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Chapter 4

Games and Large Cardinals

In this chapter, we investigate the upper bound of the consistency strength of the existence of alternating chains with length ω, which are essential objects proving projective determinacy from Woodin cardinals.

4.1 The consistency strength of the existence of alternating chains

In late 1980s, Martin and Steel [60] proved that if there are n Woodin cardinals and a measurable above them, then every $\mathbf{\Pi}^1_{n+1}$ set of reals is determined for each natural number n, where they introduced the notion of *iterations trees* which originally comes from the development of the inner model theory for strong cardinals. To build the inner model theory above one strong cardinal, one would have to iterate premice not only linearly but in more complicated way which would give us tree structures labeled with extenders that they call iteration trees. This generalization gives us another difficulty when we iterate premice more than ω times: In a limit stage, there could be many cofinal branches in the tree we have constructed and we have to choose one of them so that the direct limit through that branch will be wellfounded. This problem occurs when we reach the region of Woodin cardinals and Martin and Steel used this obstacle to prove projective determinacy by coding one second-order existential quantifier by the existence of cofinal wellfounded branch of suitable iteration trees (in their case, they arranged the iteration trees in such a way that the wellfounded branch is always unique). *Alternating chains* are the simplest iteration trees with this obstacle: They are iteration trees with length ω such that their tree structure is given as follows: For all natural numbers n, m,

$$mTn \iff m = 0 \text{ or } n - m \text{ is a positive even number.}$$

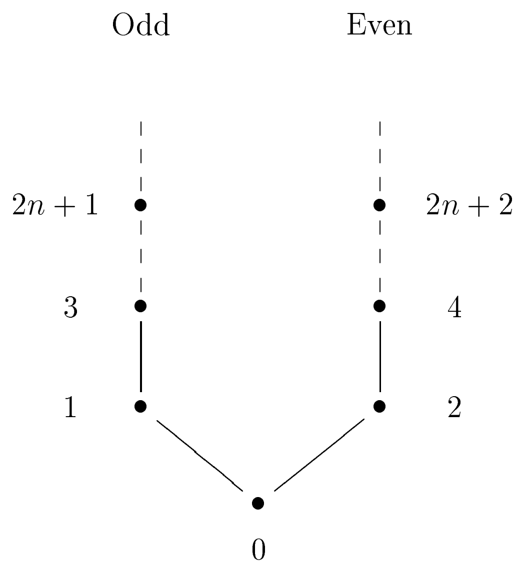

Figure 4.1: An alternating chain with length ω

This is the simplest tree structure with two cofinal branches. Let us call these two branches Even ($= \{2n \mid n \in \omega\}$) and Odd ($= \{2n+1 \mid n \in \omega\} \cup \{0\}$). Since these two branches are completely symmetric with respect to the tree structure, there is no canonical way to choose one of them so that the chosen one is wellfounded. This gives us the basic idea of how to code certain information via iteration trees. Actually, in the proof of projective determinacy, Martin and Steel replaced the odd part by ${}^{<\omega}\omega$ and ensured that the branch Even is ill-founded and that exactly one of the cofinal branches is wellfounded. This is how they code a real via a wellfounded cofinal branch.

But the above argument works only when there is only one wellfounded cofinal branch in the iteration tree. So the question is: Is there any iteration tree with length ω with more than one wellfounded branches? Martin and Steel [61] (independently by Woodin) proved that if there is a Woodin cardinal, then there are a countable transitive model M of (a large enough fragment of) ZFC and an alternating chain on M such that both branches are wellfounded. Conversely, they proved that if there is an iteration tree with limit length and two cofinal wellfounded branches, then there is a transitive model of ZF which satisfies "There is a Woodin cardinal". Hence there is a tight connection between Woodin cardinals and the existence of iteration trees with more than one cofinal wellfounded branches. In fact, what they proved is stronger:

Theorem 4.1.1 (Martin and Steel)**.** Suppose there is an iteration tree T with limit length and two cofinal branches b and c. Let δ be the supremum of the length of extenders used in T and α be an ordinal with $\alpha > \delta$ and α is in the wellfounded part of both M_b and M_c where M_b and M_c are the direct limit of models in T through b and c respectively. Then $\mathrm{L}_\alpha(V_\delta^{M_b}) \models$ "δ is Woodin".

Proof. See [62, Corollary 2.3]. $\square$

This theorem gives us more information: Note that $V_\delta^{M_b} = V_\delta^{M_c}$ and it is always a subset of the wellfounded part of both models. Since every wellfounded part of a model of KP is also a model of KP, we have the following: If one of M_b and M_c is wellfounded and θ is the least ordinal that is not in the wellfounded part of one of M_b and M_c and $\theta > \delta$, then $\mathrm{L}_\theta(V_\delta^{M_b}) \models$ "KP + δ is Woodin". Hence we get the Woodin-in-the-next-admissibleness from the assumption, here we say δ is *Woodin-in-the-next-admissible* if there is an ordinal $\theta > \delta$ such that $\mathrm{L}_\theta(V_\delta) \models$ "KP+δ is Woodin". Andretta [2] proved the following stronger converse:

Theorem 4.1.2 (Andretta)**.** Suppose δ is Woodin-in-the-next-admissible. Then for any tree order on ω with an infinite branch, there is an iteration tree such that for any infinite branch b of the tree, δ_ω is in the wellfounded part of M_b, where δ_ω is the supremum of the length of extenders in the iteration tree.

Proof. See [2, Theorem 1.3]. $\square$

Hence Woodin-in-the-next-admissible cardinals are intimately correlated to iteration trees with more than one cofinal branches. The natural question would be: What if we do not demand that δ_ω is in the wellfounded part of M_b? In this section, we partially answer this question in the case of alternating chains. In fact, we do not need Woodin-in-the-next-admissible cardinals to construct alternating chains:

Theorem 4.1.3. Suppose δ is an ordinal such that δ is Σ_2-Woodin and $V_\delta \prec_{\Sigma_2} V$. Then there is an alternating chain with length ω.

The assumption of the above theorem (which we will explain later) is much weaker than Woodin-in-the-next-admissibleness. Hence we do not need Woodin-in-the-next-admissibleness just to construct alternating chains.

Let us prepare for introducing the notions in the above theorem. For a transitive model M of ZFC and an ordinal α in M, we write $M|\alpha$ for abbreviating V_α^M. Furthermore, for a subset A of M, $\mathrm{Thy}_\Gamma(M;\in,A)$ denotes the Γ-theory of M with parameters in A where Γ is Σ_n for some natural number $n \geq 1$. Also, for a set A and an ordinal α, $A \restriction \alpha$ denotes $A \cap V_\alpha$.

Let $\kappa < \delta$ be ordinals and Γ be Σ_n for some natural number $n \geq 1$. We say κ is *$<\delta$-Γ-strong* if it is $<\delta$-A-strong where $A = \mathrm{Thy}_\Gamma(V|\delta;\in,V|\delta)$, i.e., for any ordinal $\alpha < \delta$ there is a non-trivial elementary embedding $j\colon V \to M$ with critical point κ where M is transitive such that $V_\alpha \subseteq M$, $j(\kappa) > \alpha$ and $A \restriction \alpha = j(A) \restriction \alpha$. If δ is a limit of inaccessible cardinals, such an embedding can be easily coded by an extender in V_δ. An ordinal δ is *Γ-Woodin* if it is a limit of $<\delta$-Γ-strongs.

Note that if δ is a limit of $<\delta$-strong cardinals, then δ is Σ_1-Woodin and V_δ is a Σ_1 elementary substructure of V. Hence we cannot replace Σ_2 with Σ_1 in

Theorem 4.1.3 because if we could, then we could run the argument in a mouse below $0^{\P}$ with a cardinal δ which is a limit of $<\delta$-strong cardinals, which is impossible by [73, Lemma 2.4].

Also note that Σ_n-Woodinness for a natural number n is much weaker than Woodin-in-the-next-admissibleness. In fact, if δ is Woodin-in-the-next-admissible, then for any natural number $n \geq 1$, δ is a limit of $<\delta$-strong cardinals κ such that the set of $<\kappa$-A_n-strong cardinals is stationary in κ where $A_n = \mathrm{Thy}_{\Sigma_n}(V|\delta; \in, V|\delta)$, which immediately gives us that the set of Σ_n-Woodin cardinals δ' with $V_{\delta'} \prec_{\Sigma_n} V_\kappa$ is stationary in κ. Hence the assumption of Theorem 4.1.3 is much weaker than Woodin-in-the-next-admissibleness.

Proof of Theorem 4.1.3. We will construct $\big((\kappa_n, E_n, \beta_n) \mid n < \omega\big)$ with the following properties:

$(1)_n$ $\mathrm{Thy}_{\Sigma_2}\Big(M_{2n}|\delta; \in, M_{2n}|\kappa_{2n}\Big) = \mathrm{Thy}_{\Sigma_2}\Big(M_{2n \dot{-} 1}|\beta_n; \in, M_{2n \dot{-} 1}|\kappa_{2n}\Big)$,

$(2)_n$ κ_{2n} is $<\delta$-Σ_2-strong in M_{2n},

$(3)_n$ $\mathrm{Thy}_{\Sigma_2}\Big(M_{2n+1}|\beta_{n+1} + 1; \in, M_{2n+1}|\kappa_{2n+1} + 1\Big) = \mathrm{Thy}_{\Sigma_2}\Big(M_{2n}|\delta + 1; \in, M_{2n}|\kappa_{2n+1} + 1\Big)$, and

$(4)_n$ κ_{2n+1} is $<\beta_{n+1}$-Σ_2-strong in M_{2n+1},

where $n \dot{-} 1 = \max\{n-1, 0\}$, $M_0 = V$ and $M_{n+1} = \mathrm{Ult}(M_{n \dot{-} 1}, E_n)$ for each $n \in \omega$. At the same time, we will arrange that κ_{n+1} is less than the strength and the length of E_n for each $n \in \omega$, which will ensure that each M_n is well-founded by the result of Martin and Steel [61, Theorem 3.7].

Also note that all the extenders we will use belong to V_δ. Since δ is a limit of inaccessible cardinals, δ will not move under any embedding we will consider.

Let $\beta_0 = \delta$. Then $(1)_0$ is true. Since δ is Σ_2-Woodin in V, we can pick $\kappa_0 < \delta$ such that κ_0 is $<\delta$-Σ_2-strong in V, hence $(2)_0$ is also true.

Suppose we have constructed $(\kappa_i \mid i \leq 2n), (E_i \mid i < 2n), (\beta_i \mid i \leq n)$ with the properties $(1)_n$ and $(2)_n$. We will find $\kappa_{2n+1}, E_{2n}, \beta_{n+1}, \kappa_{2n+2}$ and E_{2n+1} with the properties $(3)_n, (4)_n, (1)_{n+1}$ and $(2)_{n+1}$.

Since $\delta = \pi_{0,2n}(\delta)$ is Σ_2-Woodin in M_{2n}, we can pick $\kappa_{2n+1} > \kappa_{2n}$ such that κ_{2n+1} is $<\delta$-Σ_2-strong in M_{2n}. By $(2)_n$, κ_{2n} is $<\delta$-Σ_2-strong in M_{2n}. Hence we can pick $E_{2n} \in M_{2n}$ such that E_{2n} is an extender with critical point κ_{2n} and length and strength greater than $\kappa_{2n+1} + 3$ in M_{2n}, such that $\pi_{E_{2n}}(A) \upharpoonright (\kappa_{2n+1} + 3) =$

$A \restriction (\kappa_{2n+1}+3)$ in M_{2n}, where $A = \mathrm{Thy}_{\Sigma_2}\Big(M_{2n}|\delta; \in, M_{2n}|\delta\Big)$. Then

$$\begin{aligned}
&\mathrm{Thy}_{\Sigma_2}\Big(M_{2n+1}|\pi_{2n\dot{-}1,2n+1}(\beta_n); \in, M_{2n+1}|\kappa_{2n+1}+3\Big)\\
=&\pi_{2n\dot{-}1,2n+1}\Bigg(\mathrm{Thy}_{\Sigma_2}\Big(M_{2n\dot{-}1}|\beta_n; \in, M_{2n\dot{-}1}|\kappa_{2n}\Big)\Bigg) \restriction \kappa_{2n+1}+3\\
=&\pi_{E_{2n}}\Bigg(\mathrm{Thy}_{\Sigma_2}\Big(M_{2n}|\delta; \in, M_{2n}|\kappa_{2n}\Big)\Bigg) \restriction \kappa_{2n+1}+3\\
=&\mathrm{Thy}_{\Sigma_2}\Big(M_{2n}|\delta; \in, M_{2n}|\kappa_{2n+1}+3\Big).
\end{aligned}$$

Now the following is true in M_{2n} witnessed by $\beta = \delta$:

($*$) There is an ordinal β such that $B = \mathrm{Thy}_{\Sigma_2}\Big(V|\beta+1; \in, V|\kappa_{2n+1}+1\Big)$ and κ_{2n+1} is $<\beta$-Σ_2-strong and β is Σ_2-Woodin,

where $B = \mathrm{Thy}_{\Sigma_2}\Big(M_{2n}|\delta+1; \in, M_{2n}|\kappa_{2n+1}+1\Big)$. Note that this statement is Σ_2 in M_{2n} with parameters B and κ_{2n+1} because the statement "κ_{2n+1} is $<\beta$-Σ_2-strong and β is Σ_2-Woodin" is definable in $V|\beta$ if β is a limit of inaccessibles, which is also Σ_2 definable.

Since V_δ is a Σ_2-elementary substructure of V, $M_{2n}|\delta = M_{2n}|\pi_{0,2n}(\delta)$ is a Σ_2-elementary structure of M_{2n}. Hence ($*$) is also true in $M_{2n}|\delta$. But by the previous calculation, ($*$) is also true in $M_{2n+1}|\pi_{2n\dot{-}1,2n+1}(\beta_n)$.

Let β_{n+1} be a witness for ($*$) in $M_{2n+1}|\pi_{2n\dot{-}1,2n+1}(\beta_n)$. Then it follows that

$$\begin{aligned}
&\mathrm{Thy}_{\Sigma_2}\Big(M_{2n+1}|\beta_{n+1}+1; \in, M_{2n+1}|\kappa_{2n+1}+1\Big)\\
=\ &\mathrm{Thy}_{\Sigma_2}\Big(M_{2n}|\delta+1; \in, M_{2n}|\kappa_{2n+1}+1\Big),
\end{aligned}$$

that is $(3)_n$. Also we have that β_{n+1} is Σ_2-Woodin and κ_{2n+1} is $<\beta_{n+1}$-Σ_2-strong in M_{2n+1}, that is $(4)_n$. Since β_{n+1} is Σ_2-Woodin in M_{2n+1} and $\beta_{n+1} > \kappa_{2n+1}$, we can pick $\kappa_{2n+2} < \beta_{n+1}$ large enough and such that κ_{2n+2} is $<\beta_{2n+1}$-Σ_2-strong in M_{2n+1}.

By $(4)_n$, we can take $E_{2n+1} \in M_{2n+1}$ such that E_{2n+1} is an extender with critical point κ_{2n+1} and length and strength greater than $\kappa_{2n+2}+3$ in M_{2n+1} such that $\pi_{E_{2n+1}}(A') \restriction \kappa_{2n+2}+3 = A' \restriction \kappa_{2n+2}+3$, where $A' = \mathrm{Thy}_{\Sigma_2}\Big(M_{2n+1}|\beta_{n+1}; \in,$

$M_{2n+1}|\beta_{n+1}\Big)$. Then

$$\begin{aligned}&\mathrm{Thy}_{\Sigma_2}\Big(M_{2n+2}|\delta+1;\in M_{2n+2}|\kappa_{2n+2}+1\Big)\\ =&\pi_{2n,2n+2}\bigg(\mathrm{Thy}_{\Sigma_2}\Big(M_{2n}|\delta+1;\in,M_{2n}|\kappa_{2n+1}+1\Big)\bigg)\restriction\kappa_{2n+2}+1\\ =&\pi_{E_{2n+1}}\bigg(\mathrm{Thy}_{\Sigma_2}\Big(M_{2n+1}|\beta_{n+1}+1;\in,M_{2n+1}|\kappa_{2n+1}+1\Big)\bigg)\restriction\kappa_{2n+2}+1\\ =&\mathrm{Thy}_{\Sigma_2}\Big(M_{2n+1}|\beta_{n+1}+1;\in,M_{2n+1}|\kappa_{2n+2}+1\Big),\end{aligned}$$

and by this calculation, we obtain $\mathrm{Thy}_{\Sigma_2}\Big(M_{2n+2}|\delta;\in,M_{2n+2}|\kappa_{2n+2}\Big)=$ $\mathrm{Thy}_{\Sigma_2}\Big(M_{2n+1}|\beta_{n+1};\in,M_{2n+1}|\kappa_{2n+2}\Big)$ and κ_{2n+2} is $<\delta$-Σ_2-strong in M_{2n+2}, which are $(1)_{n+1}$ and $(2)_{n+1}$ respectively, as desired. □

Note that in the above construction, we have arranged that $\beta_{n+1}<\pi_{2n\dot{-}1,2n+1}(\beta_n)$ for each $n\in\omega$. Hence M_{Odd} is always ill-founded.

4.2 Questions

We close this chapter with asking one question.

Question 4.2.1. What is the consistency strength of the existence of alternating chains with length ω?