Downloaded from UvA-DARE, the institutional repository of the University of Amsterdam (UvA)
http://dare.uva.nl/document/175320

File ID	175320
Filename	Chapter 3: Games themselves

SOURCE (OR PART OF THE FOLLOWING SOURCE):

Type	Dissertation
Title	Games in set theory and logic
Author	D. Ikegami
Faculty	Faculty of Science
Year	2010
Pages	xiv, 150
ISBN	90-5776-208-6

FULL BIBLIOGRAPHIC DETAILS:
http://dare.uva.nl/record/341749

Copyright

It is not permitted to download or to forward/distribute the text or part of it without the consent of the author(s) and/or copyright holder(s), other then for strictly personal, individual use.

UvA-DARE is a service provided by the library of the University of Amsterdam (http://dare.uva.nl)

Chapter 3

Games themselves

In this chapter, we compare the stronger versions of determinacy of Gale-Stewart games and Blackwell games, i.e., the Axiom of Real Determinacy $\mathrm{AD}_{\mathbb{R}}$ and the Axiom of Real Blackwell Determinacy Bl-$\mathrm{AD}_{\mathbb{R}}$. In § 3.1, we show that Bl-$\mathrm{AD}_{\mathbb{R}}$ implies that $\mathbb{R}^{\#}$ exists and that the consistency of Bl-$\mathrm{AD}_{\mathbb{R}}$ is strictly stronger than that of AD. In § 3.2, we show that Bl-$\mathrm{AD}_{\mathbb{R}}$ implies that every set of reals is ∞-Borel. From this, we can derive almost all the regularity properties for every set of reals. In § 3.3, we discuss the possibility of the equivalence between $\mathrm{AD}_{\mathbb{R}}$ and Bl-$\mathrm{AD}_{\mathbb{R}}$ under ZF+DC. In § 3.4, we discuss the possibility of the equiconsistency between $\mathrm{AD}_{\mathbb{R}}$ and Bl-$\mathrm{AD}_{\mathbb{R}}$.

Throughout this chapter, we use standard notations from set theory and assume familiarity with descriptive set theory. By *reals*, we mean elements of the Cantor space and we use $\mathbb{R}$ to denote the Cantor space.

3.1 Real Blackwell Determinacy and $\mathbb{R}^{\#}$

In this section, we prove that Bl-$\mathrm{AD}_{\mathbb{R}}$ implies that $\mathbb{R}^{\#}$ exists and that the consistency of Bl-$\mathrm{AD}_{\mathbb{R}}$ is strictly stronger than that of AD.

Solovay [77] proved that $\mathrm{AD}_{\mathbb{R}}$ implies that $\mathbb{R}^{\#}$ exists. Our plan is to mimic Solovay's proof using Blackwell games. In order to do so, we analyze his proof which has two main components:

Theorem 3.1.1 (Solovay)**.** The axiom $\mathrm{AD}_{\mathbb{R}}$ implies that there is a fine normal measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$, where $\mathcal{P}_{\omega_1}(\mathbb{R})$ is the set of all countable subsets of $\mathbb{R}$.

Proof. See [77, Lemma 3.1]. □

Theorem 3.1.2 (Solovay)**.** Suppose there is a fine normal measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$ and every real has a sharp. Then $\mathbb{R}^{\#}$ exists.

Proof. See [77, Lemma 4.1 & Theorem 4.4]. □

Hence it suffices to show that there is a fine normal measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$ from Bl-$\mathrm{AD}_{\mathbb{R}}$ because Bl-$\mathrm{AD}_{\mathbb{R}}$ implies AD in $\mathrm{L}(\mathbb{R})$, which implies that every real has a sharp by the result of Harrington [31].

Theorem 3.1.3. Assume Bl-$\mathrm{AD}_{\mathbb{R}}$. Then there is a fine normal measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$.

Let us first see what is a fine normal measure. Let X be a set and κ be an uncountable cardinal. As usual, we denote by $\mathcal{P}_\kappa(X)$ the set of all subsets of X with cardinality less than κ, i.e., subsets A of X such that there are an $\alpha < \kappa$ and a surjection from α to A. Let U be a set of subsets of $\mathcal{P}_\kappa(X)$. We say that U is *κ-complete* if U is closed under intersections with $<\kappa$-many elements; we say it is *fine* if for any $x \in X$, $\{a \in \mathcal{P}_\kappa(X) \mid x \in a\} \in U$; we say that U is *normal* if for any family $\{A_x \in U \mid x \in X\}$, the diagonal intersection $\triangle_{x\in X} A_x$ is in U (where $\triangle_{x\in X} A_x = \{a \in \mathcal{P}_\kappa(X) \mid (\forall x \in a)\ a \in A_x\}$). We say that U is a *fine measure* if it is a fine κ-complete ultrafilter, and we say that it is a *fine normal measure* if it is a fine normal κ-complete ultrafilter.

Proof of Theorem 3.1.3. The following is the key point: A subset A of ${}^\omega\mathbb{R}$ is *range-invariant* if for any $\vec{x}$ and $\vec{y}$ in ${}^\omega\mathbb{R}$ with $\mathrm{ran}(\vec{x}) = \mathrm{ran}(\vec{y})$, $\vec{x} \in A$ if and only if $\vec{y} \in A$.

Lemma 3.1.4. Assume Bl-$\mathrm{AD}_{\mathbb{R}}$. Then every range-invariant subset of ${}^\omega\mathbb{R}$ is determined.

Proof of Lemma 3.1.4. Let A be a range-invariant subset of ${}^\omega\mathbb{R}$. We show that if there is an optimal strategy for player I in A, then so is a winning strategy for player I in A. The case for player II is similar and we will skip it.

Let us first introduce some notations. Given a function $f\colon {}^{<\omega}\mathbb{R} \to \mathbb{R}$, a countable set of reals a is *closed under f* if for any finite sequence s of elements in a, $f(s)$ is in a. For a strategy $\sigma\colon \mathbb{R}^{\mathrm{Even}} \to \mathbb{R}$ for player I, where $\mathbb{R}^{\mathrm{Even}}$ is the set of all finite sequences of reals with even length, a countable set of reals a is *closed under σ* if for any finite sequence s of elements in a with even length, $\sigma(s)$ is in a. For a function $F\colon {}^{<\omega}\mathbb{R} \to \mathcal{P}_{\omega_1}(\mathbb{R})$, a countable set of reals a is *closed under F* if for any finite sequence s of elements in a, $F(s)$ is a subset of a.

The following two claims are basic:

Claim 3.1.5. There is a winning strategy for player I in A if and only if there is a function $f\colon {}^{<\omega}\mathbb{R} \to \mathbb{R}$ such that if a is a countable set of reals and closed under f, then any enumeration of a belongs to A.

Proof of Claim 3.1.5. We first show the direction from left to right. Given a winning strategy σ for player I in A, let f be such that if a is closed under f, then a is closed under σ. (Since σ is a function from $\mathbb{R}^{\mathrm{Even}}$ to $\mathbb{R}$, any function from ${}^{<\omega}\mathbb{R}$ to $\mathbb{R}$ extending σ will do.) We see this f works for our purpose. Let a be a countable set of reals closed under f. Then since a is closed under σ and countable, there is a run x of the game following σ such that its range is equal to a. Since σ is winning for player I, x is in A and by the range-invariance of A, any enumeration of a is also in A.

We now show the direction from right to left. Given such an f, we can arrange a strategy σ for player I such that if x is a run of the game following σ, then the range of x is closed under f: Given a finite sequence of reals $(a_0, \cdots, a_{2n-1})$, consider the set of all finite sequences s from elements of $\{a_0, \cdots a_{2n-1}\}$ and all the values $f(s)$ from this set. What we should arrange is to choose $\sigma(a_0, \cdots, a_{2n-1})$ in such a way that the range of any run of the game via σ will cover all such values $f(s)$ when $(a_0, \cdots, a_{2n-1})$ is a finite initial segment of the run for any n in ω moves. But this is possible by a standard book-keeping argument. By the property of f, this implies that x is in A and hence σ is winning for player I.

□ (Claim 3.1.5)

Claim 3.1.6. There is a function $f\colon {}^{<\omega}\mathbb{R} \to \mathbb{R}$ such that if a is a countable set of reals and closed under f, then any enumeration of a belongs to A if and only if there is a function $F\colon {}^{<\omega}\mathbb{R} \to \mathcal{P}_{\omega_1}(\mathbb{R})$ such that if a is a countable set of reals and closed under F, then any enumeration of a belongs to A.

Proof of Claim 3.1.6. We first show the direction from left to right: Given such an f, let $F(s) = \{f(s)\}$. Then it is easy to check that this F works.

We show the direction from right to left: Given such an F, it suffices to show that there is an f such that if a is closed under f then a is also closed under F. We may assume that $F(s) \neq$ for each s. Fix a bijection $\pi\colon \mathbb{R} \to {}^{\omega}\mathbb{R}$. Let $g\colon {}^{<\omega}\mathbb{R} \to \mathbb{R}$ be such that $\mathrm{ran}\big(\pi(g(s))\big)= F(s)$ for each s (this is possible because every relation on the reals can be uniformized by a function by Theorem 1.14.9). Let $h\colon {}^{<\omega}\mathbb{R} \to \mathbb{R}$ be such that $h(s) = \pi\big(s(0)\big)(\mathrm{lh}(s) - 1)$, where $\mathrm{lh}(s)$ is the length of s when $s \neq \emptyset$, if $s = \emptyset$ let $h(s)$ be an arbitrary real.

It is easy to see that if a is closed under g and h, then so is under F: Fix a finite sequence s of reals in a. We have to show that each x in $F(s)$ is in a. Consider $g(s)$. By the closure under g, $g(s)$ is in a. By choice of g, we know that $\mathrm{ran}(\pi(g(s))) = F(s)$, so it is enough to show that x is in a for any x in $\mathrm{ran}(\pi(g(s))$. Suppose x is the nth bit of $\pi(g(s))$. Consider the finite sequence $t = \big(g(s), ..., g(s)\big)$ of length $n + 1$. Then $h(t) = \pi(t(0))(\mathrm{lh}(t) - 1) = \pi(g(s))(n) = x$. But $g(s)$ is in a and a was closed under h, so x is in a.

Now it is easy to construct an f such that if a is closed under f, then so is under g and h. □ (Claim 3.1.6)

By the above two claims, it suffices to show that there is a function $F\colon {}^{<\omega}\mathbb{R} \to \mathcal{P}_{\omega_1}(\mathbb{R})$ such that if a is a countable set of reals and closed under F, then any enumeration of a belongs to A.

Let σ be an optimal strategy for player I in A. Let F be as follows:

$$F(s) = \begin{cases} \emptyset & \text{if } \mathrm{lh}(s) \text{ is odd,} \\ \{y \in \mathbb{R} \mid \sigma(s)(y) \neq 0\} & \text{otherwise.} \end{cases}$$

Then F is as desired: If a is closed under F, then enumerate a to be $\langle a_n \mid n \in \omega\rangle$ and let player I follow σ and let player II play the Dirac measure for a_n at her nth move. Then the probability of the set $\{x \in {}^{\omega}\mathbb{R} \mid \mathrm{ran}(x) = a\}$ is 1 and since σ is optimal for player I in A, there is an x such that the range of x is a and x is in A. But by the range-invariance of A, any enumeration of a belongs to A. □ (Lemma 3.1.4)

We shall be closely following Solovay's original idea. We define a family $U \subseteq \mathcal{P}(\mathcal{P}_{\omega_1}(\mathbb{R}))$ as follows: Fix $A \subseteq \mathcal{P}_{\omega_1}(\mathbb{R})$ and consider the following game $\tilde{G}_A$: Players alternately play reals; say that they produce an infinite sequence $\vec{x} = (x_i \mid i \in \omega)$. Then player II wins the game $\tilde{G}_A$ if $\mathrm{ran}(\vec{x}) \in A$, otherwise player I wins. Since the payoff set of this game is range-invariant as a Gale-Stewart game, by Lemma 3.1.4, it is determined.

We say that $A \in U$ if and only if player II has a winning strategy in $\tilde{G}_A$. We shall show that it is a fine normal measure under the assumption of $\mathrm{Bl\text{-}AD}_{\mathbb{R}}$, thus finishing the proof of Theorem 3.1.3.

A few properties of U are obvious: For instance, we see readily that $\emptyset \notin U$ and that $\mathcal{P}_{\omega_1}(\mathbb{R}) \in U$, as well as the fact that U is closed under taking supersets. In order to see that U is a fine family, fix a real x, and let player II play x in her first move: This is a winning strategy for player II in $\tilde{G}_{\{a \mid x \in a\}}$.

We next show that for any set $A \subseteq \mathcal{P}_{\omega_1}(\mathbb{R})$, either A or the complement of A is in U. Given any such set A, suppose A is not in U. We show that the complement of A is in U. Since the game $\tilde{G}_A$ is determined, by the assumption, there is a winning strategy σ for I in $\tilde{G}_A$. Setting $\tau(s) = \sigma\big(s\restriction(\mathrm{lh}(s) - 1)\big)$ for $s \in \mathbb{R}^{\mathrm{Odd}}$, it is easy to see that τ is a winning strategy for player II in the game $\tilde{G}_{A^c}$.

We show that U is closed under finite intersections. Let A_1 and A_2 be in U. Since the payoff sets in the games $\tilde{G}_{A_1}$ and $\tilde{G}_{A_2}$ are range-invariant, by the analogue of Claim 3.1.5, there are functions $f_1 \colon {}^{<\omega}\mathbb{R} \to \mathbb{R}$ and $f_2 \colon {}^{<\omega}\mathbb{R} \to \mathbb{R}$ such that if a is closed under f_i, then a is in A_i for $i = 1, 2$. Then it is easy to find an $f \colon {}^{<\omega}\mathbb{R} \to \mathbb{R}$ such that if a is closed under f, then a is closed under both f_1 and f_2. By the analogue of Claim 3.1.5 again, this f witnesses the existence of a winning strategy for player II in the game $\tilde{G}_{A_1 \cap A_2}$.

We have shown that U is an ultrafilter on subsets of $\mathcal{P}_{\omega_1}(\mathbb{R})$. We show the ω_1-completeness of U as follows: By Theorem 1.14.8, every set of reals is Lebesgue measurable assuming Bl-AD. If there is a non-principal ultrafilter on ω, then there is a set of reals which is not Lebesgue measurable. Hence there is no non-principal ultrafilter on ω, which implies that any ultrafilter is ω_1-complete. In particular, U is ω_1-complete.

The last to show is that U is normal. Let $\{A_x \mid x \in \mathbb{R}\}$ be a family of sets in U. We show that $\triangle_{x\in\mathbb{R}} A_x$ is in U. Consider the following game $\tilde{G}$: Player I moves x, then player II passes. After that, they play the game $\tilde{G}_{A_x}$. This is Blackwell determined and player II has an optimal strategy τ since each A_x is in U. Let $F\colon {}^{<\omega}\mathbb{R} \to \mathcal{P}_{\omega_1}(\mathbb{R})$ be as follows:

$$F(s) = \begin{cases} \emptyset & \text{if } \mathrm{lh}(s) \text{ is even,} \\ \{y \in \mathbb{R} \mid \tau(s)(y) \neq 0\} & \text{otherwise.} \end{cases}$$

We claim that if a is closed under F, then a is in $\triangle_{x\in\mathbb{R}} A_x$. Then, by the analogues of Claim 3.1.5 and Claim 3.1.6, F will witness the existence of a winning strategy for player II in the game $\tilde{G}_{\triangle_{x\in\mathbb{R}} A_x}$ and we will have proved that $\triangle_{x\in\mathbb{R}} A_x \in U$.

Suppose a is closed under F. We show that $a \in A_x$ for each $x \in a$. Fix an x in a and enumerate a to be $(x_n \mid n \in \omega)$. In the game $\tilde{G}$, let player I first move x and then they play the game $\tilde{G}_{A_x}$. Let player II follow τ and player I play the Dirac measure concentrating on x_n at the nth move. Then the probability of the set $\{\vec{x} \in {}^{\omega}\mathbb{R} \mid x_0 = x \text{ and } \mathrm{ran}(\vec{x}) = a\}$ is 1 and since τ is optimal for player II in the game $\tilde{G}$, there is an $\vec{x}$ such that the range of $\vec{x}$ is a and $\vec{x}$ is a winning run for player II in $\tilde{G}$, hence a is in A_x. □ (Theorem 3.1.3)

Corollary 3.1.7. The consistency of Bl-AD$_{\mathbb{R}}$ is strictly stronger than that of AD.

Proof. Since Bl-AD$_{\mathbb{R}}$ implies Bl-AD by the first item of Proposition 1.14.2 and Bl-AD implies AD$^{\mathrm{L}(\mathbb{R})}$ by Corollary 1.14.7, Bl-AD$_{\mathbb{R}}$ implies AD$^{\mathrm{L}(\mathbb{R})}$. By Theorem 3.1.3, Bl-AD$_{\mathbb{R}}$ also implies the existence of $\mathbb{R}^{\#}$. By the property of $\mathbb{R}^{\#}$, one can construct a set-size elementary substructure of L($\mathbb{R}$). Hence AD$^{\mathrm{L}(\mathbb{R})}$ and the existence of $\mathbb{R}^{\#}$ imply the consistency of AD. Therefore, Bl-AD$_{\mathbb{R}}$ implies the consistency of AD and by Gödel's Incompleteness Theorem, the consistency of Bl-AD$_{\mathbb{R}}$ is strictly stronger than that of AD. □

3.2 Real Blackwell Determinacy and regularity properties

In this section, we show that Bl-AD$_{\mathbb{R}}$ implies almost all the regularity properties for every set of reals. Note that DC$_{\mathbb{R}}$ follows from the uniformization for every relation on the reals. Hence by Theorem 1.14.9, Bl-AD$_{\mathbb{R}}$ implies DC$_{\mathbb{R}}$. For the rest of the sections in this chapter, we freely use DC$_{\mathbb{R}}$ when we assume Bl-AD$_{\mathbb{R}}$ and we fix a fine normal measure U on $\mathcal{P}_{\omega_1}(\mathbb{R})$, which exists by Theorem 3.1.3.

We start with proving the perfect set property for every set of reals. Recall that a set of reals A has the *perfect set property* if either A is countable or A contains a perfect subset, where a perfect set of reals is a closed set without isolated points.

Theorem 3.2.1. Assume Bl-$\mathrm{AD}_{\mathbb{R}}$. Then every set of reals has the perfect set property.

Proof. The theorem follows from the following two lemmas:

Lemma 3.2.2. Assume Bl-$\mathrm{AD}_{\mathbb{R}}$. Then every relation on the reals can be uniformized by a Borel function modulo a Lebesgue null set, i.e., for any relation R on the reals, there is a Borel function f such that the set $\{x \mid (x, f(x)) \in R$ or there is no real y with $(x, y) \in R\}$ is of Lebesgue measure one.

Proof of Lemma 3.2.2. The conclusion follows by a folklore argument from Lebesgue measurability and uniformization for any relation on the reals both of which are consequences of Bl-$\mathrm{AD}_{\mathbb{R}}$ by Theorem 1.14.8 and Theorem 1.14.9).

Let R be an arbitrary relation on the reals. We may assume the domain of R is the whole space, i.e., for any real x, there is a real y such that $(x, y) \in R$. We will find a Borel function uniformizing R almost everywhere.

By the uniformization principle, there is a function g uniformizing R. For each finite binary sequence s, the set $g^{-1}([s])$ is Lebesgue measurable by Theorem 1.14.8. Hence for each s there is a Borel set B_s such that $g^{-1}([s]) \triangle B_s$ is Lebesgue null. Now define f so that the following holds: For each finite binary sequence s,

$$f(x) \in [s] \iff x \in B_s.$$

Then by the property of B_s, f is defined almost everywhere, Borel, and is equal to g almost everywhere. Hence any Borel extension of f will be the one we desired.
□ (Lemma 3.2.2)

Lemma 3.2.3 (Raisonnier and Stern)**.** Suppose every relation on the reals can be uniformized by a Borel function modulo a Lebesgue null set. Then every set of reals has the perfect set property.

Proof of Lemma 3.2.3. See [70, Theorem 5]. □

□ (Theorem 3.2.1)

Next, we show that Bl-$\mathrm{AD}_{\mathbb{R}}$ implies that every set of reals has the Baire property. We first introduce the Blackwell meager ideal as an analogue of the meager ideal. A set A of reals is *Blackwell meager* if player II has an optimal strategy in the Banach-Mazur game $G^{**}(A)$. Let I_{BM} denote the set of all Blackwell meager sets of reals.

Lemma 3.2.4. Assume Bl-AD. Then any meager set is in I_{BM}, $[s] \notin I_{\mathrm{BM}}$ for each finite binary sequence s, and I_{BM} is a σ-ideal. Moreover, every set of reals is measurable with respect to I_{BM}, i.e., for any set A of reals and finite binary sequence s, there is a finite binary sequence t extending s such that either $[t] \cap A$ or $[t] \setminus A$ is in I_{BM}.

Proof. By Theorem 1.8.3, if a set A of reals is meager, then player II has a winning strategy in the Banach-Mazur game $G^{**}(A)$ and in particular player II has an optimal strategy in $G^{**}(A)$ by Theorem 1.14.3. Hence A is Blackwell meager.

It is easy to see that $[s] \notin I_{\mathrm{BM}}$ for each finite binary sequence s by letting player I first play the Dirac measure concentrating on s in the game $G^{**}([s])$.

We show that I_{BM} is a σ-ideal. The closure of I_{BM} under subsets is immediate. We prove that it is closed under countable unions.

In order to prove this, we need to develop the appropriate *transfer technique* (as discussed and applied in [55]) for the present context. Let $\pi \subseteq \omega$ be an infinite and co-infinite set. We think of π as the set of rounds in which player I moves. We identify π with the increasing enumeration of its members, i.e., $\pi = \{\pi_i \mid i \in \omega\}$. Similarly, we write $\bar{\pi}$ for the increasing enumeration of $\omega \backslash \pi$, i.e., $\omega \backslash \pi = \{\bar{\pi}_i \mid i \in \omega\}$. For notational ease, we call π a **I-coding** if no two consecutive numbers are in π and $0 \in \pi$ (i.e., the first move is played by I). We call π a **II-coding** if no two consecutive numbers are in $\omega \backslash \pi$ and $0 \in \pi$.

Fix $A \subseteq {}^{\omega}\omega$ and define two variants of G^{**}_A with alternative orders of play as determined by π. If π is a I-coding, the game $G^{**\pi,\mathrm{I}}_A$ is played as follows:

$$\begin{array}{lllll} \mathrm{I} & s_{\pi_0} = s_0 & & s_{\pi_1} & & \dots \\ \mathrm{II} & & s_{\pi_0+1}, \dots, s_{\pi_0 - 1} & & s_{\pi_1+1}, \dots, s_{\pi_2-1} & \dots \end{array}$$

If π is a II-coding, then the game $G^{**\pi,\mathrm{II}}_A$ is played as follows:

$$\begin{array}{llllll} \mathrm{I} & s_0, \dots, s_{\bar{\pi}_0 - 1} & & s_{\bar{\pi}_0+1}, \dots, s_{\bar{\pi}_1-1} & & \dots \\ \mathrm{II} & & s_{\bar{\pi}_0} & & s_{\bar{\pi}_1} & \dots \end{array}$$

In both cases, player II wins the game if $s_0^\frown s_1^\frown \dots ^\frown s_n^\frown \dots \notin A$. Obviously, we have

$$G^{**}_A = G^{**\mathrm{Even},\mathrm{II}}_A$$

where Even is the set of even numbers.

Lemma 3.2.5. Let A be a subset of the Baire space and π be a I-coding. Then there is a translation $\sigma \mapsto \sigma_\pi$ of mixed strategies for player I such that if σ is an optimal strategy for player I in G^{**}_A, then σ_π is an optimal strategy for player I in $G^{**\pi,\mathrm{I}}_A$.

Similarly, if π is a II-coding, there is a translation $\tau \mapsto \tau_\pi$ of mixed strategies for player II such that if τ is an optimal strategy for player II in G^{**}_A, then τ_π is an optimal strategy for player II in $G^{**\pi,\mathrm{II}}_A$.

Proof of Lemma 3.2.5. We prove only the lemma for the games $G^{**\pi,\mathrm{I}}_A$, the other proof being similar. If $\vec{s} = \langle s_i \mid i \in \omega \rangle$ is an infinite sequence of finite binary sequences, we define

$$b^{\vec{s}}_i = s_{\pi_i+1}^\frown \dots ^\frown s_{\pi_{i+1}-1}.$$

Note that in order to compute $b_i^{\vec{s}}$, we only need the first π_{i+1} bits of $\vec{s}$. The idea is that now the G_A^{**}-run

$$\begin{array}{llllllll} \text{I} & s_{\pi_0} & & s_{\pi_1} & & s_{\pi_2} & & \ldots \\ \text{II} & & b_0^{\vec{s}} & & b_1^{\vec{s}} & & b_2^{\vec{s}} & \ldots \end{array} \qquad (*)$$

yields the same output in terms of the concatenation of all played finite sets as the run $\vec{s}$ in the game $G_A^{**\pi,\text{I}}$. We can define a map π^* on infinite sequences of finite binary sequences by

$$(\pi^*(\vec{s}))_i = \begin{cases} s_{\pi_k} & \text{if } i = 2k, \\ b_k^{\vec{s}} & \text{if } i = 2k+1, \end{cases}$$

and see that $s_0 \frown s_1 \frown \ldots = (\pi^*(\vec{s}))_0 \frown (\pi^*(\vec{s}))_1 \frown \ldots$.

Now, given a mixed strategy σ for player I in G_A^{**} and a run $\vec{s}$ of the game $G_A^{**\pi,\text{I}}$, we define σ_π via π^* as follows:

$$\sigma_\pi(s_0, \ldots, s_{\pi_m - 1}) = \sigma(s_{\pi_0}, b_0^{\vec{s}}, \ldots, s_{\pi_i}, b_i^{\vec{s}}, \ldots, s_{\pi_{m-1}}, b_{m-1}^{\vec{s}}).$$

Assume that σ is an optimal strategy for player I in G_A^{**} and fix an arbitrary mixed strategy τ in the game $G_A^{**\pi,\text{I}}$. We show that the payoff set for A in $G_A^{**\pi,\text{I}}$ is $\mu_{\sigma_\pi,\tau}$-measurable and $\mu_{\sigma_\pi,\tau}(A) = 1$. In order to do so, we construct a mixed strategy $\tau_{\pi^{-1}}$ for player II in G_A^{**} so that the game played by σ_π and τ is essentially the same as the game played by σ and $\tau_{\pi^{-1}}$.

Given a sequence $\vec{b}$ of moves in G_A^{**}, we need to unravel it into a sequence of moves in $G_A^{**\pi,\text{I}}$ in an inverse of the maps $\vec{s} \mapsto b_i^{\vec{s}}$ according to $(*)$, i.e., $b_{2i+1} = b_i^{\vec{s}}$. Thus, we define

$$\begin{aligned} A_{2i+1}^{\vec{b}} &= \{\vec{s} \mid b_i^{\vec{s}} = b_{2i+1}\}, \\ A_{\leq 2i+1}^{\vec{b}} &= \bigcap_{j \leq i} A_{2j+1}^{\vec{b}}. \end{aligned}$$

Note that only a finite fragment of $\vec{s}$ is needed to check whether $b_i^{\vec{s}} = b_{2i+1}$, and thus we think of $A_{\leq 2i+1}^{\vec{b}}$ as a set of $(\pi_{i+1} - (i+1))$-tuples of finite binary sequences. In the following, when we quantify over all "$\vec{s} \in A_{\leq i}^{\vec{b}}$", we think of this as a collection of finite strings of finite binary sequences. In order to pad the moves made in $G_A^{**\pi,\text{I}}$, we define the following notation: For infinite sequences $\vec{s}$ and $\vec{b}$, we write

$$x_i^{\vec{s},\vec{b}} = (b_{2i}, s_{\pi_i+1}, \ldots, s_{\pi_{i+1}-1}).$$

Compare $(*)$ to see that if $\vec{s}$ corresponds to moves in $G_A^{**\pi,\text{I}}$ and $\vec{b}$ to the moves in G_A^{**}, then these are exactly the finite sequences that player II will have to respond to in $G_A^{**\pi,\text{I}}$. Moreover, for a given sequence $\vec{z}$ of finite binary sequences, we let

$$P_\tau(z_0, \ldots, z_n) = \prod_{i \leq n, i \notin \pi} \tau(z_0, \ldots, z_{i-1})(z_i).$$

Fix a sequence $\vec{b}$ of finite binary sequences with even length and define $\tau_{\pi^{-1}}$ as follows:

$$\tau_{\pi^{-1}}(b_0, \ldots, b_{2m})(b_{2m+1}) \;=\; \frac{\sum_{\vec{s} \in A^{\vec{b}}_{\leq 2m+1}} P_\tau(x_0^{\vec{s},\vec{b}} {}^\frown \ldots {}^\frown x_m^{\vec{s},\vec{b}})}{\prod_{i=1}^{m} \tau_{\pi^{-1}}(b_0, \ldots, b_{2i-2})(b_{2i-1})}.$$

Using the two operations $\sigma \mapsto \sigma_\pi$ and $\tau \mapsto \tau_{\pi^{-1}}$, since the payoff set for G^{**}_A is invariant under π^*, it now suffices to prove for all basic open sets $[t]$ induced by a finite sequence $t = (b_0, ..., b_{\mathrm{lh}(t)-1})$ that $\mu_{\sigma,\tau_{\pi^{-1}}}([t]) = \mu_{\sigma_\pi,\tau}((\pi^*)^{-1}([t]))$. We prove this by induction on the length of t, and have to consider three different cases:

Case 1. $\mathrm{lh}(t) = 0$. This is immediate.

Case 2. $\mathrm{lh}(t) = 2m + 1$ with $m \geq 0$. By induction hypothesis, we have that $X = \mu_{\sigma,\tau_{\pi^{-1}}}([b_0, \ldots, b_{2m-1}]) = \mu_{\sigma_\pi,\tau}((\pi^*)^{-1}([b_0, \ldots, b_{2m-1}]))$. Thus,

$$\begin{aligned} \mu_{\sigma,\tau_{\pi^{-1}}}([b_0, \ldots, b_{2m}]) &= X \cdot \sigma(b_0, \ldots, b_{2m-1})(b_{2m}) \\ &= \mu_{\sigma_\pi,\tau}((\pi^*)^{-1}([b_0, \ldots, b_{2m}])). \end{aligned}$$

Case 3. $\mathrm{lh}(t) = 2m + 2$ with $m \geq 0$.

$$\begin{aligned} \mu_{\sigma,\tau_{\pi^{-1}}}(t) &= \prod_{i=0}^{m} \sigma(b_0, \ldots, b_{2i-1})(b_{2i}) \cdot \sum_{\vec{s} \in A^{\vec{b}}_{\leq 2m+1}} P_\tau(x_0^{\vec{s},\vec{b}} {}^\frown \ldots {}^\frown x_m^{\vec{s},\vec{b}}) \\ &= \mu_{\sigma_\pi,\tau}\big((\pi^*)^{-1}([b_0, \ldots, b_{2m+1}])\big). \end{aligned}$$

This calculation finishes the proof of this lemma. □ (Lemma 3.2.5)

We now show that I_{BM} is closed under countable unions. Let $\{A_n \mid n \in \omega\}$ be a family of sets in I_{BM}. Take an optimal strategy τ_n in the game $G^{**}(A_n)$ for each n. We prove that $\bigcup_{n\in\omega} A_n$ is also in I_{BM}.

Fix a bookkeeping bijection ρ from $\omega \times \omega$ to ω such that $\rho(n, m) < \rho(n, m+1)$ and $\rho(n, 0) \geq n$. We are playing infinitely many games in a diagram where the first coordinate is for the index of the game we are playing, and the second coordinate is for the number of moves. Hence the pair (n, m) stands for "mth move in the nth game". Define a II-coding $\pi_n = \omega \backslash \{2\rho(n, i) + 1 \mid i \in \omega\}$ corresponding to the following game diagram:

I	$s_0, \ldots, s_{2\rho(n,0)}$		$s_{2\rho(n,0)+2}, \ldots, s_{2\rho(n,1)}$		$\ldots$
II		$s_{2\rho(n,0)+1}$		$s_{2\rho(n,1)+1}$	$\ldots$

By Lemma 3.2.5, we know that for each $n \in \omega$, we get an optimal strategy $(\tau_n)_{\pi_n}$ for the game $G^{**\pi_n,\mathrm{II}}_{A_n}$. Let τ be the following mixed strategy

$$\tau(s_0, \ldots, s_{2\rho(n,m)}) = (\tau_n)_{\pi_n}(s_0, \ldots, s_{2\rho(n,m)}).$$

The properties of ρ make sure that this strategy is well-defined. We shall now prove that τ is an optimal strategy for player II in $G^{**}_{\bigcup_{n\in\omega} A_n}$.

Pick any mixed strategy σ for player I in $G^{**}_{\bigcup_{n\in\omega} A_n}$ and define strategies σ_n for $G^{**\pi_n,\mathrm{II}}_{A_n}$. Let $m = \rho(k,\ell)$, then

$$\begin{aligned} \sigma_n(s_0,\ldots,s_{2m-1}) &= \sigma(s_0,\ldots,s_{2m-1}), \text{ and} \\ \sigma_n(s_0,\ldots,s_{2m}) &= (\tau_k)_{\pi_k}(s_0,\ldots,s_{2m}) \text{ (if } k \neq n). \end{aligned}$$

Note that for each $n \in \omega$, $\mu_{\sigma,\tau} = \mu_{\sigma_n,(\tau_n)_{\pi_n}}$.

The payoff set (for player II) in $G^{**}_{\bigcup_{n\in\omega} A_n}$ is $A = \{\vec{s} \mid s_0^\frown s_1^\frown \ldots \notin \bigcup_{n\in\omega} A_n\}$. We show that $\mu_{\sigma,\tau}(A) = 1$. Since $A = \bigcap_{n\in\omega}\{\vec{s} \mid s_0^\frown s_1^\frown \ldots \notin A_n\}$, it suffices to check that the sets $B_n = \{\vec{s} \mid s_0^\frown s_1^\frown \ldots \notin A_n\}$ has $\mu_{\sigma,\tau}$-measure 1. But $\mu_{\sigma,\tau}(B_n) = \mu_{\sigma_n,(\tau_n)_{\pi_n}}(B_n) = 1$. Thus we have shown that I_{BM} is a σ-ideal.

We finally show that every set A of reals is measurable with respect to I_{BM}, i.e., for any finite binary sequence s, there is a finite binary sequence t extending s such that either $[t] \cap A$ or $[t] \setminus A$ is in I_{BM}. Fix such A and s. If $[s] \cap A$ is in I_{BM}, we are done. So suppose not. Then player II does not have an optimal strategy in the game $G^{**}([s] \cap A)$. By Bl-AD, there is an optimal strategy σ for player I in the game $G^{**}([s] \cap A)$. Let t be any s' with $\sigma(\emptyset)(s') \neq 0$. Then since σ is optimal, t extends s and the strategy σ easily gives us an optimal strategy for player II in the game $G^{**}([t] \setminus A)$. Hence $[t] \setminus A$ is in I_{BM}. □ (Lemma 3.2.4)

Recall the notions of Stone space $\mathrm{St}(\mathbb{P})$ and $\mathbb{P}$-Baireness for a partial order $\mathbb{P}$ from chapter 2. The based set of $\mathrm{St}(\mathbb{P})$ was the set of all ultrafilters on $B_{\mathbb{P}}$ where $B_{\mathbb{P}}$ is a completion of $\mathbb{P}$. Without the Axiom of Choice, it might be empty if $\mathbb{P}$ is big. But in this chapter, we only consider partial orders $\mathbb{P}$ which are elements of $\mathcal{H}_{\omega_1}$ in V, i.e., the transitive closure of $\mathbb{P}$ is countable in V. If $\mathbb{P}$ is an element of $\mathcal{H}_{\omega_1}$, then $\mathrm{St}(\mathbb{P})$ is essentially the same as $\mathrm{St}(\mathbb{C})$ where $\mathbb{C}$ is Cohen forcing, hence the Cantor space ${}^\omega\omega$

Since every meager set is Blackwell meager as we have seen in Lemma 3.2.4, if $\mathbb{P}$ is in $\mathcal{H}_{\omega_1}$, then one can consider Blackwell meagerness for subsets of $\mathrm{St}(\mathbb{P})$ by identifying $\mathrm{St}(\mathbb{P})$ with the Cantor space.

We are now ready to prove the Baire property for every set of reals from $\mathrm{Bl\text{-}AD}_{\mathbb{R}}$.

Theorem 3.2.6. Assume $\mathrm{Bl\text{-}AD}_{\mathbb{R}}$. Then every set of reals has the Baire property.

Proof. Take any set A of reals. We show that A has the Baire property. Let $\mathcal{A}^2_A$ be the second-order arithmetic structure with A as a unary predicate. Since any relation on the reals can be uniformized by a function by Theorem 1.14.9, we can construct a Skolem function F for $\mathcal{A}^2_A$ and by a simple coding of finite sequences of reals and formulas via reals, we regard it as a function from the reals to themselves. Let $\Gamma_F = \{(x,s) \in \mathbb{R} \times {}^{<\omega}2 \mid F(x) \supseteq s\}$. The following are the key objects for the proof (they are called *term relations*): Recall from

Lemma 2.1.2 that for a $\mathbb{P}$-name τ for a real, f_τ is the Baire measurable function (which is continuous on a comeager set) corresponding to τ.

$$\begin{aligned}
\tau_A &= \{(\mathbb{P}, p, \sigma) \in \mathcal{H}_{\omega_1} \mid \sigma \text{ is a } \mathbb{P}\text{-name for a real and} \\
&\qquad \big(\forall^\infty G \in \mathrm{St}(\mathbb{P})\big)\ p \in G \implies f_\sigma(G) \in A\}, \\
\tau_{A^c} &= \{(\mathbb{P}, p, \sigma) \in \mathcal{H}_{\omega_1} \mid \sigma \text{ is a } \mathbb{P}\text{-name for a real and} \\
&\qquad \big(\forall^\infty G \in \mathrm{St}(\mathbb{P})\big)\ p \in G \implies f_\sigma(G) \in A^c\}, \\
\tau_{\Gamma_F} &= \{(\mathbb{P}, p, \sigma, s) \in \mathcal{H}_{\omega_1} \mid \sigma \text{ is a } \mathbb{P}\text{-name for a real and} \\
&\qquad \big(\forall^\infty G \in \mathrm{St}(\mathbb{P})\big)\ p \in G \implies \big(f_\sigma(G), s\big) \in \Gamma_F\}, \\
\tau_{\Gamma_F{}^c} &= \{(\mathbb{P}, p, \sigma, s) \in \mathcal{H}_{\omega_1} \mid \sigma \text{ is a } \mathbb{P}\text{-name for a real and} \\
&\qquad \big(\forall^\infty G \in \mathrm{St}(\mathbb{P})\big)\ p \in G \implies \big(f_\sigma(G), s\big) \in \Gamma_{F^c}\},
\end{aligned}$$

where $\big(\forall^\infty G \in \mathrm{St}(\mathbb{P})\big)$ means "for all G modulo a Blackwell meager set in $\mathrm{St}(\mathbb{P})\ldots$". Let $M = \mathrm{HOD}^{\mathrm{L}[\tau_A, \tau_{A^c}, \tau_{\Gamma_F}, \tau_{\Gamma_F{}^c}]}_{\tau_A, \tau_{A^c}, \tau_{\Gamma_F}, \tau_{\Gamma_F{}^c}}$ and for $G \in \mathrm{St}(\mathbb{P})$, let $A_G = \{f_\sigma(G) \mid (\exists p \in G)\ (\mathbb{P}, p, \sigma) \in \tau_A \cap M\}$. Note that for any countable ordinal α, $\mathcal{P}(\alpha) \cap M$ is countable: Since M is a transitive model of ZFC, if $\mathcal{P}(\alpha) \cap M$ was uncountable, then there would be an uncountable sequence of distinct reals which would contradict Lebesgue measurability for every set of reals. Hence for any $\mathbb{P} \in \mathcal{H}_{\omega_1} \cap M$, the set of $\mathbb{P}$-generic filters over M is comeager, in particular Blackwell comeager (i.e., its complement is Blackwell meager). Therefore, when we discuss statements starting from $\big(\forall^\infty G \in \mathrm{St}(\mathbb{P})\big)$, we may assume that G is $\mathbb{P}$-generic over M.

Claim 3.2.7.

1. Let $\mathbb{P}$ be a partial order in M. Then $\big(\forall^\infty G \in \mathrm{St}(\mathbb{P})\big)\ A_G = A \cap M[G] \in M[G]$ and $M[G]$ is closed under F.

2. Let $\mathbb{P} = \mathrm{Coll}(\omega, 2^\omega)^M$, where $\mathrm{Coll}(\omega, 2^\omega)$ is the forcing collapsing the cardinal 2^ω into countable with finite conditions. Then $\big(\forall^\infty G \in \mathrm{St}(\mathbb{P})\big)\ A_G$ has the Baire property in $M[G]$.

Proof. We first show that $A_G = A \cap M[G]$ for Blackwell comeager many G. Since I_{BM} is a σ-ideal, for Blackwell comeager many G, G is $\mathbb{P}$-generic over M and if $(\mathbb{P}, p, \sigma) \in \tau_A \cap M$ (resp., $\tau_{A^c} \cap M$) and $p \in G$, then $f_\sigma(G) = \sigma^G \in A$ (resp., A^c). We show that $A_G = A \cap M[G]$ for any such G.

Fix such a G. We first prove that $A_G \subseteq A \cap M[G]$. Take any real x in A_G. Then there is a $p \in G$ and a σ such that $(\mathbb{P}, p, \sigma) \in \tau_A \cap M$ and $\sigma^G = x$. Then by the property of G, $x = \sigma^G = f_\sigma(G) \in A$, as desired. We show that $A \cap M[G] \subseteq A_G$. Let x be a real in $M[G]$ which is not in A_G. We prove that x is also not in A. Since x is in $M[G]$, there is a $\mathbb{P}$-name σ for a real in M such that $\sigma^G = x$. Since A is measurable with respect to I_{BM} by Lemma 3.2.4, the set $\{p \in \mathbb{P} \mid \text{ either } (\mathbb{P}, p, \sigma) \in \tau_A \cap M \text{ or } (\mathbb{P}, p, \sigma) \in \tau_{A^c} \cap M\}$ is dense and it is in M. Since G is $\mathbb{P}$-generic over M, there is a $p \in G$ such that either $(\mathbb{P}, p, \sigma) \in \tau_A$ or $(\mathbb{P}, p, \sigma) \in \tau_{A^c}$. But $(\mathbb{P}, p, \sigma) \in \tau_A$ cannot hold because it would imply $x = \sigma^G \in A_G$. Hence $(\mathbb{P}, p, \sigma) \in \tau_{A^c}$ and $x = \sigma^G = f_\sigma(G) \in A^c$ by the property of G, as desired.

Let $\rho_A = \{(\sigma, p) \mid (\mathbb{P}, p, \sigma) \in \tau_A \cap M\}$. Since the comprehension axioms with τ_A as a unary predicate hold in M, ρ_A is a $\mathbb{P}$-name for a set of reals in M and $\rho_A^G = A_G \in M[G]$. Hence $A_G = A \cap M[G] \in M[G]$ for Blackwell comeager many G, as desired.

Next, we show that $M[G]$ is closed under F for Blackwell comeager many G. We prove this for any G which is $\mathbb{P}$-generic over M such that if $(\mathbb{P}, p, \sigma, s) \in \tau_{\Gamma_F}$ (resp., $\tau_{\Gamma_{F^c}}$) and p is in G, then $F(\sigma^G) \supseteq s$ (resp., $F(\sigma^G) \not\supseteq s$). Fix such a G and let x be a real in $M[G]$. We show that $F(x)$ is also in $M[G]$. Since x is in $M[G]$, there is a $\mathbb{P}$-name σ for a real in M such that $\sigma^G = x$. Since every subset of $\mathrm{St}(\mathbb{P})$ is measurable with respect to I_{BM}, the function $G' \mapsto F\big(f_\sigma(G')\big)$ is continuous modulo a Blackwell meager set in $\mathrm{St}(\mathbb{P})$. Hence for any finite binary sequence s, the set of all $p \in \mathbb{P}$ such that either $\big(\forall^\infty G' \in \mathrm{St}(\mathbb{P})\big)\ p \in G' \implies F\big(f_\sigma(G')\big) \supseteq s$ or $\big(\forall^\infty G' \in \mathrm{St}(\mathbb{P})\big)\ p \in G' \implies F\big(f_\sigma(G')\big) \not\supseteq s$ is dense and is in M. By the genericity and the property of G, for any s, there is a $p \in G$ such that $F(\sigma^G) \supseteq s$ if and only if $\big(\forall^\infty G' \in \mathrm{St}(\mathbb{P})\big)\ p \in G' \implies F\big(f_\sigma(G')\big) \supseteq s$ if and only if $(\mathbb{P}, p, \sigma, s) \in \tau_{\Gamma_F} \cap M$. Hence $F(x) = F(\sigma^G) = \bigcup\{s \mid (\exists p \in G)\ (\mathbb{P}, p, \sigma, s) \in \tau_{\Gamma_f} \cap M\}$, which is in $M[G]$, as desired.

Finally, we show that A_G has the Baire property in $M[G]$ for Blackwell comeager many G when $\mathbb{P} = \mathrm{Coll}(\omega, 2^\omega)^M$. Actually, we show that A_G has the Baire property in $M[G]$ for any $\mathbb{P}$-generic G over M. Let s be a finite binary sequence. We show that there is a t extending s such that either $[t] \cap A_G$ or $[t] \setminus A_G$ is meager in $M[G]$. Let $\dot{c}$ be a canonical name for a Cohen real. Since one can embed Cohen forcing into $\mathrm{Coll}(\omega, 2^\omega)^M$ in a natural way in M, we may regard $\dot{c}$ as a $\mathbb{P}$-name for a Cohen real. Since 2^ω in M is countable in $M[G]$, the set of Cohen reals over M is comeager in $M[G]$. Take any Cohen real c over M with $s \subseteq c$ in $M[G]$. We may assume c is in A_G (the case $c \notin A_G$ can be dealt with in the same way). Recall that $\rho^G = A_G$ and hence by the forcing theorem, there is a $p \in G$ and a σ such that $M \vDash p \Vdash$ "$\dot{c} = \sigma \supseteq \check{s}$" and $(\mathbb{P}, p, \sigma) \in \tau_A \cap M$, which implies $(\mathbb{P}, p, \dot{c}) \in \tau_A \cap M$, namely $(\dot{c}, p) \in \rho_A$. But the value of $\dot{c}$ will be decided within Cohen forcing and by the definition of τ_A, we may assume that p is a condition of Cohen forcing extending s. Hence for any Cohen real c' over M with $p \subseteq c$ in $M[G]$, c is in A_G. Since the set of all Cohen reals over M is comeager in $M[G]$, this is what we desired. □ (Claim 3.2.7)

We now finish the proof of Theorem 3.2.6 by showing that A has the Baire property. Let G be such that the conclusions of Claim 3.2.7 hold. By the first item of Claim 3.2.7, the structure $(\omega, {}^\omega\omega \cap M[G], \mathrm{app}, +, \cdot, =, 0, 1, A_G)$ is an elementary substructure of $\mathcal{A}_A^2$. Since the Baire property for A can be described in the structure $\mathcal{A}_A^2$ in this language and A_G has the Baire property in $M[G]$, A also has the Baire property, as desired. □ (Theorem 3.2.6)

Next, we show that every set of reals is ∞-Borel assuming $\mathrm{Bl\text{-}AD}_{\mathbb{R}}$. For that purpose, we introduce the Vopěnka algebra and its variant, which is a main tool for our argument. The original motivation for the Vopěnka algebra is to make every set to be generic over HOD, the class of all the hereditarily ordinal definable sets, i.e., any element of the transitive closure of a given set is ordinal definable. HOD is an important inner model of ZFC containing all the (possible) important inner models with large cardinals and it is close to V in the sense that any set in V can be generic over HOD via the Vopěnka algebra.

We define the Vopěnka algebra and its variant for HOD_X, where X is an arbitrary set, OD_X is the class of all sets ordinal definable with a parameter X, and HOD_X is the class of sets a where any element of the transitive closure of a is in OD_X.

Take any arbitrary set X and fix an ordinal definable injection $i_X \colon \mathrm{OD}_X \to \mathrm{HOD}_X$. Then consider the *Vopěnka algebra* $\mathbb{P}_{V,X}$ in HOD_X as follows: $\mathbb{P}_{V,X} = \{i_X(A) \mid A \in \mathrm{OD}_X \text{ and } A \subseteq \mathcal{P}(\omega)\}$. For $p, q \in \mathbb{P}_{V,X}$, $p \leq q$ if $i_X^{-1}(p) \subseteq i_X^{-1}(q)$. It is easy to see that the definition of $\mathbb{P}_{V,X}$ does not depend on the choice of i_X, i.e., if there are two such injections, then the corresponding two partial orders are isomorphic in HOD_X. Vopěnka [87] proved that $\mathbb{P}_{V,\emptyset}$ is a complete Boolean algebra in HOD (when $X = \emptyset$) and each real in V can be seen as a $\mathbb{P}_{V,\emptyset}$-generic filter over HOD in the following way: For each real x in V, the set $G_x = \{p \in \mathbb{P}_{V,\emptyset} \mid x \in i_\emptyset^{-1}(p)\}$ is a $\mathbb{P}_{V,\emptyset}$-generic filter over HOD and $\mathrm{HOD}[x] = \mathrm{HOD}[G_x]$. Conversely, if G is a $\mathbb{P}_{V,\emptyset}$-generic filter over HOD, then the set $\bigcap\{i_\emptyset^{-1}(p) \mid p \in G\}$ is a singleton. We call the element of the singleton a *Vopěnka real over HOD* and denote it y_G. Then $y_{G_x} = x$ for each real x in V. The analogue of the above results holds for HOD_X for arbitrary set X.

We now introduce a variant of the Vopěnka algebra, namely the *Vopěnka algebra with ∞-Borel codes.* Given a set X, consider the following partial order $\mathbb{P}^*_{V,X}$ in HOD_X: Conditions of $\mathbb{P}^*_{V,X}$ are ∞-Borel codes in HOD_X where the ordinals used in their trees are below Θ in HOD_X and for ϕ, ψ in $\mathbb{P}^*_{V,X}$, $\phi \leq \psi$ if $B_\phi \subseteq B_\psi$.[1] Then we can prove the analogue of Vopěnka's theorem in exactly the same way:

Theorem 3.2.8 (ZF)**.** (Folklore) Let X be an arbitrary set.

1. $\mathbb{P}^*_{V,X}$ is a complete Boolean algebra in HOD_X.

2. For each real x in V, the set $G_x = \{\phi \in \mathbb{P}^*_{V,X} \mid x \in B_\phi\}$ is $\mathbb{P}^*_{V,X}$-generic over HOD_X and $\mathrm{HOD}_X[x] = \mathrm{HOD}_X[G_x]$. Conversely, if G is a $\mathbb{P}^*_{V,X}$-generic filter over HOD_X, then the set $\bigcap\{B_\phi \mid \phi \in G\}$ is a singleton and we call the real in the singleton a *Vopěnka real over* HOD_X and denote it y_G. Then $\mathrm{HOD}_X[y_G] = \mathrm{HOD}_X[G]$ and $y_{G_x} = x$ for each G and x.

Proof. The proof is exactly the same as for the Vopěnka algebra which can be found, e.g., in Jech's textbook [37, Theorem 15.46]. □

[1] For any ∞-Borel code ϕ in HOD_X, there is an ∞-Borel code ψ where the ordinals used in the tree of ψ is less than Θ in HOD_X such that $\phi \leq \psi$ and $\psi \leq \phi$. Hence the restriction of ordinals for ∞-Borel codes will not affect the structure of this partial order.

The difference between $\mathbb{P}_{V,X}$ and $\mathbb{P}^*_{V,X}$ is that y_G might not recover G from HOD_X for $\mathbb{P}_{V,X}$ while $\mathrm{HOD}_X[y_G] = \mathrm{HOD}_X[G]$ for $\mathbb{P}^*_{V,X}$. This is because the injection i_X is not in HOD_X in general while the definition of $\mathbb{P}^*_{V,X}$ does not refer to OD. For our purpose, we will use $\mathbb{P}^*_{V,X}$.

Theorem 3.2.9. Assume Bl-$\mathrm{AD}_{\mathbb{R}}$. Then every set of reals is ∞-Borel.

Proof. We modify the argument for the following theorem by Woodin:

Theorem 3.2.10 (Woodin)**.** Assume AD and that every relation on the reals can be uniformized. Then every set of reals is ∞-Borel.

Let A be an arbitrary set of reals. We show that A is ∞-Borel.

By Theorem 3.2.6, every set of reals has the Baire property. Hence every subset of $\mathrm{St}(\mathbb{P})$ has the Baire property for any $\mathbb{P} \in \mathcal{H}_{\omega_1}$. We freely use this fact later. We fix a simple coding of elements of $\mathcal{H}_{\omega_1}$ by reals and if we say "a real x codes…", then we refer to this coding.

Let τ_A and R_A be as follows:

$$\begin{aligned}\tau_A = \{(\mathbb{P}, p, \sigma) \in \mathcal{H}_{\omega_1} \mid{}& \sigma \text{ is a } \mathbb{P}\text{-name for a real and} \\ & \big(\forall^\infty G \in \mathrm{St}(\mathbb{P})\big)\ p \in G \implies f_\sigma(G) \in A\}, \\ R_A = \{(x, y) \mid{}& \text{if } x \text{ codes a } (\mathbb{P}, p, \sigma) \in \tau_A, \text{ then } y \text{ codes a } (D_i \mid i < \omega) \\ & \text{such that } (\forall i)\ D_i \text{ is dense in } \mathbb{P} \text{ and} \\ & \big(\forall G \in \mathrm{St}(\mathbb{P})\big)\ \big(p \in G, (\forall i)\ G \cap D_i \neq \emptyset \implies f_\sigma(G) \in A\big)\},\end{aligned}$$

where "$\big(\forall^\infty G \in \mathrm{St}(\mathbb{P})\big) \ldots$" means "For comeager many G in $\mathrm{St}(\mathbb{P}) \ldots$". Note that the term relation τ_A defined here is different from the one in Theorem 3.2.6 in the sense that now we use comeagerness for the quantifier $\forall^\infty$ instead of Blackwell comeagerness.

Let F_A uniformize R_A and Γ_A be the graph of F_A, i.e., $\Gamma_A = \{(x, s) \mid s \in {}^{<\omega}\omega, F_A(x) \supseteq s\}$. Define τ_{Γ_A} as follows:

$$\begin{aligned}\tau_{\Gamma_A} = \{(\mathbb{P}, p, \sigma, s) \in \mathcal{H}_{\omega_1} \mid{}& \sigma \text{ is a } \mathbb{P}\text{-name for a real and} \\ & \big(\forall^\infty G \in \mathrm{St}(\mathbb{P})\big)\ p \in G \implies \big(f_\sigma(G), s\big) \in \Gamma_A\},\end{aligned}$$

here we also use comeagerness for the quantifier $\forall^\infty$.

Let A^{c} be the complement of A and define and construct $\tau_{A^{\mathrm{c}}}, R_{A^{\mathrm{c}}}, F_{A^{\mathrm{c}}}, \Gamma_{A^{\mathrm{c}}}$, and $\tau_{\Gamma_{A^{\mathrm{c}}}}$ as above.

The following is the key point:

Claim 3.2.11 (Woodin)**.** Let M be a transitive subset of $\mathcal{H}_{\omega_1}$ and $(M, \in, \tau_A, \tau_{\Gamma_A})$ is a model of ZFC.[2] Let $(\mathbb{P}, p, \sigma) \in M \cap \tau_A$. Then for every $\mathbb{P}$-generic filter G over M, if p is in G, then $\sigma^G \in A$. The same holds for A^{c}.

[2]Here it satisfies Comprehension scheme and Replacement scheme for formulas in the language of set theory with predicates for τ_A and τ_{Γ_A}.

Proof of Claim 3.2.11. Let $\mathbb{Q} = \mathrm{Coll}\big(\omega, \mathrm{TC}(\mathbb{P})\big)$, where $\mathrm{Coll}\big(\omega, \mathrm{TC}(\mathbb{P})\big)$ is the standard forcing collapsing $\mathrm{TC}(\mathbb{P})$ into a countable set with finite sets as conditions. Since $\mathbb{P}, p, \sigma$ are countable in $M^{\mathbb{Q}}$, there is a $\mathbb{Q}$-name σ' for a real in M coding the triple $(\mathbb{P}, p, \sigma)$.

Subclaim 3.2.12. There is a $\mathbb{Q}$-name ρ for a real in M such that in V, for comeager many H in $\mathrm{St}(\mathbb{Q})$, $f_\rho(H) = F_A(f_{\sigma'}(H))$.

Proof of Subclaim 3.2.12. First note that the map $f\colon H \mapsto F_A\big(f_{\sigma'}(H)\big)$ is continuous on a comeager set in $\mathrm{St}(\mathbb{Q})$, i.e., Baire measurable. This is because every subset of $\mathrm{St}(\mathbb{Q})$ has the Baire property in $\mathrm{St}(\mathbb{Q})$ and we can do the same argument as the one in Proposition 3.2.2 to uniformize a relation almost everywhere (since we use open sets in $\mathrm{St}(\mathbb{Q})$ to approximate subsets in $\mathrm{St}(\mathbb{Q})$ in this case, we get a continuous function instead of a Borel function).

Let $\rho = \tau_f$ where the notation τ_f is from Lemma 2.1.2. Then ρ is a $\mathbb{Q}$-name for a real because the map f is Baire measurable as we observed. Moreover, ρ is in M because

$$\big((m,n)\check{}, q\big) \in \rho \iff (\exists s \in {}^{<\omega}2)\ \big(s(m) = n \text{ and } \big(\mathbb{Q}, q, (\sigma, s)\big) \in \tau_{\Gamma_A}\big)$$

and the right hand side of the equivalence is definable in $(M, \tau_A, \tau_{\Gamma_A})$, which is a model of ZFC by assumption. Finally, by Lemma 2.1.2, it is easy to see that for comeager many H in $\mathrm{St}(\mathbb{Q})$, $f_\rho(H) = F_A(f_{\sigma'}(H))$. □ (Subclaim 3.2.12)

Now let G be a $\mathbb{P}$-generic filter over M with $p \in G$. We show that $f_\sigma(G) \in A$. Take a $\mathbb{Q}$-generic filter H over $M[G]$ with $\rho^H = F_A(\sigma'^H)$. This is possible by Subclaim 3.2.12 and that $M[G] \subseteq \mathcal{H}_{\omega_1}$. Then G is also a $\mathbb{P}$-generic filter over $M[H]$ and $F_A(\sigma'^H) = \rho^H \in M[H]$. But by the definition of F_A, $F_A(\sigma'^H)$ codes a sequence $(D_i \mid i \in \omega)$ such that D_i is a dense subset of $\mathbb{P}$ in $M[H]$ for each $i \in \omega$ and for any G' in $\mathrm{St}(\mathbb{P})$, if $G' \cap D_i \neq$ for each i, then $f_\sigma(G') \in A$. But G is a $\mathbb{P}$-generic filter over $M[H]$ and each D_i is in $M[H]$. Hence $G \cap D_i \neq \emptyset$ for each $i \in \omega$ and $f_\sigma(G) \in A$, as desired. □ (Claim 3.2.11)

Let $X = (A, \tau_A, \tau_{\Gamma_A}, \tau_{A^c}, \tau_{\Gamma_{A^c}})$. Recall that U is the fine normal measure on $\mathcal{P}_{\omega_1}$ we fixed at the beginning of this section. Let $M = \mathrm{L}(X, \mathbb{R})[U]$. Since the statement "a real is in the decode of an ∞-Borel code" is absolute between transitive models of ZF as in § 1.13 and M contains all the reals, if A is ∞-Borel in M, so is in V.

From now on, we work in M and prove that A is ∞-Borel in M, which completes the proof of this theorem. The benefit of working in M is that we have DC in M because $\mathrm{DC}_{\mathbb{R}}$ implies DC in M while DC might fail in V in general. Note that $U \cap M$ is a fine normal measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$ in M and we use U to denote $U \cap M$ from now on.

We find a set of ordinals S and a formula ϕ such that for any real x,

$$x \in A \iff \mathrm{L}[S, x] \vDash \phi(x). \tag{3.1}$$

By Fact 1.13.2, this implies that A is ∞-Borel.

For a in $\mathcal{P}_{\omega_1}(\mathbb{R})$, let $M_a, \mathbb{Q}^*_a$, and b_a be as follows:

$$\begin{aligned} M_a &= \mathrm{HOD}_X^{\mathrm{L}_{\omega_1}[X](a)}, \\ \mathbb{Q}^*_a &= \mathbb{P}^*_{V,X} \text{ in } M_a, \\ b_a &= \sup\{q \in \mathbb{Q}^*_a \mid (\mathbb{Q}^*_a, q, \dot{y_G}) \in \tau_A\} \text{ in } M_a, \end{aligned}$$

where $\dot{y_G}$ is a canonical $\mathbb{Q}^*_a$-name for a Vopěnka real given in Theorem 3.2.8.

Note that M_a is a transitive subset of $\mathcal{H}_{\omega_1}$ and $(M_a, \tau_A, \tau_{\Gamma_A})$ and $(M_a, \tau_{A^c}, \tau_{\Gamma_{A^c}})$ are models of ZFC because $\mathrm{L}_{\omega_1}[X](a)$ is a transitive model of ZF (to check the power set axiom, we use the condition that there is no uncountable sequence of distinct reals ensured by Lebesgue measurability). Note also that b_a is well-defined because $\mathbb{Q}^*_a$ is a complete Boolean algebra in M_a by Theorem 3.2.8.

Then we claim that for each $a \in \mathcal{P}_{\omega_1}(\mathbb{R})$ and real x which induces the filter G_x that is $\mathbb{P}^*_{V,X}$-generic filter over M_a, $x \in A \iff b_a \in G_x$. Fix a and x. Assume $b_a \in G_x$. We show that $x \in A$. If we apply Claim 3.2.11 to $M = M_a, (\mathbb{P}, p, \tau) = (\mathbb{Q}^*_a, b_a, \dot{y_G})$, and $G = G_x$, then we get $x \in A$ because $y_{G_x} = x$ as in Theorem 3.2.8. For the converse, we assume b_a is not in G_x and prove that x is not in A. Let ${b_a}'$ be the one corresponding to b_a for A^c instead of for A, i.e.,

$${b_a}' = \sup\{q \in \mathbb{Q}^*_a \mid (\mathbb{Q}^*_a, q, \dot{y_G}) \in \tau_{A^c}\}.$$

Then $b_a \vee {b_a}' = \mathbf{1}$. This is because $f^{-1}_{\dot{y_G}}(A)$ has the Baire property in $\mathrm{St}(\mathbb{Q}^*_a)$. Since $b_a \notin G_x$ and G_x is $\mathbb{P}^*_{V,X}$-generic over M_a, ${b_a}'$ is in G_x. Hence we can apply Claim 3.2.11 to $M_a, A^c, (\mathbb{Q}^*_a, {b_a}', \dot{y_G})$, and G_x and we get $x \in A^c$, i.e., x is not in A, as desired.

Fix an $a \in \mathcal{P}_{\omega_1}(\mathbb{R})$. Note that since $\mathbb{P}^*_{V,X}$ is the Vopěnka algebra with ∞-Borel codes defined in M_a, any real in $\mathrm{L}_{\omega_1}[X](a)$ is $\mathbb{P}^*_{V,X}$-generic over M_a. Hence for any real x in $\mathrm{L}_{\omega_1}[X](a)$, $x \in A \iff b_a \in G_x$.

Now we use this local equivalence in $\mathrm{L}_{\omega_1}[X](a)$ to get the global equivalence (3.1) by taking the ultraproduct of M_a via U. Let $M_\infty, \mathbb{Q}_\infty, b_\infty$ be as follows:

$$M_\infty = \prod_U M_a,\ \mathbb{Q}_\infty = \prod_U \mathbb{Q}^*_a,\ b_\infty = \prod_U b_a.$$

Note that Łoś's theorem holds for M_∞ because there is a canonical function mapping a to a well-order on M_a.[3] By DC (in M), M_∞ is wellfounded. So we may assume M_∞ is transitive. Hence, M_∞ is a transitive model of ZFC, $\mathbb{Q}_\infty$ is a partial order consisting of ∞-Borel codes, and $b_\infty \in \mathbb{Q}_\infty$.

We claim that for each real x, $x \in A \iff x \in B_{b_\infty}$. This will establish the equivalence (3.1) because the pair $(\mathbb{Q}_\infty, b_\infty)$ can be seen as a set of ordinals since they are objects in the transitive model M_∞ of ZFC.

[3]Łoś's theorem fails for $\prod_U \mathrm{L}_{\omega_1}[X](a)$. This is because $\mathrm{L}_{\omega_1}[X](a)$ is not a model of ZFC for almost all a and we cannot assign a well-order on $\mathrm{L}_{\omega_1}[X](a)$ to each a as we did for $\prod_U M_a$.

Let us fix a real x. By the fineness of U, $x \in a$ for almost all a w.r.t. U. Then

$$\begin{aligned} x \in A &\iff b_a \in G_x \text{ for almost all } a \\ &\iff x \in B_{b_a} \text{ for almost all } a \\ &\iff x \in B_{b_\infty}, \end{aligned}$$

where the first equivalence is by the local equivalence we have seen and the third equivalence follows from Łoś's theorem for $\prod_U M_a[x]$ (note that $M_a[x]$ is a generic extension of M_a given by G_x and we can prove Łoś's theorem for $\prod_U M_a[x]$ in the same way as for $\prod_U M_a$). This completes the proof. □

Together with the non-existence of uncountable sequences of distinct reals, the ∞-Borelness for every set of reals gives us almost all the regularity properties we introduced in chapter 2 for every set of reals. Recall that $\mathbb{P}$-measurability for a strongly arboreal forcing $\mathbb{P}$ was the regularity property we introduced in Definition 2.1.7. Also recall that strongly proper forcings are strengthening of proper forcings for projective forcings.

Proposition 3.2.13. Assume that there is no uncountable sequence of distinct reals and every set of reals is ∞-Borel. Then every set of reals is $\mathbb{P}$-measurable for any strongly arboreal, strongly proper forcing $\mathbb{P}$.

Proof. The results for Cohen forcing, random forcing, and Mathias forcing are well-known and the proof is the same as the one in Case 1 in Theorem 2.4.2. We just replace $\mathrm{L}[a]$ in Theorem 2.4.2 with $\mathrm{L}[S]$, where S codes a given set of reals and a given partial order $\mathbb{P}$. The fact that the set of all dense subsets of $\mathbb{P}$ in $\mathrm{L}[S]$ is countable follows from the non-existence of uncountable sequences of distinct reals (because $\mathrm{L}[S]$ is a ZFC model) and the fact that $\mathrm{L}[S]$ correctly computes $\mathbb{P}$ follows from that S codes $\mathbb{P}$. The rest is exactly the same as in Case 1 in Theorem 2.4.2. □

Corollary 3.2.14. Assume Bl-$\mathrm{AD}_{\mathbb{R}}$. Then every set of reals is $\mathbb{P}$-measurable for any strongly arboreal, strongly proper forcing $\mathbb{P}$.

3.3 Toward $\mathrm{AD}_{\mathbb{R}}$ from Bl-$\mathrm{AD}_{\mathbb{R}}$

In this section, we discuss the following conjecture:

Conjecture 3.3.1 (DC)**.** $\mathrm{AD}_{\mathbb{R}}$ and Bl-$\mathrm{AD}_{\mathbb{R}}$ are equivalent.

Since $\mathrm{AD}_{\mathbb{R}}$ implies Bl-$\mathrm{AD}_{\mathbb{R}}$ by Theorem 1.14.3, the question is whether Bl-$\mathrm{AD}_{\mathbb{R}}$ implies $\mathrm{AD}_{\mathbb{R}}$ in ZF+DC. Woodin proved the following:

Theorem 3.3.2 (Woodin)**.** Assume AD and DC. Then the following are equivalent:

1. Every set of reals is Suslin,
2. The axiom $\mathsf{AD}_{\mathbb{R}}$ holds, and
3. Every relation on the reals can be uniformized.

Hence, to prove Conjecture 3.3.1, it suffices to show that every set of reals is Suslin from Bl-$\mathsf{AD}_{\mathbb{R}}$: If every set of reals is Suslin, then by Theorem 1.14.5, AD holds. Now by Theorem 3.3.2 and Theorem 1.14.9, $\mathsf{AD}_{\mathbb{R}}$ holds assuming Bl-$\mathsf{AD}_{\mathbb{R}}$ and DC. Note that Martin's Conjecture (i.e., Bl-AD implies AD) implies Conjecture 3.3.1 by Theorem 3.3.2. Hence it is interesting to see whether this is Conjecture is true or not.

We try to mimic the arguments for the implication from uniformization to Suslinness in Theorem 3.3.2 and reduce Conjecture 3.3.1 to a small conjecture. Throughout this section, we fix U as a fine normal measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$, which exists by Theorem 3.1.3.

First, we show that every set of reals is strong ∞-Borel assuming Bl-$\mathsf{AD}_{\mathbb{R}}$. Before giving a definition of strong ∞-Borel codes, we start with a small lemma:

Lemma 3.3.3. Assume Bl-$\mathsf{AD}_{\mathbb{R}}$ and DC. Let $j : V \to \mathrm{Ult}(V, U)$ be the ultrapower map via U. Then $j(\omega_1) = \Theta$.

Proof. We first show that $j(\omega_1) \geq \Theta$. Let α be an ordinal less than Θ and R be a prewellorder on the reals with length α. Define $f \colon \mathcal{P}_{\omega_1}(\mathbb{R}) \to \omega_1$ be as follows: For $a \in \mathcal{P}_{\omega_1}(\mathbb{R})$, $f(a)$ is the length of the prewellorder $R \cap (a \times a)$ on a. Since a is countable, $f(a)$ is also countable. Hence $f \in_U c_{\omega_1}$, where $\in_U$ is the membership relation for $\mathrm{Ult}(V, U)$ and c_{ω_1} is the constant function on $\mathcal{P}_{\omega_1}(\mathbb{R})$ with value ω_1.

We show that the structure $([f]_U, \in)$ is isomorphic to $(\alpha, \in)$ and hence $[f]_U = \alpha$, which implies $\alpha < j(\omega_1)$ because $f \in_U c_{\omega_1}$. For any $a \in \mathcal{P}_{\omega_1}(\mathbb{R})$, let $\pi(a)$ be the transitive collapse of $\bigl(a, R \cap (a \times a)\bigr)$ into $\bigl(f(a), \in\bigr)$. Then by Łoś's Theorem for simple formulas, $[\pi]_U$ is an isomorphism between $\bigl([\mathrm{id}]_U, j(R) \cap ([\mathrm{id}]_U \times [\mathrm{id}]_U)\bigr)$ and $([f]_U, \in)$, where id is the identity function on $\mathcal{P}_{\omega_1}(\mathbb{R})$.

Claim 3.3.4. The identity function id represents $\mathbb{R}$, i.e., $[\mathrm{id}]_U = \mathbb{R}$.

Proof of Claim 3.3.4. By the fineness of U, for any real x, $\{a \mid x \in a\} \in U$. Hence $[c_x]_U \in [\mathrm{id}]_U$. By the countable completeness of U, $[c_x]_U = x$ and hence $x \in [\mathrm{id}]_U$ for any real x. Suppose f is a function on $\mathcal{P}_{\omega_1}(\mathbb{R})$ with $f \in_U \mathrm{id}$. Then by the normality of U, there is a real x such that $\{a \mid x = f(a)\} \in U$, i.e., $c_x =_U f$. Hence $[f]_U = x$ and $[f]_U$ is a real, which finishes the proof. □ (Claim 3.3.4)

By Claim 3.3.4, we have $[\mathrm{id}]_U = \mathbb{R}$ and $j(R) \cap \bigl([\mathrm{id}]_U \times [\mathrm{id}]_U\bigr) = R$. Since $\bigl([\mathrm{id}]_U, j(R) \cap ([\mathrm{id}]_U \times [\mathrm{id}]_U)\bigr)$ and $([f]_U, \in)$ are isomorphic, $([f]_U, \in)$ is isomorphic to $(\mathbb{R}, R)$, which is isomorphic to $(\alpha, \in)$, as desired. Hence $\alpha < j(\omega_1)$ and $j(\omega_1) \geq \Theta$.

Next, we show that $j(\omega_1) \leq \Theta$. Let f be a function from $\mathcal{P}_{\omega_1}(\mathbb{R})$ to ω_1. We show that $[f]_U < \Theta$. By uniformization for every set of reals, there is a function

e from the reals to themselves such that if a real x codes an $a \in \mathcal{P}_{\omega_1}(\mathbb{R})$, then $e(x)$ codes $f(a)$. Let S be an ∞-Borel code for the graph Γ_e of e which exists by Theorem 3.2.9.

Claim 3.3.5. For all $a \in \mathcal{P}_{\omega_1}(\mathbb{R})$, $f(a) < \Theta^{\mathrm{L}[S](a)}$.

Proof of Claim 3.3.5. Note that $\mathcal{P}(x) \cap \mathrm{L}[S](a)$ is countable in V for any $x \in \mathcal{H}_{\omega_1} \cap \mathrm{L}[S](a)$. Hence there is a $\mathrm{Coll}(\omega, a)$-generic g over $\mathrm{L}[S](a)$ in V. Fix such a g. Let x_g be a real coding a from g. Then since S is an ∞-Borel code for Γ_e, one can compute whether $e(x_g) \supseteq s$ for each finite binary sequence s or not in $\mathrm{L}[S](a, g)$, hence $e(x_g) \in \mathrm{L}[S](a, g)$. Therefore $f(a)$ is countable in $\mathrm{L}[S](a, g)$. But $\Theta^{\mathrm{L}[S](a)}$ stays an uncountable cardinal in $\mathrm{L}[S](a, g)$. Hence $f(a) < \Theta^{\mathrm{L}[S](a)}$, as desired. □

By the normality of U, the following choice principle holds: For any function $F\colon \mathcal{P}_{\omega_1}(\mathbb{R}) \to V$ such that $\emptyset \neq F(a) \in \mathrm{L}[S](a)$ for almost a with respect to U, then there is a function $f\colon \mathcal{P}_{\omega_1}(\mathbb{R}) \to V$ such that $f(a) \in F(a)$ for almost all a with respect to U. This implies Łoś's Theorem for the ultraproduct $\prod_U \mathrm{L}[S](a)$.

Let $S^* = j(S)$. Then $\left(\prod_U \mathrm{L}[S](a), \in_U\right)$ is isomorphic to $\left(\mathrm{L}[S^*](\mathbb{R}), \in\right)$ by looking at the map $g \mapsto j(g)(\mathbb{R})$. (Note that $\mathrm{Ult}(V, U)$ is wellfounded by DC.) Hence

$$[f]_U < [a \mapsto \Theta^{\mathrm{L}[S](a)}]_U = \Theta^{\mathrm{L}[S^*](\mathbb{R})} \leq \Theta^V,$$

as desired. □

We now introduce strong ∞-Borel codes. An ∞-Borel code S is *strong* if the tree of S is a tree on γ for some $\gamma < \Theta$ and for any $f\colon {}^{<\omega}\mathbb{R} \to \mathbb{R}$ and surjection $\pi\colon \mathbb{R} \to \gamma$, there is an $a \in \mathcal{P}_{\omega_1}$ such that a is closed under f, $S \restriction \pi[a]$ is an ∞-Borel code, and $B_{S \restriction \pi[a]} \subseteq B_S$. Note that the choice of γ does not depend on the definition of strong ∞-Borel codes. A set of reals A is *strong ∞-Borel* if $A = B_S$ for some strong ∞-Borel code S. There is a finer version of Fact 1.13.2 as follows:

Fact 3.3.6.

1. Let S be a strong ∞-Borel code and $\gamma < \Theta$ be such that S is a tree on β for some $\beta < \gamma$ and $\mathrm{L}_\gamma[S, x] \vDash$ "$\mathrm{KP} + \Sigma_1$-Separation" for any real x. Let $\phi(S, x)$ be a Σ_1-formula expressing "$x \in B_S$". Then for any function $f\colon {}^{<\omega}\mathbb{R} \to \mathbb{R}$ and surjection $\pi\colon \mathbb{R} \to \gamma$, there is an $a \in \mathcal{P}_{\omega_1}(\mathbb{R})$ such that a is closed under f and for any real x, if $\mathrm{L}_{\bar{\gamma}}[\bar{S}, x] \vDash \phi(\bar{S}, x)$, then $\mathrm{L}_\gamma[S, x] \vDash \phi(S, x)$, where $\mathrm{L}_{\bar{\gamma}}[\bar{S}]$ is the transitive collapse of the Skolem hull of $\pi[a] \cup \{S\}$ in $\mathrm{L}_\gamma[S]$.

2. Let γ be an ordinal with $\gamma < \Theta$, ϕ be a Σ_1-formula, and S be a bounded subset of γ such that $\mathrm{L}_\gamma[S, x] \vDash$ "$\mathrm{KP} + \Sigma_1$-Separation" for any real x. Set $A = \{x \in \mathbb{R} \mid \mathrm{L}_\gamma[S, x] \vDash \phi(S, x)\}$. Assume that for any function $f\colon {}^{<\omega}\mathbb{R} \to \mathbb{R}$ and surjection $\pi\colon \mathbb{R} \to \gamma$, there is an $a \in \mathcal{P}_{\omega_1}(\mathbb{R})$ such that a is closed under f and for any real x, if $\mathrm{L}_{\bar{\gamma}}[\bar{S}, x] \vDash \phi(\bar{S}, x)$, then $\mathrm{L}_\gamma[S, x] \vDash \phi(S, x)$, where $\mathrm{L}_{\bar{\gamma}}[\bar{S}]$ is the

transitive collapse of the Skolem hull of $\pi[a] \cup \{S\}$ in $\mathrm{L}_\gamma[S]$. Then A is strong ∞-Borel.

Proof. This can be done by closely looking at the argument for Fact 1.13.2 in [80]. □

Theorem 3.3.7. Assume Bl-$\mathrm{AD}_{\mathbb{R}}$ and DC. Then every set of reals is strong ∞-Borel.

Proof. Fix a set of reals A. We show that A is strong ∞-Borel. Let $\big((M_a, \mathbb{Q}^*_a, b_a) \mid a \in \mathcal{P}_{\omega_1}(\mathbb{R})\big)$ and $(M_\infty, \mathbb{Q}^*_\infty, b_\infty)$ be as in the proof of Theorem 3.2.9, but we construct them in V, not in M. Since we have DC now, we can prove the following equivalences in exactly the same way as in Theorem 3.2.9: For all $a \in \mathcal{P}_{\omega_1}(\mathbb{R})$ and all real x inducing the filter G_x which is $\mathbb{Q}^*_a$-generic over M_a,

$$x \in A \iff b_a \in G_x \text{ (in } \mathbb{Q}^*_a).$$

Also,

$$(\forall x \in \mathbb{R})\ x \in A \iff b_\infty \in G_x \text{ (in } \mathbb{Q}^*_\infty).$$

For any a, let D_a be the set of all dense subsets of $\mathbb{Q}^*_a$ in M_a and let $D_\infty = \prod_U D_a$. Let ϕ be a Σ_1-formula such that for all a,

$$\begin{aligned}
\phi(\mathbb{Q}^*_a, b_a, D_a, x) \iff & \ x \text{ determines the filter } G_x \subseteq \mathbb{Q}^*_a \text{ such that} \\
& (\forall D \in D_a)\ G_x \cap D \neq \emptyset \text{ and } b_a \in G_x, \\
\phi(\mathbb{Q}^*_\infty, b_\infty, D_\infty, x) \iff & \ x \text{ determines the filter } G_x \subseteq \mathbb{Q}^*_\infty \text{such that} \\
& (\forall D \in D_\infty)\ G_x \cap D \neq \emptyset \text{ and } b_\infty \in G_x.
\end{aligned}$$

Let S_a and S_∞ be sets of ordinals coding the two triples $(\mathbb{Q}^*_a, b_a, D_a)$ and $(\mathbb{Q}^*_\infty, b_\infty, D_\infty)$ respectively. For an $a \in \mathcal{P}_{\omega_1}(\mathbb{R})$, let α_a be the least ordinal α such that S_a is a bounded subset of α and for all $x \in a$, $\mathrm{L}_\alpha[S_a, x]$ is a model of KP+Σ_1-Separation and let α_∞ be the least ordinal α such that S_∞ is a bounded subset of α and for all $x \in \mathbb{R}$, $\mathrm{L}_\alpha[S_\infty, x]$ is a model of KP+Σ_1-Separation. Note that by Łoś's Theorem, $(\prod_U \mathrm{L}_{\alpha_a}[S_a, x], \in_U)$ is isomorphic to $(\mathrm{L}_{\alpha_\infty}[S_\infty, x], \in)$ for every real x. Since each α_a is countable, by Lemma 3.3.3, $\alpha_\infty < \Theta$. Also, by the above equivalences, for all $a \in \mathcal{P}_{\omega_1}(\mathbb{R})$ and all reals x,

$$\begin{aligned}
x \in A &\iff \mathrm{L}_{\alpha_a}[S_a, x] \vDash \phi\ (S_a, x) \\
x \in A &\iff \mathrm{L}_{\alpha_\infty}[S_\infty, x] \vDash \phi\ (S_\infty, x).
\end{aligned}$$

By the second item of Fact 3.3.6, it suffices to show the following: For any function $f \colon {}^{<\omega}\mathbb{R} \to \mathbb{R}$ and surjection $\pi \colon \mathbb{R} \to \alpha_\infty$, there is an $a \in \mathcal{P}_{\omega_1}(\mathbb{R})$ such that a is closed under f and for any real x, if $\mathrm{L}_{\bar{\alpha_\infty}}[\bar{S_\infty}, x] \vDash \phi(\bar{S_\infty}, x)$, then $\mathrm{L}_{\alpha_\infty}[S_\infty, x] \vDash \phi(S_\infty, x)$, where $\mathrm{L}_{\bar{\alpha_\infty}}[\bar{S_\infty}]$ is the transitive collapse of the Skolem hull of $\pi[a] \cup \{S_\infty\}$ in $\mathrm{L}_{\alpha_\infty}[S_\infty]$.

Let us fix $f \colon {}^{<\omega}\mathbb{R} \to \mathbb{R}$ and $\pi \colon \mathbb{R} \to \alpha_\infty$. Since $x \in A \iff \mathrm{L}_{\alpha_b}[S_b, x] \vDash \phi\ (S_b, x)$ for each real x and $b \in \mathcal{P}_{\omega_1}(\mathbb{R})$, the following claim completes the proof:

Claim 3.3.8. There are a and b in $\mathcal{P}_{\omega_1}(\mathbb{R})$ such that a is closed under f and $(X_a, \in)$ is isomorphic to $(\mathrm{L}_{\alpha_b}[S_b], \in)$, where X_a is the Skolem hull of $\pi[a] \cup \{S_\infty\}$ in $\mathrm{L}_{\alpha_\infty}[S_\infty]$.

Proof of Claim 3.3.8. Let $\Gamma_f = \{(x, s) \in \mathbb{R} \times {}^{<\omega}2 \mid f(x) \supseteq s\}$. For each b, consider the following game $\hat{G}_b$ in $\mathrm{L}[S_b, S_\infty, \Gamma_f, \pi]$: In ω rounds,

1. Player I and II produce a countable elementary substructure X of $\mathrm{L}_{\alpha_b}[S_b]$,
2. Player II produces an $a \in \mathcal{P}_{\omega_1}(\mathbb{R})$ which is closed under f, and
3. Player II tries to construct an isomorphism between $(X, \in)$ and $(X_a, \in)$, where X_a is the Skolem hull of $\pi[a] \cup \{S_\infty\}$ in $\mathrm{L}_{\alpha_\infty}[S_\infty]$.

Player II wins if she succeeds to construct an isomorphism between $(X, \in)$ and $(X_a, \in)$. This is an open game on some set of the form $T_b \times \mathbb{R}$ where T_b is wellorderable. Hence by $\mathrm{DC}_{\mathbb{R}}$, it is determined.

Subclaim 3.3.9. There is a $b \in \mathcal{P}_{\omega_1}(\mathbb{R})$ such that player II has a winning strategy in the game $\hat{G}_b$.

Proof of Subclaim 3.3.9. To derive a contradiction, suppose there is no b such that player II has a winning strategy in the game $\hat{G}_b$ in $\mathrm{L}[S_b, S_\infty, \Gamma_f, \pi]$. By the determinacy of the game $\hat{G}_b$, player I has a winning strategy in the game $\hat{G}_b$. Let $j \colon V \to \mathrm{Ult}(V, U)$ be the ultrapower map. Then by Łoś's Theorem, $\prod_U \big(\mathrm{L}[S_b, S_\infty, \Gamma_f, \pi], \in_U, \Gamma_f, \pi\big)$ is isomorphic to $\big(\mathrm{L}[S_\infty, j(S_\infty), \Gamma_f, j(\pi)], \in, \Gamma_f, j(\pi)\big)$. Then the game $\hat{G}_\infty = \prod_U \hat{G}_b$ is an open game on some set of the form $T_\infty \times \mathbb{R}$ where T_∞ is wellorderable in $\mathrm{L}[S_\infty, j(S_\infty), \Gamma_f, j(\pi)]$ such that in ω rounds,

1. Players I and II produce a countable elementary substructure Y of $\mathrm{L}_{\alpha_\infty}[S_\infty]$,
2. Player II produces an $a \in \mathcal{P}_{\omega_1}(\mathbb{R})$ which is closed under f, and
3. Player II tries to construct an isomorphism between $(Y, \in)$ and $(Y_a, \in)$, where Y_a is the Skolem hull of $j(\pi)[a] \cup \{j(S_\infty)\}$ in $\mathrm{L}_{j(\alpha_\infty)}[j(S_\infty)]$.

Player II wins if she succeeds to construct an isomorphism between Y and Y_a. By Łoś's Theorem, player I has a winning strategy σ in $\mathrm{L}[S_\infty, j(S_\infty), \Gamma_f, j(\pi)]$. By Theorem 1.12.6, σ is also winning in V. In V, let player II move in such a way that she can arrange that a is closed under f, $j[Y] = Y_a$, and $j \restriction Y$ is the candidate for the isomorphism. This is possible by a bookkeeping argument. But then player II wins because $j \restriction Y$ is an isomorphism between Y and $j[Y]$ and defeats the strategy σ, contradiction! □ (Subclaim 3.3.9)

Hence there is a $b \in \mathcal{P}_{\omega_1}(\mathbb{R})$ such that player II has a winning strategy τ in the game $\hat{G}_b$ in $\mathrm{L}[S_b, S_\infty, \Gamma_f, \pi]$. By Theorem 1.12.6, τ is also winning in V. Since $\mathrm{L}_{\alpha_b}[S_b]$ is countable in V, we can let player I move in such a way that $X = \mathrm{L}_{\alpha_b}[S_b]$ and let player II follow τ. Since τ is winning in V, there is an $a \in \mathcal{P}_{\omega_1}(\mathbb{R})$ such that a is closed under f and $\mathrm{L}_{\alpha_b}[S_b] = X$ is isomorphic to X_a, as desired. □ (Claim 3.3.8)

□

We are now ready to prove the key statement toward Conjecture 3.3.1: Recall that for a natural number n with $n \geq 1$ and a subset A of $\mathbb{R}^{n+1}$, $\exists^{\mathbb{R}} A = \{x \in \mathbb{R}^n \mid (\exists y \in \mathbb{R})\ (x, y) \in A\}$.

Theorem 3.3.10. Assume Bl-AD$_{\mathbb{R}}$ and DC. Let A be a subset of $\mathbb{R}^3$ and assume $\exists^{\mathbb{R}} A$ is a strict well-founded relation on a set of reals. Suppose A has a strong ∞-Borel code S and let γ be an ordinal less than Θ such that the tree of S is on γ. Then the length of $\exists^{\mathbb{R}} A$ is less than γ^+.

Proof. Let A, S, and γ be as in the assumptions. We show that the length of $\exists^{\mathbb{R}} A$ is less than γ^+. Fix a surjection $\pi \colon \mathbb{R} \to \gamma$. Let us start with the following lemma:

Lemma 3.3.11. There is a function $f \colon {}^{<\omega}\mathbb{R} \to \mathbb{R}$ such that if a is closed under f, then $S \restriction \pi[a]$ is an ∞-Borel code and $B_{S\restriction \pi[a]} \subseteq B_S$.

Note that the assertion of the above lemma is the strengthening of the definition of strong ∞-Borel codes.

Proof of Lemma 3.3.11. Let us consider the following game: Player I and II choose reals one by one and produce an ω-sequence x of reals. Setting $a = \mathrm{ran}(f)$, player I wins if $S \restriction \pi[a]$ is an ∞-Borel code and $B_{S\restriction\pi[a]} \subseteq B_S$. Since S is a strong ∞-Borel code, player I can defeat any strategy for player II because strategies can be seen as functions from ${}^{<\omega}\mathbb{R}$ to $\mathbb{R}$ by Claim 3.1.5. Since the payoff set of this game is range-invariant, by Lemma 3.1.4, this game is determined. Hence player I has a winning strategy and by Claim 3.1.5, there is a function f as desired. □ (Lemma 3.3.11)

We fix an f_0 satisfying the conclusion of Lemma 3.3.11 for the rest of this proof. Recall that U is the fine normal measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$ we fixed at the beginning of this section. Using π, we can transfer this measure to a fine normal measure on $\mathcal{P}_{\omega_1}(\gamma)$ as follows: Let $\pi_* \colon \mathcal{P}_{\omega_1}(\mathbb{R}) \to \mathcal{P}_{\omega_1}(\gamma)$ be such that $\pi_*(a) = \pi[a]$ for each $a \in \mathcal{P}_{\omega_1}(\mathbb{R})$. For $A \subseteq \mathcal{P}_{\omega_1}(\gamma)$, $A \in U_\pi$ if $\pi_*^{-1}(A) \in U$. It is easy to check that U_π is a fine normal measure on $\mathcal{P}_{\omega_1}(\gamma)$.

We now prove the key lemma for this theorem:

Lemma 3.3.12. Let G be $\mathrm{Coll}(\omega, \gamma)$-generic over V. Then in $V[G]$, there is an elementary embedding $j\colon \mathrm{L}(\mathbb{R}, S, f_0, \pi) \to \mathrm{L}\big(j(\mathbb{R}), j(S), j(f_0), j(\pi)\big)$ such that all the reals in $V[G]$ are contained in $\mathrm{L}\big(j(\mathbb{R}), j(S), j(f_0), j(\pi)\big)$.

Proof of Lemma 3.3.12. The argument is based on the result of Kechris and Woodin [47, Theorem 6.2]. We first introduce the notion of weakly meager sets. A subset B of ${}^{\omega}\gamma$ is *weakly meager* if there is an $X \in U_\pi$ such that $(\forall b \in X)\,{}^{\omega}b \cap B$ is meager in the space ${}^{\omega}b$. Since b is countable, the space ${}^{\omega}b$ is homeomorphic to the Baire space in most cases. Note that if B is a meager set in the space ${}^{\omega}\gamma$, then it is weakly meager. A subset B of ${}^{\omega}\gamma$ is *weakly comeager* if its complement is weakly meager. Let I be the set of weakly meager sets.

Sublemma 3.3.13.

1. The ideal I is a σ-ideal on ${}^{\omega}\gamma$.

2. For any $s \in {}^{<\omega}\gamma$, $[s]$ is not weakly meager.

3. If a subset B of ${}^{\omega}\gamma$ is not weakly meager, then there is an $s \in {}^{<\omega}\gamma$ such that $[s] \setminus B$ is weakly meager.

4. Let g be a function from ${}^{\omega}\gamma$ to On. Then for any B which is not weakly meager, there is a $B' \subseteq B$ which is not weakly meager such that for all x and y in B', if $\mathrm{ran}(x) = \mathrm{ran}(y)$, then $g(x) = g(y)$.

Proof. The first statement follows from the σ-completeness of U_π. The second statement follows from the fineness of U_π.

For the third statement, suppose B is not weakly meager. Then since U_π is an ultrafilter, there is an $X \in U_\pi$ such that $(\forall b \in X)\ {}^{\omega}b \cap B$ is not meager in ${}^{\omega}b$. We may assume that each b in X is infinite because the set of finite subsets of γ is measure zero with respect to U_π by the fineness of U_π. Take any b in X. Since the space ${}^{\omega}b$ is homeomorphic to the Baire space, the set ${}^{\omega}b \cap B$ has the Baire property in ${}^{\omega}b$. Hence there is an $s_b \in {}^{<\omega}b$ such that $[s_b] \setminus B$ is meager in ${}^{\omega}b$. By normality of U_π, there is a $Y \in U_\pi$ such that $Y \subseteq X$ and there is an $s \in {}^{<\omega}\gamma$ such that $s_b = s$ for any $b \in Y$. Hence $[s] \setminus B$ is weakly meager.

For the last statement, let g be such a function and B be not weakly meager. Then there is an $X \in U_\pi$ such that $\forall b \in X$, ${}^{\omega}b \cap B$ is not meager in ${}^{\omega}b$. Since ${}^{\omega}b \cap B$ has the Baire property in ${}^{\omega}b$, there is an $s_b \in {}^{<\omega}b$ such that $[s_b] \setminus B$ is meager in ${}^{\omega}b$. By normality of U_π, there are a $Y \subseteq X$ and $s_0 \in {}^{<\omega}\gamma$ such that $Y \in U_\pi$ and $s_b = s_0$ for every $b \in Y$. We use the following fact:

Fact 3.3.14 (Folklore)**.** Assume every set of reals has the Baire property. Then the meager ideal in the Baire space is closed under any wellordered union.

Take any $b \in Y$. Since $[s_0] \cap {}^\omega b$ is homeomorphic to the Baire space, we can apply Fact 3.3.14 to the space $[s_0] \cap {}^\omega b$ and hence there is an α_b such that $[s_0] \cap {}^\omega b \cap g^{-1}(\alpha_b)$ is not meager in $[s_0] \cap {}^\omega b$. Since the set $[s_0] \cap {}^\omega b \cap g^{-1}(\alpha_b)$ has the Baire property in $[s_0] \cap {}^\omega b$, there is an $s_b \in {}^{<\omega} b$ such that $s_b \supseteq s_0$ and $[s_b] \setminus g^{-1}(\alpha_b)$ is meager in ${}^\omega b$. By normality of U_π, there are a $Z \in U_\pi$ with $Z \subseteq Y$ and an $s_1 \supseteq s_0$ such that $[s_1] \setminus g^{-1}(\alpha_b)$ is meager in ${}^\omega b$ for each $b \in Z$. Then $B' = B \cap [s_1] \cap \{x \mid g(x) = \alpha_{\mathrm{ran}(x)}\}$ is as desired. □ (Sublemma 3.3.13)

Now we prove Lemma 3.3.12. Let G be $\mathrm{Coll}(\omega, \gamma)$-generic over V. Consider the Boolean algebra $\mathcal{P}({}^\omega\gamma)/I$. Then it is naturally forcing equivalent to $\mathrm{Coll}(\omega, \gamma)$: In fact, for $s \in {}^{<\omega}\gamma$, let $i(s) = [s]/I$. Then by the third item of Sublemma 3.3.13, i is a dense embedding from $\mathrm{Coll}(\omega, \gamma)$ to $\mathcal{P}({}^\omega\gamma)/I \setminus \{\mathbf{0}\}$. Define U' as follows: For a subset B of ${}^\omega\gamma$ in V, B is in U' if there is a $p \in G$ such that $[p] \setminus B$ is weakly meager. By the genericity of G and the third item of Sublemma 3.3.13, U' is an ultrafilter on $({}^\omega\gamma)^V$ and U' contains all the weakly comeager sets. Take an ultrapower $\mathrm{Ult}\big(\mathrm{L}(\mathbb{R}, S, f_0, \pi), U'\big) = \big({}^{({}^\omega\gamma)^V}\mathrm{L}(\mathbb{R}, S, f_0, \pi) \cap V\big)/U'$ and let j be the ultrapower map. (Note that we consider $\mathrm{L}(\mathbb{R}, S, f_0, \pi)$-valued functions in V which are not necessarily in $\mathrm{L}(\mathbb{R}, S, f_0, \pi)$.)

We show that j is the desired map. We first check Łoś's Theorem for this ultrapower. It is enough to show that for any $B \in U'$ and a function F from B to $\mathrm{L}(\mathbb{R}, S, f_0, \pi)$ such that all the values of F are nonempty, then there is a function f on B in V such that $f(x) \in F(x)$ for all x in B'. Since there is a surjection from $\mathbb{R} \times \mathrm{On}$ to $\mathrm{L}(\mathbb{R}, S, f_0, \pi)$, we may assume that the values of F are sets of reals. But then by uniformization for every relation on the reals by Theorem 1.14.9, we get the desired f.

Next, we check the well-foundedness of $\mathrm{Ult}\big(\mathrm{L}(\mathbb{R}, S, f_0, \pi), U'\big)$. By DC, we know that the ultrapower $\mathrm{Ult}(V, U_\pi)$ is wellfounded. Hence it suffices to show the following: For a function $f \colon \mathcal{P}_{\omega_1}(\gamma) \to \mathrm{On}$, let $g_f \colon {}^\omega\gamma \to \mathrm{On}$ be as follows: $g_f(x) = f\big(\mathrm{ran}(x)\big)$.

Sublemma 3.3.15. The map $[f]_{U_\pi} \mapsto [g_f]_{U'}$ is an isomorphism from $\big(({}^{\mathcal{P}_{\omega_1}(\gamma)}\mathrm{On} \cap V)/U_\pi, \in_{U_\pi}\big)$ to $\big(({}^{{}^\omega\gamma}\mathrm{On} \cap V)/U', \in_{U'}\big)$.

Proof of Sublemma 3.3.15. We first show that if $f_1 \in_{U_\pi} f_2$, then $g_{f_1} \in_{U'} g_{f_2}$. Since $f_1 \in_{U_\pi} f_2$, there is an $X \in U_\pi$ such that for any b in X, $f_1(b) \in f_2(b)$. Fix a b in X. Since the set $\{x \in {}^\omega b \mid \mathrm{ran}(x) = b\} \cap {}^\omega b$ is comeager in ${}^\omega b$, the set $\{x \in {}^\omega b \mid f_1\big(\mathrm{ran}(x)\big) \in f_2\big(\mathrm{ran}(x)\big)\}$ is comeager in ${}^\omega b$. Hence for every $b \in X$, the set $\{x \in {}^\omega b \mid g_{f_1}(x) \in g_{f_2}(x))\}$ is comeager in ${}^\omega b$ and the set $\{x \in {}^\omega\gamma \mid g_{f_1}(x) \in g_{f_2}(x)\}$ is weakly comeager and hence is in U'. Therefore, $g_{f_1} \in_{U'} g_{f_2}$. In the same way, one can prove that if $f_1 =_{U_\pi} f_2$, then $g_{f_1} =_{U'} g_{f_2}$.

Next, we show that the map is surjective. Take any function $g \colon {}^\omega\gamma \to \mathrm{On}$ in V. We show that there is an $f \colon \mathcal{P}_{\omega_1}(\gamma) \to \mathrm{On}$ in V such that $g_f =_{U'} g$. By the last item of Sublemma 3.3.13 and the genericity of G, there is an Y in U' such that if x and y are in Y with the same range, then $g(x) = g(y)$. Since Y is in U', there is a $p \in G$ such that $[p] \setminus Y$ is weakly meager, hence there is an X in U_π such that for all b in X, $([p] \setminus Y) \cap {}^\omega b$ is meager in ${}^\omega b$. This means that g is constant on a comeager set in $[p] \cap {}^\omega b$ for each $b \in X$. Let α_b be the constant value for each $b \in X$ and f be such that $f(b) = \alpha_b$ if b is in Y and $f(b) = 0$ otherwise. Then it is easy to check that $g_f =_{U'} g$, as desired. □ (Sublemma 3.3.15)

We have shown that j is elementary and we may assume that the target model of j is transitive. Then j is an elementary embedding from $\mathrm{L}(\mathbb{R}, S, f_0, \pi)$ to $\mathrm{L}\big(j(\mathbb{R}), j(S), j(f_0), j(\pi)\big)$. Let $M = \mathrm{L}\big(j(\mathbb{R}), j(S), j(f_0), j(\pi)\big)$. We finally check that all the reals in $V[G]$ are in M. Let x be a real in $V[G]$ and τ be a $\mathbb{P}$-name for a real in V such that $\tau^G = x$. We claim that $[f_\tau]_{U'} = x$, where f_τ is the Baire measurable function from $\mathrm{St}\big(\mathrm{Coll}(\omega, \gamma)\big)$ to the reals induced by τ from Lemma 2.1.2, which completes the proof.

Take any natural number n and set $m = x(n)$. We show that $[f_\tau]_{U'}(n) = m$. Since $x(n) = m$, there is a $p \in G$ such that $p \Vdash \tau(\check{n}) = \check{m}$. By the definition of f_τ, for any $x \in [p]$, $f_\tau(x)(n) = m\}$. Since p is in G, by the definition of U', the set $\{x \mid f_\tau(x)(n) = m$ is in U', as desired. □ (Lemma 3.3.12)

We now finish the proof of Theorem 3.3.10. Let us keep using M to denote $\mathrm{L}\big(j(\mathbb{R}), j(S), j(f_0), j(\pi)\big)$. We first claim that S and $j[S]$ are in M. Since γ is countable in $V[G]$, there is a real x coding S in $V[G]$. But by Lemma 3.3.12, such an x is in M. Hence S is also in M. Since γ is countable in $V[G]$, there is an $a \in \mathcal{P}_{\omega_1}(\mathbb{R})$ such that $\pi[a] = S$ and hence $j(\pi)[a] = j[S]$ in $V[G]$. But since $j(\pi) \in M$ and $a \in M$ by Lemma 3.3.12, $j[S] = j(\pi)[a]$ is also in M, as desired. By Lemma 3.3.11 and elementarity of j, the following is true in M: For any a closed under $j(f)$, $j(S) \restriction a$ is an ∞-Borel code and $B_{j(S)\restriction a} \subseteq B_{j(S)}$. Also, by elementarity of j, $\exists^{\mathbb{R}} B_{j(S)}$ is a well-founded relation on a set of reals in M. Set $a = j[S]$. Since a is closed under $j(f)$, in M, $j(S) \restriction a$ is an ∞-Borel code, $B_{j(S)\restriction a} \subseteq B_{j(S)}$, and $\exists^{\mathbb{R}} B_{j[S]}$ is also a wellfounded relation on a set of reals in M. Since $j[S]$ is countable in M, the relation $\exists^{\mathbb{R}} B_{j[S]}$ is $\mathbf{\Sigma}^1_1$ and hence by Kunen-Martin Theorem (see [66, 2G.2]), its rank is less than ω_1 in M which is the same as γ^+ in V. Finally, since S and $j[S]$ are equivalent as Borel codes, $\exists^{\mathbb{R}} B_S$ has length less than ω_1 in M and since M has more reals than V, $\big(\exists^{\mathbb{R}} B_S\big)^V \subseteq \big(\exists^{\mathbb{R}} B_S\big)^M$. Therefore, the length of $\big(\exists^{\mathbb{R}} B_S\big)^V$ is less than $\omega_1^M = (\gamma^+)^V$, as desired. □

Becker proved the following:

Theorem 3.3.16 (Becker)**.** Assume AD, DC, and the uniformization for every relation on the reals. Suppose that the conclusion of Theorem 3.3.10 holds, i.e., let A be a subset of $\mathbb{R}^3$ and assume $\exists^{\mathbb{R}} A$ is a well-founded relation on a set of reals. Suppose A has a strong ∞-Borel code S and let γ be an ordinal less than Θ such that the tree of S is on γ. Then the length of $\exists^{\mathbb{R}} A$ is less than γ^+. Then every set of reals is Suslin.

Proof. See [9]. □

We try to simulate Becker's argument, make a small conjecture, and reduce Conjecture 3.3.1 to the small conjecture.

As preparation, we prove a weak version of Moschovakis' Coding Lemma. Let us introduce some notions for that. Let A be a set of reals. Let $\mathrm{IND}(A)$ be the set of all $\mathrm{pos}\Sigma^1_n(A)$-inductive sets of reals for some natural number $n \geq 1$. For the definition of $\mathrm{pos}\Sigma^1_n(A)$-inductive sets, see [66, 7C]. All we need is as follows:

Fact 3.3.17. For any set of reals A, $\mathrm{IND}(A)$ is the smallest Spector pointclass containing A and closed under $\exists^{\mathbb{R}}$ and $\forall^{\mathbb{R}}$.

Proof. The argument is the same as [66, 7C.3]. □

Theorem 3.3.18 (Weak version of Moschovakis' Coding Lemma)**.** Assume Bl-AD. Let $<$ be a strict wellfounded relation on a set A of reals with rank function $\rho\colon A \to \gamma$ onto and let Γ be a Spector pointclass containing $<$ and closed under $\exists^{\mathbb{R}}$ and $\forall^{\mathbb{R}}$. Then for any subset S of γ, there is a set of reals $C \in \boldsymbol{\Gamma}$ such that $\rho[C] = S$.

By Fact 3.3.17, $\mathrm{IND}(<)$ satisfies the conditions for Γ.

Proof. The argument is based on Moschovakis' original argument [66, 7D.5].

Let S be a subset of γ. We show that for any $\alpha \leq \gamma$, there is a set of reals $C_\alpha \in \boldsymbol{\Gamma}$ with $\rho[C_\alpha] = S \cap \alpha$ by induction on α.

It is trivial when $\alpha = 0$ and it is also easy when α is a successor ordinal because $\boldsymbol{\Gamma}$ is a boldface pointclass. So assume α is a limit ordinal and the above claim holds for each $\xi < \alpha$. We show that there is a $C \in \boldsymbol{\Gamma}$ with $\rho[C] = S \cap \alpha$.

Since Γ is ω-parametrized and closed under recursive substitutions, we have $\{G^n \subseteq \mathbb{R} \times \mathbb{R}^n \mid n \geq 1\}$ given in Lemma 1.7.1. Let $G^2_a = \{x \in \mathbb{R} \mid (a, x) \in G^2\}$ for each real a. For a real a, we say G^2_a *codes a subset* S' *of* S if $G^2_a \subseteq A$ and $\rho[G^2_a] = S'$.

Let us consider the following game $\mathcal{G}_\alpha$: Player I and II choose 0 or 1 one by one and they produce reals a and b separately and respectively. Player II wins if either (G^2_a does not code $S \cap \xi$ for any $\xi < \alpha$) or (G^2_a codes $S \cap \xi$ for some $\xi < \alpha$ and G^2_b codes $S \cap \eta$ for some $\eta < \alpha$ with $\eta > \xi$). By Bl-AD, one of the players has an optimal strategy in this game.

Case 1: Player I has an optimal strategy σ in $\mathcal{G}_\alpha$.

For a real b, let τ_b be the mixed strategy for player II such that player II produces b with probability 1 no matter how player I plays. Since σ is optimal for player I, for each real b, for μ_{σ,τ_b}-measure one many reals a, G^2_a codes $S \cap \xi$ for some $\xi < \alpha$. Fix a real b. We use the following fact analogous to Fact 3.3.14:

Fact 3.3.19 (Folklore)**.** Let μ be a Borel probability measure on the Baire space and assume every set of reals is μ-measurable. Then the set of μ-null sets is closed under wellordered unions.

Since every set of reals is Lebesgue measurable by Theorem 1.14.8, every set of reals is μ_{σ,τ_b}-measurable. By Fact 3.3.19, there is a unique $\xi_b < \alpha$ such that for μ_{σ,τ_b}-positive measure many reals a, G^2_a codes $S \cap \xi_b$ and the set of reals a such that G^2_a codes $S \cap \xi$ for some $\xi < \xi_b$ is μ_{σ,τ_b}-measure zero. Let C be the following: A real x is in C if there is a real b such that for μ_{σ,τ_b}-positive measure many reals a, they code the same subset S' of γ, and no proper subsets of S' can be coded by μ_{σ,τ_b}-positive measure many reals, and $x \in G^2_a$ for some real a such that G^2_a codes S'. Since Γ is closed under $\exists^{\mathbb{R}}$ and $\forall^{\mathbb{R}}$, C is in $\Gamma(\sigma)$. By induction hypothesis, for any $\xi < \alpha$, there is a real b such that G^2_b codes $S \cap \xi$. Since σ is optimal, C codes $S \cap \alpha$, as desired.

Case 2: Player II has an optimal strategy τ in $\mathcal{G}_\alpha$.

Let $(a, x) \mapsto \{a\}(x)$ be the partial function from $\mathbb{R}\times\mathbb{R}$ to $\mathbb{R}$ which is universal for all the partial functions from $\mathbb{R}$ to itself that are $\boldsymbol{\Gamma}$-recursive on their domain. For reals a and w, define a set of reals $A_{a,w}$ as follows: a real x is in $A_{a,w}$ if there exists $z < w$ such that $\{a\}(z)$ is defined and $\big(\{a\}(z), x\big) \in G^2$. It is easy to see that $A_{a,w}$ is in $\boldsymbol{\Gamma}$. By Lemma 1.7.1, there is a Γ-recursive function $\pi\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ such that $A_{a,w} = G^2_{\pi(a,w)}$ for each a and w.

For each real a and w, define a set of reals $C_{a,w}$ as follows: A real x is in $C_{a,w}$ if for $\mu_{\sigma_{\pi(a,w)},\tau}$-positive measure many b, they code the same subset S' of γ, no proper subsets of S' can be coded by μ_{σ,τ_b}-positive measure many reals, and x is in G^2_b for some real b such that G^2_b codes S'. It is easy to see that $C_{a,w}$ is in $\boldsymbol{\Gamma}$. Hence by Lemma 1.7.1, there is a Γ-recursive function $\pi'\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ such that $C_{a,w} = G^2_{\pi'(a,w)}$ for each a and w.

Since the function $(a, w) \mapsto \pi'(a, w)$ is Γ-recursive in τ and total, by Recursion Theorem 1.7.3, we can find a fixed a^* such that for all w, $\{a^*\}(w) = \pi'(a^*, w)$. Let $g(w) = \{a^*\}(w)$.

Claim 3.3.20. For each $w \in A$ with $\rho(w) < \alpha$, there is some $\eta(w) < \alpha$ with $\rho(w) < \eta(w)$ such that $G^2_{g(w)}$ codes $S \cap \eta(w)$.

Proof of Claim 3.3.20. We show the claim by induction on w. Suppose it is done for all $x < w$. Then $A_{a^*,w}$ codes $S \cap \xi$ where $\xi = \sup\{\eta(x) \mid x < w\} \geq \rho(w)$. Since τ is optimal for II, $C_{a^*,w}$ codes $S \cap \eta$ for some $\eta > \xi$. Since $G^2_{g(w)} = C_{a^*,w}$, setting $\eta(w) = \eta$, $\eta(w) > \rho(w)$ and $G^2_{g(w)}$ codes $S \cap \eta(w)$. □ (Claim 3.3.20)

Let $C = \bigcup_{w\in A,\rho(w)<\alpha} G^2_{g(w)}$. Then by Claim 3.3.20, C codes $S \cap \alpha$ and C is in $\boldsymbol{\Gamma}$, as desired. □

We also need a weak version of Wadge's Lemma: Let A be a set of reals. For a natural number $n \geq 1$, a set of reals B is $\boldsymbol{\Sigma}^1_n$ *in* A if B is definable by a Σ^1_n formula in the structure $\mathcal{A}^2_A$ that is the second order structure with A as an unary predicate with a parameter x for some real x. A set of reals B is *projective in* A if B is $\boldsymbol{\Sigma}^1_n(A)$ for some $n \geq 1$.

Lemma 3.3.21 (Weak version of Wadge's Lemma)**.** Assume Bl-AD. Then for any two sets of reals A and B, either A is $\mathbf{\Sigma}^1_2$ in B or B is $\mathbf{\Sigma}^1_2$ in A.

Proof. Recall the Wadge game $G_{\mathrm{W}}(A, B)$ from § 1.15. By Bl-AD, one of the players has an optimal strategy in $G_{\mathrm{W}}(A, B)$. Assume player II has an optimal strategy τ in $G_{\mathrm{W}}(A, B)$. Then for any real x,

$$x \in A \iff \mu_{\sigma_x,\tau}\big(\{(x', y) \mid x' = x \text{ and } y \in B\}\big) = 1.$$

It is easy to see that the right hand side of the equivalence is $\mathbf{\Sigma}^1_2$ in B. If player I has an optimal strategy in $G_{\mathrm{W}}(A, B)$, then one can prove that B is $\mathbf{\Sigma}^1_2$ in A^{c} in the same way and hence B is $\mathbf{\Sigma}^1_2$ in A. □

For the rest of this section, we assume Bl-AD$_{\mathbb{R}}$ and DC. We fix a set of reals A and give a scenario to prove that A is Suslin. We fix a simple surjection ρ from the reals to $\{0, 1\}$, e.g., $x \mapsto x(0)$.

Claim 3.3.22. There is a sequence $\big((\Gamma_n, <_n, \gamma_n,) \mid n < \omega\big)$ such that for all n,

1. Γ_n is a Spector pointclass closed under $\exists^{\mathbb{R}}$ and $\forall^{\mathbb{R}}$, $\Gamma_n \subseteq \Gamma_{n+1}$, and $A \in \Gamma_0$,
2. every relation on the reals which is projective in a set in $\mathbf{\Gamma}_n$ can be uniformized by a function in $\mathbf{\Gamma}_{n+1}$,
3. $<_n$ is in $\mathbf{\Gamma}_n$ and a strict wellfounded relation on the reals with length γ_n and every set of reals which is projective in a set in $\mathbf{\Gamma}_n$ has a strong ∞-Borel code whose tree is on γ_{n+1}.

Proof of Claim 3.3.22. We construct them by induction on n. For $n = 0$, let Γ_0 be any Spector pointclass closed under $\exists^{\mathbb{R}}$ and $\forall^{\mathbb{R}}$ containing A which exists by Fact 3.3.17, and $<_0$ be any strict wellfounded relation on the reals in Γ_0. Then they satisfy all the items above.

Suppose we have constructed $(\Gamma_n, <_n, \gamma_n)$ with the above properties. We construct $\Gamma_{n+1}, <_{n+1}$, and γ_{n+1} . First note that there is a set B_n of reals which is not projective in any set in $\mathbf{\Gamma}_n$ by uniformization for every relation on the reals. Then by Lemma 3.3.21, every set projective in a set in $\mathbf{\Gamma}_n$ is $\mathbf{\Sigma}^1_2$ in B_n. Let H_n and H'_n be universal sets for $\mathbf{\Sigma}^1_2(B_n)$ sets of reals and $\mathbf{\Sigma}^1_2(B_n)$ subsets of $\mathbb{R}^2$, respectively. By uniformization, there is a function f_n uniformizing H'_n. By Theorem 3.3.7, there is a $\gamma < \Theta$ such that H_n has a strong ∞-code whose tree is on γ. Let $\gamma_{n+1} = \gamma$, $<_{n+1}$ be a strict wellfounded relation on the reals with length γ_{n+1}, and let Γ_{n+1} be a Spector pointclass closed under $\exists^{\mathbb{R}}$ and $\forall^{\mathbb{R}}$ containing $\Gamma_n \cup \{H_n, H'_n, f_n, <_{n+1}\}$. We show that they satisfy all the items above for $n+1$. The first item is trivial. The second item is easy by noting that if f_n uniformizes H'_n then $(f_n)_a$ uniformizes $(H'_n)_a$ for any real a. The third item follows from that if H_n has a strong ∞-code whose tree is on γ_{n+1}, then $(H_n)_a$ has a strong ∞-code whose tree is on $\gamma_{n+1}$for every real a. □ (Claim 3.3.22)

Note that in the proof of Claim 3.3.22, we have essentially used DC.

We fix $\big((\Gamma_n, <_n, \gamma_n) \mid n < \omega\big)$ as above and let $\Gamma^{\mathrm{I}}_n = \Gamma_{2n}, \Gamma^{\mathrm{II}}_n = \Gamma_{2n+1}, <^{\mathrm{I}}_n$ be induced by ρ, $<^{\mathrm{II}}_n = <_{2n+1}$, $\gamma^{\mathrm{I}}_n = \omega$ and $\gamma^{\mathrm{II}}_n = \gamma_{2n+1}$, Let $\rho^{\mathrm{I}}_n = \rho$ and ρ^{II}_n be the surjection between the reals onto ${}^n\gamma_{2n+1}$ induced by $<_{2n+1}$. Let π^{II}_n be the function $a \mapsto \rho^{\mathrm{II}}_n[G^n_a]$ where G^n is a universal set for $\mathbf{\Gamma}^{\mathrm{II}}_n$ sets of reals (we do not use π^{I}_n). Then by Theorem 3.3.18, π^{II}_n is a surjection from the reals onto ${}^n\gamma^{\mathrm{II}}_n$. Consider the following game $\hat{G}_A$: Player I plays 0 or 1 and player II plays reals one by one in turn and they produce a real z and a sequence $t \in {}^\omega\mathbb{R}$, respectively. Setting $T_n = \pi^{\mathrm{II}}_n\big(t(n)\big)$, player II wins if for all $n < m$, $T_{n+1}\restriction n \subseteq T_n$, $T_{n+1}\restriction n = T_m\restriction n$, and $z \in A \iff \bigcup_{n\in\omega} T_{n+1}\restriction n$ is illfounded, where $T_m\restriction n = \{s\restriction n \mid s \in T_m\}$. This is an integer-real game in the sense player I chooses integers and player II chooses reals.

We introduce an integer-integer game $\tilde{G}_A$ simulating the game $\hat{G}_A$. In the game $\tilde{G}_A$, players choose pairs of 0 or 1 one by one and produce a pair of reals (x_0, y_0) and (a_0, b_0) in ω rounds respectively. From (x_0, y_0) and (a_0, b_0), we "decode" a real z and an ω-sequence of reals t respectively as follows: For each pointclass Γ above, we fix a set U^Γ universal for relations in $\mathbf{\Gamma}$. Setting $F_0 = U^{\Gamma^{\mathrm{I}}_0}_{x_0}$, F_0 is a function from the reals to perfect sets of reals (or codes of them) (otherwise player I loses). Let $P_{x_0} = F(x_0)$. Then y_0 is an element of P_{x_0} (otherwise player I loses) and is identified with a triple (u_0, x_1, y_1) of reals by looking at a canonical homeomorphism between P_{x_0} and $\mathbb{R}^3$. Then setting $F_1 = U^{\Gamma^{\mathrm{I}}_1}_{x_1}$, F_1 is a function from the reals to perfect trees on 2 (or codes of trees) (otherwise player I loses). Let $P_{x_1} = F(x_1)$. Then y_1 is an element of P_{x_1} (otherwise player I loses) and is identified with a triple (u_1, x_2, y_2) of reals by looking at a canonical homeomorphism between P_{x_1} and $\mathbb{R}^3$. Continuing this process, one can unwrap (x_n, y_n) and obtain (u_n, x_{n+1}, y_{n+1}) for each n and get an ω-sequence $(u_n \mid n < \omega)$. Let $z(n) = \rho(u_n)$. In the same way, one can obtain an ω-sequence $(t_n \mid n < \omega)$ of reals from (a_0, b_0). Setting $T_n = \pi^{\mathrm{II}}_n\big(t(n)\big)$, player II wins if for all $n < m$, $T_{n+1}\restriction n \subseteq T_n$, $T_{n+1}\restriction n = T_m\restriction n$, and $z \in A \iff \bigcup_{n\in\omega} T_{n+1}\restriction n$ is illfounded.

Becker proved the following:

Lemma 3.3.23.

1. If player I has a winning strategy in the game $\tilde{G}_A$, then player I has a winning strategy σ in the game $\hat{G}_A$ such that σ is a countable union of sets in $\mathbf{\Gamma}^{\mathrm{II}}_n$ for some n as a set of reals.

2. If player II has a winning strategy in the game $\tilde{G}_A$, then player II has a winning strategy in the game $\hat{G}_A$.

Proof. See [9, Lemma A & B]. □

We show and conjecture the following: Let $B \subseteq {}^\omega\mathbb{R}$. A mixed strategy σ for player I is *weakly optimal in* B if for any $s \in \mathbb{R}^{\mathrm{Even}}$, the set $\{x \mid \sigma(s)(x) \neq 0\}$ is finite and for any ω-sequence y of reals, $\mu_{\sigma,\tau_y}(B) > 1/2$. One can introduce the weak optimality for mixed strategies for player II in the same way. Note that if player I has an optimal strategy in some payoff set, then player I has a weakly optimal strategy in the same payoff set. The same holds for player II.

Lemma 3.3.24. If player I has an optimal strategy in the game $\tilde{G}_A$, then player I has a weakly optimal strategy σ in the game $\hat{G}_A$ such that σ is a countable union of sets in $\mathbf{\Gamma}^{\mathrm{II}}_n$ for some n as a set of reals.

Conjecture 3.3.25. If player II has an optimal strategy in the game $\tilde{G}_A$, then player II has a weakly optimal strategy in the game $\hat{G}_A$.

Proof of Lemma 3.3.24. We first topologize the set $\mathrm{Prob}(\mathbb{R})$ of all Borel probabilities on the reals. Consider the following map $\iota\colon \mathrm{Prob}(\mathbb{R}) \to {}^{<\omega}2[0,1]$: Given a Borel probability μ on the reals, for any finite binary sequence s, $\iota(\mu)(s) = \mu([s])$. We topologize ${}^{<\omega}2[0,1]$ by the product topology where each coordinate $[0,1]$ is equipped with the relative topology of the real line and we identify $\mathrm{Prob}(\mathbb{R})$ with its image via ι and topologize it with the relative topology of ${}^{<\omega}2[0,1]$. Then the space $\mathrm{Prob}(\mathbb{R})$ is compact.

Claim 3.3.26. For any set B of reals, the map $\mu \mapsto \mu(B)$ is a continuous map from $\mathrm{Prob}(\mathbb{R})$ to $[0,1]$.

Proof of Claim 3.3.26. This is easy when B is closed or open. In general, it follows from the following equations: For any $\mu \in \mathrm{Prob}(\mathbb{R})$,

$$\begin{aligned}\mu(B) &= \sup\{\mu(C) \mid C \subseteq B \text{ and } C \text{ is closed}\}\\ &= \inf\{\mu(O) \mid O \supseteq B \text{ and } O \text{ is open}\}.\end{aligned}$$

□ (Claim 3.3.26)

Next, we introduce a complete metric d on $\mathrm{Prob}(\mathbb{R})$ compatible with the topology we consider. Let $(s_n \mid n \in \omega)$ be an injective enumeration of finite binary sequences. For μ and μ' in $\mathrm{Prob}(\mathbb{R})$, $d(\mu,\mu') = \sum_{n\in\omega} |\mu([s_n]) - \mu'([s_n])|/2^{n+1}$. Then d is a complete metric compatible with our topology. Since $\mathrm{Prob}(\mathbb{R})$ is compact, the map $\mu \mapsto \mu(A)$ is uniformly continuous with the metric d. Hence there is an $\epsilon > 0$ such that if $d(\mu,\mu') < \epsilon$, then $|\mu(A) - \mu'(A)| < 1/2$. Let us fix a sequence $(\epsilon_n \mid n \in \omega)$ of positive real numbers such that $\sum_{n\in\omega} \epsilon_n/2^{n+1} < \epsilon$. For any finite binary sequence s', let $n_{s'}$ be the natural number such that $s_{n'_s} = s'$.

Let σ be an optimal strategy for player I in the game $\tilde{G}_A$. We show that there is a weakly optimal strategy $\tilde{\sigma}$ for player I in the game $\hat{G}_A$. Given a real a. Consider the function $F^0_a\colon \mathbb{R} \to {}^2[0,1]$ as follows: Given a real b, $F^0_a(b)(i) = \mu_{\sigma,\tau_{(a,b)}}\big(\{(x_0,y_0) \mid \rho(u_0) = i\}\big)$ for $i = 0,1$, where y_0 is identified with (u_0,x_1,y_1) as discussed. Since every set of reals has the Baire property, F^0_a is continuous on a comeager set. Then there is a perfect set P of reals such that for any b and b'

in P, $|F^0_a(b)(i) - F^0_a(b')(i)| < \epsilon_{n_{\langle i \rangle}}$. Since the set $X_0 = \{(a, P) \mid (\forall b, b' \in P)\ (\forall i < 2)\ |F^0_a(b)(i) - F^0_a(b')(i)| < \epsilon_{n_{\langle i \rangle}}\}$ is projective in Γ^{I}_0, there is a real a_0 such that the function $f_0 = U^{\Gamma^{\mathrm{I}_0}}_{a_0}$ uniformizes X_0. Let $\tilde{\sigma}(\emptyset)(0) = \max\{F^0_{a_0}(b)(0) \mid b \in f_0(a_0)\}$ and $\tilde{\sigma}(\emptyset)(1) = 1 - \tilde{\sigma}(\emptyset)(0)$. We have specified $\tilde{\sigma}$ for the first round.

Next, suppose player II played a real t_0 for her first round. We decide the probability $\tilde{\sigma}(t_0)$ on 2. Let a be a real. Consider the function $F^1_a \colon \mathbb{R} \to {}^2[0,1]$ as follows: For a real b, $F^1_a(b)(i) = \mu_{\sigma, \tau_{(a_0,(t_0,a,b))}}\big(\{(x_0, y_0) \mid \rho(u_1) = i\}\big)$ for $i = 0, 1$, where $y_1 = (t_1, x_2, y_2)$ as discussed. Then the function F^1_a is continuous on a comeager set. Then there is a perfect set P of reals such that for any b and b' in P, $|F^1_a(b)(i) - F^1_a(b')(i)| < \min\{\epsilon_{n_{s^\frown \langle i \rangle}} \mid s \in {}^1 2\}$ for $i = 0, 1$. Since the set $X_1 = \{(a, P) \mid (\forall b, b' \in P)\ (\forall i < 2)\ |F^1_a(b)(i) - F^1_a(b')(i)| < \min\{\epsilon_{n_{s^\frown \langle i \rangle}} \mid s \in {}^1 2\}\}$ is projective in $\mathbf{\Gamma}^{\mathrm{I}_1}$, there is a real a_1 such that the function $f_1 = U^{\Gamma^{\mathrm{I}_1}}_{a_1}$ uniformizes X_1. Let $\tilde{\sigma}(t_0)(0) = \max\ \{F^1_{a_1}(b)(i) \mid b \in f_1(a_1)\}$ and $\tilde{\sigma}(t_0)(1) = 1 - \tilde{\sigma}(t_0)(0)$.

Continuing this process, we can specify $\tilde{\sigma}$ with the following property: For any natural number m and m-tuple reals $(t_0, \ldots, t_{m-1})$, $|\tilde{\sigma}(t_0, \ldots, t_{m-1})(i) - F^m_{a_m}(b)(i)| < \min\{\epsilon_{n_{s^\frown \langle i \rangle}} \mid s \in {}^m 2\}$ for each $b \in f_m(a_m)$. Also we have specified the reals a_m and b_m for all $m < \omega$.

We show that $\tilde{\sigma}$ is weakly optimal in the game $\hat{G}_A$. Let $(t_n \mid n < \omega)$ be an ω-sequence of reals such that the tree $\bigcup_{n<\omega} T_{n+1} \restriction n$ is illfounded. We show that the probability of the payoff set via $\mu_{\tilde{\sigma}, \tau_{(t_n \mid n < \omega)}}$ is greater than $1/2$. (The case when the tree is wellfounded is dealt with in the same way.)

First note that together with $(t_n \mid n < \omega)$, $\tilde{\sigma}$ produces a Borel probability μ on the reals such that for any finite binary sequence s, $\mu([s]) = \prod_{i<m} \tilde{\sigma}(t_j \mid j < i)\big(s(j)\big)$, where m is the length of s. Since the tree from $(t_n \mid n < \omega)$ is illfounded, it suffices to show that $\mu(A) > 1/2$. On the other hand, the measure $\mu_{\sigma, \tau_{(a_0, b_0)}}$ induces a Borel probability measure ν on the reals as follows: For a finite binary sequence s, $\nu([s]) = \mu_{\sigma, \tau_{(a_0, b_0)}}\big(\{(x_0, y_0) \mid (\forall i < m)\ \rho(t_i) = s(i)\}\big)$, where m is the length of s. By the property of $\tilde{\sigma}$, $d(\mu, \nu) < \epsilon$. Hence $|\mu(A) - \nu(A)| < 1/2$. Since σ is optimal for player I in the game $\tilde{G}_A$ and the tree from $(t_n \mid n < \omega)$ is illfounded, $\nu(A) = 1$. Therefore, $\mu(A) > 1/2$, as desired. □

From Lemma 3.3.24 together with Theorem 3.3.10, one can conclude the following:

Lemma 3.3.27. There is no optimal strategy for player I in the game $\tilde{G}_A$.

Proof. To derive a contradiction, suppose player I has an optimal strategy in the game $\tilde{G}_A$. Then by Lemma 3.3.24, player I has a weakly optimal strategy σ in the game $\hat{G}_A$ such that σ is in a countable union of sets in $\mathbf{\Gamma}^{\mathrm{I}}_n$ for some n as a set of reals.

Consider the following set:

$$X = \{(t, s) \in {}^\omega\mathbb{R} \times {}^{<\omega}\mathbb{R} \mid \mu_{\sigma, \tau_t}\big(\{(z, t') \mid t' = t \text{ and } z \in A\}\big) > 1/2 \text{ and } (\forall i < \mathrm{s})\ \big(|s(0)|_{<^{\mathrm{II}}_0}, \ldots, |s(i)|_{<^{\mathrm{II}}_i}\big) \in T_{i+1} \restriction i\},$$

where $|s(i)|_{<^{\mathrm{II}}_i}$ is the rank of $s(i)$ with respect to the wellfounded relation $<^{\mathrm{II}}_i$ and $T_i = \rho^{\mathrm{II}}_i\big(t(i)\big)$. For (t,s) and (t',s') in X, $(t,s) < (t',s')$ if t and t' code the same tree T and s codes a node in T extending a node coded by s'. Note that for any (t,s) in X, if T is the tree coded by t, T is wellfounded because σ is weakly optimal in the game $\hat{G}_A$. Hence $(X,<)$ is a strict wellfounded relation on X. Let $\gamma_\omega = \sup\{\gamma^{\mathrm{II}}_n \mid n \in \omega\}$. By DC, the cofinality of Θ is greater than ω. Hence $\gamma_\omega < \Theta$. Note that for any ordinal $\alpha < \gamma^+_\omega$, there is a wellfounded tree T coded by some real t as in the definition of X such that the length of T is α. Hence the length of $(X,<)$ is γ^+_ω.

Since σ is a countable union of sets in $\mathbf{\Gamma}^{\mathrm{I}}_n$ for some n as a set of reals, the set $<$ on X is in $\exists^{\mathbb{R}} \bigwedge^\omega \bigvee^\omega \bigcup_{n\in\omega} \mathbf{\Gamma}^{\mathrm{I}}_n$, i.e., it is a projection of a countable intersection of countable unions of sets in $\mathbf{\Gamma}^{\mathrm{I}}_n$ for some n. Since every set in $\mathbf{\Gamma}^{\mathrm{I}}_n$ has a strong ∞-Borel code whose tree is on γ^{II}_n for every n, every set in $\bigwedge^\omega \bigvee^\omega \bigcup_{n\in\omega} \mathbf{\Gamma}^{\mathrm{I}}_n$ has a strong ∞-Borel code whose tree is on γ^+_ω. By Theorem 3.3.10, the length of $<$ must be less than γ^+_ω, which is not possible because it was equal to γ^+_ω. Contradiction! □

We close this section by proving that Conjecture 3.3.25 implies Conjecture 3.3.1.

Proof of Conjecture 3.3.1 from Conjecture 3.3.25. By Lemma 3.3.27, player I does not have an optimal strategy in the game $\tilde{G}_A$. Hence by Bl-AD, player II has an optimal strategy in the game $\tilde{G}_A$. By Conjecture 3.3.25, player II has a weakly optimal strategy τ in the game $\hat{G}_A$. Note that τ can be seen as a real because each measure on the reals given by τ is with finite support by the weak optimality of τ. For each finite binary sequence s with length n, let $t_s = \{u \in {}^n\mathbb{R} \mid (\forall i < n)\ \tau\Big(\big(s{\restriction}i\big) * \big(u{\restriction}(i-1)\big)\Big)\big(s(i)\big) \neq 0\}$, where $(s{\restriction}i) * \big(u{\restriction}(i-1)\big)$ is the concatenation of $s{\restriction}i$ and $u{\restriction}(i-1)$ bit by bit. For each finite binary sequence s, we identify t_s with a set of n-tuples of natural numbers via a map π_s by using the isomorphisms between $(a, <_{\mathbb{R}})$ and $(n, \in)$ for a finite set of reals a and a natural number, where $<_{\mathbb{R}}$ is a standard total order on the reals. For any real x, $t_x = \bigcup_{n\in\omega} t_{x{\restriction}n}$ is a tree on natural numbers and $(\pi_s \mid s \in {}^{<\omega}\omega)$ induces a homeomorphism π_x between $[t_x]$ and $[\{t' \in {}^{<\omega}\mathbb{R} \mid \mu_{\sigma_x,\tau}([t']) \neq 0\}]$. Consider the following tree:

$$T = \{(s,t,u) \in \bigcup_{n\in\omega} ({}^n2 \times {}^n\omega \times {}^n\gamma_\omega) \mid\ t \in \pi_s(t_s) \text{ and } \big(\forall i < \mathrm{lh}(s)\big)\ u(i) = |x_i|_{<^{\mathrm{II}}_i}\},$$

where x_i is the $t(i)$th real of the set of successors of $(x_j \mid j < i)$ in $t_s{\restriction}i$. Then by the weak optimality of τ, the following holds: Setting $B = \{(x,y) \in \mathbb{R} \times {}^\omega\omega \mid (\exists f \in {}^\omega\gamma_\omega)\ (x,y,f) \in [T]\}$, for any real x,

$$\begin{aligned} x \in A &\iff \mu_{\sigma_x,\tau}\big(\pi_x[B_x]\big) > 1/2 \\ &\iff (\exists T' : \text{ a tree on } 2)\ [T'] \subseteq B_x \text{ and } \mu_{\sigma_x,\tau}\big(\pi_x\big[[T']\big]\big) > 1/2. \end{aligned}$$

Since B is Suslin, the set $\{(x, T') \mid [T'] \subseteq B_x\}$ is also Suslin. Hence A is Suslin, as desired.

We have shown that every set of reals is Suslin. Then by Theorem 1.14.5, AD holds. Now by Theorem 3.3.2 and Theorem 1.14.9, $\mathrm{AD}_{\mathbb{R}}$ holds. □

3.4 Toward the equiconsistency between $\mathrm{AD}_{\mathbb{R}}$ and Bl-$\mathrm{AD}_{\mathbb{R}}$

In the last section, we have discussed the possibility of the equivalence between $\mathrm{AD}_{\mathbb{R}}$ and Bl-$\mathrm{AD}_{\mathbb{R}}$ under AD+DC. Solovay proved the following:

Theorem 3.4.1 (Solovay)**.** If we have $\mathrm{AD}_{\mathbb{R}}$ and DC, then we can prove the consistency of $\mathrm{AD}_{\mathbb{R}}$. Hence the consistency of $\mathrm{AD}_{\mathbb{R}}$+DC is strictly stronger than that of $\mathrm{AD}_{\mathbb{R}}$.

Proof. See [78]. □

Hence assuming DC to see the equivalence between $\mathrm{AD}_{\mathbb{R}}$ and Bl-$\mathrm{AD}_{\mathbb{R}}$ is not optimal. One can ask whether they are equivalent without DC. So far we do not have any scenario to answer this question. Instead, one could ask the equiconsistency between $\mathrm{AD}_{\mathbb{R}}$ and Bl-$\mathrm{AD}_{\mathbb{R}}$. In this section, we discuss the following conjecture:

Conjecture 3.4.2. $\mathrm{AD}_{\mathbb{R}}$ and Bl-$\mathrm{AD}_{\mathbb{R}}$ are equiconsistent.

Woodin conjectured the following:

Conjecture 3.4.3 (Woodin)**.** Assume the following:

1. The principle $\mathrm{DC}_{\mathbb{R}}$ holds,
2. Every Suslin & co-Suslin set of reals is determined, and
3. There is a fine normal measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$.

Then either there is an inner model of $\mathrm{AD}_{\mathbb{R}}$ or there is an inner model M of AD^+ such that M contains all the reals and $\Theta^M = \Theta^V$.

We show that Conjecture 3.4.3 implies Conjecture 3.4.2.

Proof of Conjecture 3.4.2 from Conjecture 3.4.3. First note that the assumptions in Conjecture 3.4.3 hold if we assume Bl-$\mathrm{AD}_{\mathbb{R}}$. Hence by Conjecture 3.4.3, there is an inner model of $\mathrm{AD}_{\mathbb{R}}$ or there is an inner model M of AD^+ such that M contains all the reals and $\Theta^M = \Theta^V$. If there is an inner model of $\mathrm{AD}_{\mathbb{R}}$, then we are done. Hence we assume that there is an inner model M of AD^+ such that M contains all the reals and $\Theta^M = \Theta^V$.

We show that $\mathrm{AD}_{\mathbb{R}}$ holds in V. First we claim that M contains all the sets of reals. Suppose not. Then there is a set of reals A which is not in M. Then by Lemma 3.3.21, every set of reals in M is $\boldsymbol{\Sigma}^1_2(A)$. Then Θ^M must be less than Θ^V because one can code all the prewellorderings by reals using A in V, which contradicts the condition of M. Hence every set of reals is in M. Since we have uniformization for every relation on the reals in V, it is also true in M. We use the following fact:

Fact 3.4.4. Assume AD^+. Then the following are equivalent:

1. The axiom $\mathrm{AD}_{\mathbb{R}}$ holds, and
2. Every relation on the reals can be uniformized.

By Fact 3.4.4, since every relation on the reals can be uniformized in M, M satisfies $\mathrm{AD}_{\mathbb{R}}$. Since $\mathcal{P}(\mathbb{R}) \cap M = \mathcal{P}(\mathbb{R})$, $\mathrm{AD}_{\mathbb{R}}$ holds in V, as desired. □

3.5 Questions

We close this chapter by raising questions.

The equivalence between $\mathrm{AD}_{\mathbb{R}}$ and Bl-$\mathrm{AD}_{\mathbb{R}}$ under ZF+DC As discussed in § 3.3, it is enough to show Conjecture 3.3.25 to prove the equivalence between $\mathrm{AD}_{\mathbb{R}}$ and Bl-$\mathrm{AD}_{\mathbb{R}}$. In the proof of Lemma 3.3.24, in each round, we shrank the reals into a perfect set sufficiently enough so that the strategy we constructed gives us a measure on the reals which is close enough to the measure derived from a given optimal strategy and the opponent's moves, which yields the weak optimality of the strategy. But the same argument does not work for Conjecture 3.3.25 because one cannot shrink the reals into a perfect set to get the continuity of a given function from $\mathbb{R}$ to ${}^{\mathbb{R}}[0,1]$. Nonetheless, we can proceed the similar argument to the coded space $\prod_{n\in\omega} \mathcal{P}({}^n\gamma^{\mathrm{II}}_n)$ from the space ${}^\omega\mathbb{R}$ by using the fact that the meager ideal on the reals is closed under any wellordered union and deciding the probability on the space $\prod_{n\in\omega} \mathcal{P}({}^n\gamma^{\mathrm{II}}_n)$ is enough to determine the probability of the payoff set. Although the details of the argument seem complicated and it is not yet done, we believe it is possible and it is not so difficult.

The equiconsistency between $\mathrm{AD}_{\mathbb{R}}$ and Bl-$\mathrm{AD}_{\mathbb{R}}$ By the argument in § 3.4, it is enough to show Conjecture 3.4.3 to prove the equiconsistency between $\mathrm{AD}_{\mathbb{R}}$ and Bl-$\mathrm{AD}_{\mathbb{R}}$. It seems possible because Bl-$\mathrm{AD}_{\mathbb{R}}$ gives us a generic embedding similar to the one obtained by an ω_1-dense ideal on ω_1, CH and "The restriction of the generic embedding given by the ideal to On is definable in V". Let us see more details. If one takes a generic filter G of the partial order ${}^{<\omega}\mathbb{R}$ ordered by reverse inclusion, then this filter generates an ultrafilter U' extending the dual filter of the meager ideal in ${}^{\omega}\mathbb{R}$ in the same way as we have seen in Lemma 3.3.12. If one takes the generic ultrapower of V via U' and lets M be the target model of the ultrapower embedding j, then Łoś's Theorem holds for M if the cofinality of Θ is ω, the reals in V belongs to M as an element (as a real), M contains all the reals in $V[G]$ and $j\restriction\mathrm{On}$ is definable in V (the last statement is ensured by the existence of a fine normal measure U in Theorem 3.1.2, in fact, the ultrapower embedding via U' agrees with j on ordinals as we have seen). In general, M is not well-founded (in the case $\mathrm{cof}(\Theta) = \omega$). But Θ is always in the well-founded part of M. Together with the determinacy of Suslin & co-Suslin sets of reals, this seems enough to proceed the Core Model Induction up to $\Theta = \Theta_\omega$, i.e., a minimal model of $\mathrm{AD}_{\mathbb{R}}$.

A stronger weak Moschovakis' Lemma As we have seen in § 3.3, a weak version of Moschovakis's Lemma 3.3.18 holds assuming Bl-AD. One can ask whether one can prove a stronger version of Moschovakis's Lemma formulated in [66, 7D.5] from Bl-AD. If this is possible, it would be plausible to show that the set of strong partition cardinals is unbounded in Θ and that every Suslin set of reals is determined from Bl-AD.