}S'\Phi^{-1}SD^{\zeta}FI_{22}\right\} \\
&= \sum\nolimits_{\zeta,\kappa=0}^{\infty} tr\left\{\left(V^{-1}PV\right)\Lambda^{\kappa}\Lambda^{2}\Lambda^{\zeta}\right\} tr\left\{\left(V^{-1}FI_{22}F'V^{-1\prime}\right)\Lambda^{\kappa}\left(V'S'\Phi^{-1}SV\right)\Lambda^{\zeta}\right\} \\
&= \sum\nolimits_{\zeta,\kappa=0}^{\infty} tr\left\{A_4\Lambda^{\kappa}\Lambda^{2}\Lambda^{\zeta}\right\} tr\left\{A_1\Lambda^{\kappa}A_5\Lambda^{\zeta}\right\} \\
&= \sum\nolimits_{\zeta,\kappa=0}^{\infty} tr\left\{\left(A_4\otimes A_1\right)\left(I-\Lambda\otimes\Lambda\right)^{-1}\left(\Lambda^{2}\otimes A_5\right)\left(I-\Lambda\otimes\Lambda\right)^{-1}\right\} \\
&= \sum\nolimits_{i=1}^{n} \left(A_4\right)_{ii} \upsilon_i^{2} tr\left\{A_1A_{9i}A_5A_{9i}\right\}
\end{aligned}
$$

$$
\begin{aligned}
t8' &= \sum\nolimits_{\zeta,\eta,\lambda=0}^{\infty} tr\left\{C'_{\lambda}\Phi^{-1}C_{\eta}C'_{\lambda+\zeta+\eta+2}\Phi^{-1}C_{\zeta}I_{22}\right\} \\
&= \sum\nolimits_{\zeta,\lambda=0}^{\infty} tr\left\{FI_{22}F'D^{\lambda\prime}PD^{\lambda+\zeta+2\prime}S'\Phi^{-1}SD^{\zeta}\right\} \\
&= \sum\nolimits_{\zeta,\lambda=0}^{\infty} tr\left\{V^{-1}FI_{22}F'V^{-1\prime}\Lambda^{\kappa}V'PV^{-1\prime}\Lambda^{\kappa}\Lambda^{2}\Lambda^{\zeta\prime}V'S'\Phi^{-1}SV\Lambda^{\zeta}\right\} \\
&= \sum\nolimits_{\zeta,\lambda=0}^{\infty} tr\left\{\left(V^{-1}FI_{22}F'V^{-1\prime}\right)\Lambda^{\kappa}\left(V'PV^{-1\prime}\right)\Lambda^{\kappa}\left(\Lambda^{2}\right)\Lambda^{\zeta\prime}\left(V'S'\Phi^{-1}SV\right)\Lambda^{\zeta}\right\} \\
&= \sum\nolimits_{\zeta,\lambda=0}^{\infty} tr\left\{A_1\Lambda^{\kappa}A'_4\Lambda^{\kappa}\left(\Lambda^{2}\right)\Lambda^{\zeta\prime}A_5\Lambda^{\zeta}\right\} \\
&= tr\left\{A_1\left(A'_4\oslash\left(ll'-\Lambda^{co}\Lambda^{ro}\right)\right)\left(\Lambda^{2}\right)\left(A_5\oslash\left(ll'-\Lambda^{co}\Lambda^{ro}\right)\right)\right\} \\
&= tr\left\{A_1\left(A'_4\oslash A_8\right)\left(\Lambda^{2}\right)\left(A_5\oslash A_8\right)\right\}
\end{aligned}
$$

$$
\begin{aligned}
t9' &= \sum\nolimits_{\kappa,\eta,\alpha=0}^{\infty} tr\left\{C'_{\kappa}\Phi^{-1}C_{\alpha+\eta+1}C'_{\eta}\Phi^{-1}C_{\alpha+\kappa+1}I_{22}\right\} \\
&= \sum\nolimits_{\kappa,\alpha=0}^{\infty} tr\left\{FI_{22}F'D^{\kappa\prime}S\Phi^{-1}SD^{\alpha+1}P'D^{\alpha+\kappa+1}\right\} \\
&= \sum\nolimits_{\kappa,\alpha=0}^{\infty} tr\left\{\Lambda^{\kappa}\left(V^{-1}FI_{22}F'V^{-1\prime}\right)\Lambda^{\kappa}\left(V'S\Phi^{-1}SV\Lambda\right)\Lambda^{\alpha}\left(V^{-1}P'V\right)\Lambda^{\alpha}\Lambda\right\} \\
&= \sum\nolimits_{\kappa,\alpha=0}^{\infty} tr\left\{\Lambda^{\kappa}A_1\Lambda^{\kappa}A_3\Lambda^{\alpha}A_4\Lambda^{\alpha}\Lambda\right\} \\
&= tr\left\{\left(A_1\oslash A_8\right)A_3\left(A_4\oslash A_8\right)\Lambda\right\}
\end{aligned}
$$

$$
\begin{aligned}
t10' &= \sum\nolimits_{\kappa,\zeta,\alpha=0}^{\infty} tr\left\{C'_{\alpha+\kappa+1}\Phi^{-1}C_{\zeta+\alpha+1}C'_{\zeta}\Phi^{-1}C_{\kappa}I_{22}\right\} \\
&= \sum\nolimits_{\kappa,\alpha=0}^{\infty} tr\left\{FI_{22}F'D^{\alpha+\kappa+1\prime}S\Phi^{-1}SD^{\alpha+1}P'D^{\kappa}\right\} \\
&= \sum\nolimits_{\kappa,\alpha=0}^{\infty} tr\left\{\Lambda^{\kappa}\left(V^{-1}FI_{22}F'V^{-1\prime}\right)\Lambda^{\kappa}\Lambda\Lambda^{\alpha}\left(V'S\Phi^{-1}SV\Lambda\right)\Lambda^{\alpha}\left(V^{-1}P'V\right)\right\} \\
&= \sum\nolimits_{\kappa,\alpha=0}^{\infty} tr\left\{\Lambda^{\kappa}A_1\Lambda^{\kappa}\Lambda\Lambda^{\alpha}A_3\Lambda^{\alpha}A_4\right\} \\
&= tr\left\{\left(A_1 \oslash A_8\right)\Lambda\left(A_3 \oslash A_8\right)A_4\right\}
\end{aligned}
$$

Adding the ten terms up, we obtain the expression in theorem 10:

$$
\begin{aligned}
\daleth &= tr\left\{A_1\left(A_2 \oslash A_8\right)A_3\left(A_4 \oslash A_8\right)\right\} \\
&+ 2\sum\nolimits_{i=1}^{n}\left(A_2\right)_{ii} tr\left\{A_1\left(A_2 \oslash A_8\right)A_5A_{9i}\right\} \\
&+ \sum\nolimits_{i,j=1}^{n}\left(A_2\right)_{ii}\left(A_2\right)_{jj} tr\left\{A_1A_{9i}A_5A_{9j}\right\} \\
&+ \sum\nolimits_{i,j,k,m=1}^{n}\frac{\left(A_1\right)_{ij}\left(A_2\right)_{jk}\left(A_3\right)_{km}\left(A_4\right)_{mi}}{(1-\upsilon_j\upsilon_m)(1-\upsilon_i\upsilon_k)} \\
&+ 2\sum\nolimits_{i,j,k,m=1}^{n}\frac{\left(A_1\right)_{ij}\left(A_2\right)_{jk}\left(A_2\right)_{km}\left(A_5\right)_{mi}}{(1-\upsilon_j\upsilon_m)(1-\upsilon_i\upsilon_k)} \\
&+ \sum\nolimits_{i,j=1}^{n}\left(A_2\right)_{ji}\left(A_2\right)_{ij} tr\left\{A_1A_{9i}A_5A_{9j}\right\} \\
&- 2\sum\nolimits_{i=1}^{n}\left(A_4\right)_{ii}\upsilon_i^2 tr\left\{A_1A_{9i}A_5A_{9i}\right\} \\
&- 2tr\left\{A_1\left(A'_4 \oslash A_8\right)\left(\Lambda^2\right)\left(A_5 \oslash A_8\right)\right\} \\
&- 2tr\left\{\left(A_1 \oslash A_8\right)A_3\left(A_4 \oslash A_8\right)\Lambda\right\} \\
&- 2tr\left\{\left(A_1 \oslash A_8\right)\Lambda\left(A_3 \oslash A_8\right)A_4\right\}
\end{aligned}
$$

A.13 Proof of theorem 11

Theorem 11 is a special case of 10 with $S = I$. Inserting this in the expressions in table 3 we see that $P = I$ and furthermore that $A_2 = \Lambda, A_3 = A_6\Lambda, A_4 = I, A_5 = A_6$ and for any diagonal matrix G, $G \oslash A_8 = GA_7$. We substitute this in the ten terms of $\daleth$ in the last expression:

$$
\begin{aligned}
t1' &= tr\left\{A_1\left(A_2 \oslash A_8\right)A_3\left(A_4 \oslash A_8\right)\right\} \\
&= tr\left\{A_1\Lambda A_7A_6\Lambda A_7\right\}
\end{aligned}
$$

$$
\begin{aligned}
t2' &= \sum\nolimits_{i=1}^{n}\left(A_2\right)_{ii} tr\left\{A_1\left(A_2 \oslash A_8\right)A_5A_{9i}\right\} \\
&= \sum\nolimits_{i=1}^{n}\upsilon_i tr\left\{A_1\Lambda A_7A_6A_{9i}\right\}
\end{aligned}
$$

$$
\begin{aligned}
t3' &= \sum\nolimits_{i,j=1}^{n}\left(A_2\right)_{ii}\left(A_2\right)_{jj} tr\left\{A_1A_{9i}A_5A_{9j}\right\} \\
&= \sum\nolimits_{i,j=1}^{n}\upsilon_i\upsilon_j tr\left\{A_1A_{9i}A_6A_{9j}\right\}
\end{aligned}
$$

$$
\begin{aligned}
t4' &= \sum\nolimits_{\beta,\kappa=0}^{\infty} tr\left\{A_1\Lambda^{\beta}A_2\Lambda^{\kappa}A_3\Lambda^{\beta}A_4\Lambda^{\kappa}\right\} \\
&= \sum\nolimits_{\beta,\kappa=0}^{\infty} tr\left\{A_1\Lambda^{\beta+\kappa+1}A_6\Lambda^{\beta+\kappa+1}\right\} \\
&= tr\left\{\left(\left(\Lambda A_1\Lambda\right)\oslash A_8\right)\left(A_6\oslash A_8\right)\right\}
\end{aligned}
$$

$$
\begin{aligned}
t5' &= \sum\nolimits_{\beta,\lambda=0}^{\infty} tr\left\{A_1\Lambda^{\kappa}A_2\Lambda^{\lambda}A_2\Lambda^{\kappa}A_5\Lambda^{\lambda}\right\} \\
&= \sum\nolimits_{\beta,\lambda=0}^{\infty} tr\left\{A_1\Lambda^{2\kappa+\lambda+2}A_6\Lambda^{\lambda}\right\} \\
&= tr\left\{A_1A_7\Lambda^2\left(A_6\oslash A_8\right)\right\}
\end{aligned}
$$

$$
\begin{aligned}
t6' &= \sum\nolimits_{i,j=1}^{n}\left(A_2\right)_{ji}\left(A_2\right)_{ij} tr\left\{A_1A_{9i}A_5A_{9j}\right\} \\
&= \sum\nolimits_{i=1}^{n}\upsilon_i^2 tr\left\{A_1A_{9i}A_6A_{9i}\right\}
\end{aligned}
$$

$$
\begin{aligned}
t7' &= \sum\nolimits_{i=1}^{n}\left(A_4\right)_{ii}\upsilon_i^2 tr\left\{A_1A_{9i}A_5A_{9i}\right\} \\
&= \sum\nolimits_{i=1}^{n}\upsilon_i^2 tr\left\{A_1A_{9i}A_6A_{9i}\right\}
\end{aligned}
$$

$$
\begin{aligned}
t8' &= tr\left\{A_1\left(A_4'\oslash A_8\right)\left(\Lambda^2\right)\left(A_5\oslash A_8\right)\right\} \\
&= tr\left\{A_1A_7\Lambda^2\left(A_6\oslash A_8\right)\right\}
\end{aligned}
$$

$$
\begin{aligned}
t9' &= \sum\nolimits_{\kappa,\alpha=0}^{\infty} tr\left\{\Lambda^{\kappa}A_1\Lambda^{\kappa}A_3\Lambda^{\alpha}A_4\Lambda^{\alpha}\Lambda\right\} \\
&= \sum\nolimits_{\kappa,\alpha=0}^{\infty} tr\left\{A_1\Lambda^{\kappa}A_6\Lambda^{\kappa+2\alpha+2}\right\} \\
&= tr\left\{\Lambda^{\kappa+2\alpha+2}A_6\Lambda^{\kappa}A_1\right\} \\
&= tr\left\{A_1A_7\Lambda^2\left(A_6\oslash A_8\right)\right\}
\end{aligned}
$$

$$
\begin{aligned}
t10' &= tr\left\{\left(A_1\oslash A_8\right)\Lambda\left(A_3\oslash A_8\right)A_4\right\} \\
&= tr\left\{\left(\Lambda A_1\Lambda\oslash A_8\right)\left(A_6\oslash A_8\right)\right\}
\end{aligned}
$$

Noting that in this case $t_5' = t_8' = t_9', t_4' = t_{10}'$ and $t_6' = t_7'$ and adding up we find the result in theorem 11:

$$
\begin{aligned}
\daleth &= tr\left\{A_1\Lambda A_7A_6\Lambda A_7\right\} \\
&+ 2\sum\nolimits_{i=1}^{n}\upsilon_i tr\left\{A_1\Lambda A_7A_6A_{9i}\right\} \\
&+ \sum\nolimits_{i,j=1}^{n}\upsilon_i\upsilon_j tr\left\{A_1A_{9i}A_6A_{9j}\right\} \\
&- tr\left\{\left(\left(\Lambda A_1\Lambda\right)\oslash A_8\right)\left(A_6\oslash A_8\right)\right\} \\
&- 2tr\left\{A_1A_7\Lambda^2\left(A_6\oslash A_8\right)\right\} \\
&- \sum\nolimits_{i=1}^{n}\upsilon_i^2 tr\left\{A_1A_{9i}A_6A_{9i}\right\}
\end{aligned}
$$