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# Chapter 1

# Introduction

*Games* have been used in many areas of mathematics, especially mathematical logic as well as theoretical computer science. It was the Polish school of mathematicians who connected infinite games with analysis (e.g., Lebesgue measurability) and topology (e.g., the Baire property) and obtained many results. In this thesis, we give several results on games in set theory and logic or obtained by application of games.

## 1.1 Outline

In this thesis, we discuss the following topics. All the definitions and the notions given in this outline can be found in the later sections of this chapter.

In Chapter 2, entitled 'Games and Regularity Properties', we characterize almost all the known regularity properties for sets of reals via the Baire property for some topological spaces and use Banach-Mazur games to prove the general equivalence theorems between regularity properties, forcing absoluteness, and the transcendence properties over some canonical inner models. With the help of these equivalence results, we answer some open questions from set theory of the reals. Almost all the results in this chapter are contained in my paper [35].

In Chapter 3, entitled 'Games themselves', we compare the Axiom of Real Determinacy ($\mathsf{AD}_{\mathbb{R}}$) and the Axiom of Real Blackwell Determinacy ($\mathsf{Bl}\text{-}\mathsf{AD}_{\mathbb{R}}$). We show that the consistency strength of $\mathsf{Bl}\text{-}\mathsf{AD}_{\mathbb{R}}$ is strictly greater than that of the Axiom of Determinacy (AD) in § 3.1 and that $\mathsf{Bl}\text{-}\mathsf{AD}_{\mathbb{R}}$ implies almost all the known regularity properties for every set of reals in § 3.2. In § 3.3, we discuss the possibility of the equivalence between $\mathsf{AD}_{\mathbb{R}}$ and $\mathsf{Bl}\text{-}\mathsf{AD}_{\mathbb{R}}$ under ZF+DC. In § 3.4, we discuss the possibility of the equiconsistency between $\mathsf{AD}_{\mathbb{R}}$ and $\mathsf{Bl}\text{-}\mathsf{AD}_{\mathbb{R}}$. The results in § 3.1 are joint work with David de Kloet and Benedikt Löwe [36]. The results in § 3.2, § 3.3, and § 3.4 are joint work with Hugh Woodin.

In Chapter 4, entitled 'Games and Large Cardinals', we work on the connection between the determinacy of Gale-Stewart games and large cardinals. We investigate the upper bound of the consistency strength of the existence of alternating chains with length $\omega$, which are essential objects to prove projective determinacy from Woodin cardinals. This is joint work with Ralf Schindler.

In Chapter 5, entitled 'Wadge reducibility for the real line', we study the Wadge reducibility for the real line. Unlike the Wadge order for the Baire space, the Wadge order for the real line cannot be characterized by infinite games. We show that the Wadge Lemma for the real line fails and the Wadge order for the real line is ill-founded and we investigate more properties of the Wadge order for the real line. All the results in this chapter are joint work with Philipp Schlicht and Hisao Tanaka.

In Chapter 6, entitled 'Fixed-Point Logic and Product Closure', we define a product construction of an event model and a Kripke model and discuss the product closure of modal fixed point logics. We show that PDL, the modal $\mu$-calculus, and the continuous fragment of the modal $\mu$-calculus are product closed. Most of the results are joint work with Johan van Benthem [12].

In the remaining sections of this chapter, we give the mathematical background and results used in this thesis.

## 1.2 Choice principles

We use the following two types of choice principles in this thesis.

The first one is the family of the Choice Principles $\mathrm{AC}_X(Y)$. Let $X, Y$ be nonempty sets. The *Choice Principle* $\mathrm{AC}_X(Y)$ states that for any family $\{A_x \mid x \in X\}$ of nonempty subsets of $Y$, there is a function $f\colon X \to Y$ such that $f(x) \in A_x$ for every $x \in X$. The *Axiom of Choice* AC states that $\mathrm{AC}_X(Y)$ holds for all nonempty sets $X$ and $Y$. The following is easy to see:

**Remark 1.2.1.** Let $X, Y_1, Y_2$ be nonempty sets and suppose there is a surjection from $Y_2$ to $Y_1$. Then $\mathrm{AC}_X(Y_2)$ implies $\mathrm{AC}_X(Y_1)$.

Furthermore, we consider the Dependent Choice Principles $\mathrm{DC}_X$. Let $X$ be a nonempty set. The *Dependent Choice Principle* $\mathrm{DC}_X$ states that for any relation $R$ on $X$ (i.e., $R \subseteq X \times X$), if $(\forall x \in X)\ (\exists y \in X)\ (x, y) \in R$, then there is a function $f\colon \omega \to X$ such that $\big(f(n), f(n+1)\big) \in R$ for every $n \in \omega$. The *Axiom of Dependent Choice* DC states that $\mathrm{DC}_X$ holds for every nonempty set $X$.

Throughout this thesis, we work in $\mathrm{ZF} + \mathrm{AC}_\omega(\mathbb{R})$, where ZF is the axiom system of Zermelo-Fraenkel set theory. When we need more choice principles, we explicitly mention them (especially at the beginning of each chapter).

## 1.3 Trees

*Trees* are basic objects in mathematical logic, especially descriptive set theory and recursion theory. We fix some notation and introduce definitions about trees.

If $f$ is a function from $X$ to $Y$ and $A$ is a subset of $X$, then $f \restriction A$ denotes the *restriction of $f$ to $A$*, i.e., $f \restriction A = \{(a, f(a)) \mid a \in A\}$. For a relation $R$ between $X$ and $Y$ (i.e., $R \subseteq X \times Y$), $\mathrm{dom}(R) = \{x \in X \mid (\exists y)\ (x, y) \in R\}$ and $\mathrm{ran}(R) = \{y \in Y \mid (\exists x)\ (x, y) \in R\}$.

Given a nonempty set $X$, ${}^{<\omega}X$ denotes the set of all finite sequences of elements in $X$ and a nonempty subset $T$ of ${}^{<\omega}X$ is a *tree* on $X$ if it is closed under initial segments, i.e., if $s$ is in $T$ and $t$ is a subsequence of $s$ (i.e., $t = s \restriction n$ for some $n$), then $t$ is in $T$. For a finite sequence $t$ of elements in $X$, $\mathrm{lh}(t)$ denotes the length of $t$.

By *nodes*, we mean elements of trees. For a tree $T$ on $X$, two nodes $s, t$ of $T$ are *incompatible* (denoted by $s \perp t$) if there is an $n$ in $\mathrm{dom}(s) \cap \mathrm{dom}(t)$ such that $s(n) \neq t(n)$. Note that $s, t$ are incompatible if and only if there is no $u$ in $T$ such that $s, t \subseteq u$. For a node $t$ of $T$ and an element $x$ of $X$, $t^\frown\langle x \rangle$ denotes the one-step extension of $t$ with $x$, i.e., $t^\frown\langle x \rangle = t \cup \{(\mathrm{lh}(t), x)\}$.

A tree $T$ on $X$ is called *perfect* if for any node $s$ in $T$, there are two nodes $t_1, t_2$ of $T$ such that $s \subseteq t_i$ for $i = 1, 2$ and $t_1 \perp t_2$. For a tree $T$ on $X$, $[T]$ denotes the set of all infinite paths through $[T]$, i.e., $[T] = \{x \in {}^{\omega}X \mid (\forall n \in \omega)\ x \restriction n \in T\}$. For a tree $T$ on $X$ and a node $t$ in $T$, $t$ is called *splitting in $T$* if there are $x$ and $y$ in $X$ such that $x \neq y$ and both $t^\frown\langle x \rangle$ and $t^\frown\langle y \rangle$ are in $T$. For a tree $T$, the *stem of $T$* (denoted by $\mathrm{stem}(T)$) is the minimal splitting node in $T$ if it exists.

If $T$ is a tree on $X$ and $X$ is of the form $Y \times Z$, then we often identify a node $s$ of $T$ with the pair $(t_1, t_2)$ where $t_i = (s(0)_i, \ldots s(n-1)_i)$ for $i = 1, 2$, $n = \mathrm{dom}(s)$, and $s(j) = (s(j)_1, s(j)_2)$ for $j < n$. The same identification will be applied in case $X$ is of the form $Y_1 \times \ldots \times Y_m$ for a finite natural number $m \geq 1$.

## 1.4 General topology

*Topological spaces* are fundamental objects in mathematics. Throughout this thesis, we assume the basic theory of topological spaces which can be found in, e.g., [49]. We mainly use the following three types of topological spaces:

**The spaces ${}^{\omega}X$.** Let $X$ be a nonempty set. The set ${}^{\omega}X$ is the set of all $\omega$-sequences of elements in $X$ and we topologize it via the product topology where $X$ is always regarded as the discrete space. Hence for each finite sequence $s$ of elements in $X$, the set $[s] = \{x \in {}^{\omega}X \mid x \supseteq s\}$ (i.e., the set of all $\omega$-sequences of elements in $X$ extending $s$) is a basic open set in this topology and any open set is a union of basic open sets of this form.

Our main interest is when $X = 2$ (i.e., $\{0, 1\}$) or $\omega$. The space ${}^{\omega}2$ is called the *Cantor space* and the space ${}^{\omega}\omega$ is called the *Baire space*.

One of the special properties of this type of topological spaces is that closed sets have a *tree representation*: A subset $A$ of ${}^{\omega}X$ is closed if and only if there is a tree $T$ on $X$ such that $A = [T]$. Also, there is a one-to-one correspondence between perfect subsets of ${}^{\omega}X$ and perfect trees on $X$, where a subset $A$ of ${}^{\omega}X$ is *perfect* if it is closed and it has no isolated points: A subset $A$ of ${}^{\omega}X$ is perfect if and only if there is a perfect tree $T$ on $X$ such that $A = [T]$.

A subset $A$ of the Baire space or the Cantor space has the *perfect set property* if either it is countable or it contains a perfect set. It is easy to see that for any perfect set $C$, there is a bijection between $C$ and the Cantor space. Hence sets $A$ with the perfect set property satisfy Cantor's Continuum Hypothesis (CH), i.e., either $A$ is countable or there is a bijection between $A$ and the Cantor space. For this reason, it is interesting to see what kind of sets have the perfect set property. We discuss this in § 1.11.

**The spaces** $\mathrm{St}(\mathbb{P})$**.** *Stone spaces* are fundamental topological spaces not only in mathematical logic but also in general mathematics. We give basic definitions and the basic properties of Stone spaces of partial orders in our context.

Let $\mathbb{P}$ and $\mathbb{Q}$ be partial orders. A map $i\colon \mathbb{P} \to \mathbb{Q}$ is called a *dense embedding* if it satisfies the following:

- $i$ preserves the order, i.e., if $p_1 \leq p_2$ in $\mathbb{P}$, then $i(p_1) \leq i(p_2)$ in $\mathbb{Q}$,
- $i$ preserves the incompatibility, i.e., given two elements $p_1, p_2$ of $\mathbb{P}$, if there is no $p$ in $\mathbb{P}$ with $p \leq p_1$ and $p \leq p_2$, then there is no $q$ in $\mathbb{Q}$ with $q \leq i(p_1)$ and $q \leq i(p_2)$, and
- the image of $i$ is dense, i.e., for any $q$ in $\mathbb{Q}$ there is a $p$ in $\mathbb{P}$ such that $i(p) \leq q$.

Dense embeddings are important in forcings in the sense that if there is a dense embedding from $\mathbb{P}$ to $\mathbb{Q}$, then forcing with $\mathbb{P}$ and forcing with $\mathbb{Q}$ are essentially the same. (See § 1.9 about forcing.)

It is well known that if $\mathbb{P}$ is a partial order, then there is a complete Boolean algebra $\mathbb{B}$ and a dense embedding $i$ from $\mathbb{P}$ to $\mathbb{B}$. Moreover, the pair $(\mathbb{B}, i)$ is unique up to isomorphism in the sense that if there are two such pairs $(\mathbb{B}_1, i_1)$ and $(\mathbb{B}_2, i_2)$, then there is an isomorphism $i$ between $\mathbb{B}_1$ and $\mathbb{B}_2$ as complete Boolean algebras such that $i \circ i_1 = i_2$. We call such a pair $(\mathbb{B}, i)$ a *completion of* $\mathbb{P}$ and write $(B_{\mathbb{P}}, i_{\mathbb{P}})$ for $(\mathbb{B}, i)$.

Let $\mathbb{P}$ be a partial order. A nonempty subset $u$ of $\mathbb{P}$ is a *filter* on $\mathbb{P}$ if it is upward closed (i.e., if $p \in u$ and $p \leq q$, then $q$ is also in $u$) and any two elements of $u$ have an extension in $u$ (i.e., if $p$ and $q$ are in $u$, then there is an $r$ in $u$ such that $r \leq p$ and $r \leq q$). A filter $u$ on $\mathbb{P}$ is an *ultrafilter* if $u \neq \mathbb{P}$ and $u$ is maximal with respect to inclusions (i.e., if $v$ is a filter containing $u$, then $v = u$ or $v = \mathbb{P}$).

We now define Stone spaces of partial orders. Given a partial order $\mathbb{P}$, the set $\mathrm{St}(\mathbb{P})$ is the collection of all ultrafilters on $B_{\mathbb{P}}$. For each $b \in B_{\mathbb{P}}$, we define the set $O_b = \{u \in \mathrm{St}(\mathbb{P}) \mid u \ni b\}$ and the *Stone space* of $\mathbb{P}$ (also denoted by $\mathrm{St}(\mathbb{P})$) is the topology on the set $\mathrm{St}(\mathbb{P})$ generated by the set $\{O_b \mid b \in B_{\mathbb{P}}\}$.

For example, if $\mathbb{P}$ is the pair $({}^{<\omega}\omega, \supseteq)$, i.e., the set of all finite sequences of natural numbers ordered by reverse inclusion, then the Stone space of $\mathbb{P}$ is homeomorphic to the Cantor space ${}^{\omega}2$.

There are two advantages for taking ultrafilters on $B_{\mathbb{P}}$ rather than on $\mathbb{P}$ itself as a definition of the Stone space of $\mathbb{P}$: The first one is that it has several nice properties as topological spaces (e.g., it is a compact Hausdorff zero-dimensional space). The second is that it does not depend on the representation of $\mathbb{P}$, i.e., if there is a dense embedding from $\mathbb{P}$ to $\mathbb{Q}$, then $\mathrm{St}(\mathbb{P})$ and $\mathrm{St}(\mathbb{Q})$ are homeomorphic.

**The real line $\mathbb{R}$.** We use $\mathbb{R}$ to denote the set of all real numbers except in Chapter 2, where we use it for Mathias forcing (we use $\mathbb{R}$ for Mathias forcing because it is closely related to the Ramsey property). As usual, the topology of the real line is generated by open intervals $(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$ for $a, b \in \mathbb{R}$.

## 1.5 Borel sets, projective sets, and definability in the second-order arithmetics

Let $X$ be a topological space. Starting from open sets (or closed sets), we form the two hierarchies of sets of subsets of $X$. One is called the *Borel hierarchy* and the other is called the *projective hierarchy*:

**Definition 1.5.1.** Let $X$ be a topological space. The *Borel hierarchy* of $X$ $\left(\mathbf{\Sigma}^0_\xi, \mathbf{\Pi}^0_\xi, \mathbf{\Delta}^0_\xi \mid 1 \leq \xi < \omega_1\right)$ is defined as follows:

**Case 1**: $\xi = 1$.

By $\mathbf{\Sigma}^0_1$, we mean the set of all open subsets of $X$ and $\mathbf{\Pi}^0_1$ denotes the set of all closed subsets of $X$. The set of all clopen subsets of $X$ is denoted by $\mathbf{\Delta}^0_1$.

**Case 2**: $\xi > 1$.

By $\mathbf{\Sigma}^0_\xi$, we mean the set of all countable unions of sets in $\bigcup_{\eta<\xi} \mathbf{\Pi}^0_\eta$, and $\mathbf{\Pi}^0_\xi$ denotes the set of all countable intersections of sets in $\bigcup_{\eta<\xi} \mathbf{\Sigma}^0_\eta$. The intersection of $\mathbf{\Sigma}^0_\xi$ and $\mathbf{\Pi}^0_\xi$ is denoted by $\mathbf{\Delta}^0_\xi$.

Elements of $\mathbf{\Sigma}^0_\xi, \mathbf{\Pi}^0_\xi$ and $\mathbf{\Delta}^0_\xi$ are called *$\mathbf{\Sigma}^0_\xi$ sets, $\mathbf{\Pi}^0_\xi$ sets and $\mathbf{\Delta}^0_\xi$ sets* respectively. We set $\mathbf{B} = \bigcup_{\xi<\omega_1} \mathbf{\Sigma}^0_\xi$ and elements of $\mathbf{B}$ are called *Borel sets*.

It is immediate that $\mathbf{\Delta}^0_\xi = \mathbf{\Sigma}^0_\xi \cap \mathbf{\Pi}^0_\xi$ for each $1 \leq \xi < \omega_1$. By induction on $\xi$, it is easy to show that $\mathbf{\Pi}^0_\xi = \{X \setminus A \mid A \in \mathbf{\Sigma}^0_\xi\}$ for each $1 \leq \xi < \omega_1$. With the help of $\mathrm{AC}_\omega(\mathbb{R})$, it is easy to show that $\omega_1$ is a regular cardinal and hence that the set of all the Borel sets $\mathbf{B}$ is closed under complements and countable unions and it contains the empty set. Such a family of subsets of $X$ is called a *$\sigma$-algebra on $X$*. Note that the set of all the Borel subsets of $X$ is the smallest $\sigma$-algebra on $X$ containing all the open sets.

**Theorem 1.5.2** (Lebesgue)**.** Let $X$ be the Cantor space, the Baire space, or the real line. Then the following strict inclusions hold for each $1 \leq \xi < \omega_1$:

$$\begin{tikzcd}
 & \mathbf{\Sigma}^0_\xi \arrow[dr, phantom, sloped, "\subsetneq"] & \\
\mathbf{\Delta}^0_\xi \arrow[ur, phantom, sloped, "\subsetneq"] \arrow[dr, phantom, sloped, "\subsetneq"] & & \mathbf{\Delta}^0_{\xi+1} \\
 & \mathbf{\Pi}^0_\xi \arrow[ur, phantom, sloped, "\subsetneq"] &
\end{tikzcd}$$

*Proof.* See, e.g., [45, Theorem 22.4]. □

**Definition 1.5.3.** Let $X$ be a topological space. The *projective hierarchy* of $X$ $\big(\mathbf{\Sigma}^1_n, \mathbf{\Pi}^1_n, \mathbf{\Delta}^1_n \mid 1 \leq n < \omega\big)$ is defined as follows:

**Case 1**: $n = 1$.

By $\mathbf{\Sigma}^1_1$, we mean the set of all subsets $A$ of $X$ such that there is a closed subset $C$ of $X \times {}^\omega\omega$ such that $A$ is the first projection of $C$, i.e., $A = \mathrm{dom}(C)$, where $X \times {}^\omega\omega$ is topologized as the product space of $X$ and ${}^\omega\omega$. The set of all subsets $A$ of $X$ whose complements are in $\mathbf{\Sigma}^1_1$ is denoted $\mathbf{\Pi}^1_1$. The intersection between $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$ is denoted $\mathbf{\Delta}^1_1$.

**Case 2**: $n > 1$.

By $\mathbf{\Sigma}^1_n$, we mean the set of all subsets $A$ of $X$ such that there is a subset $C$ of $X \times {}^\omega\omega$ in $\mathbf{\Pi}^1_{n-1}$ such that $A$ is the first projection of $C$. The set of all subsets $A$ of $X$ whose complements are in $\mathbf{\Sigma}^1_n$ is denoted $\mathbf{\Pi}^1_n$. The intersection between $\mathbf{\Sigma}^1_n$ and $\mathbf{\Pi}^1_n$ is denoted $\mathbf{\Delta}^1_n$.

Elements of $\mathbf{\Sigma}^1_n, \mathbf{\Pi}^1_n$, and $\mathbf{\Delta}^1_n$ are called $\mathbf{\Sigma}^1_n$ *sets,* $\mathbf{\Pi}^1_n$ *sets and* $\mathbf{\Delta}^1_n$ *sets* respectively. Sets in $\mathbf{\Sigma}^1_n$ for some $n$ are called *projective sets.*

Elements of $\mathbf{\Sigma}^1_1$ are also called *analytic sets*, and *co-analytic sets* are the same as $\mathbf{\Pi}^1_1$ sets. It is immediate that $\mathbf{\Delta}^1_n = \mathbf{\Sigma}^1_n \cap \mathbf{\Pi}^1_n$ for each $n$ and that $\mathbf{\Pi}^1_n = \{X \setminus A \mid A \in \mathbf{\Sigma}^1_n\}$ for each $n$.

**Theorem 1.5.4** (Suslin)**.** Let $X$ be the Cantor space, the Baire space, or the real line. Then $\mathbf{B} = \mathbf{\Delta}^1_1$.

*Proof.* See, e.g., [45, Theorem 14.11]. □

**Theorem 1.5.5** (Lusin)**.** Let $X$ be the Cantor space, the Baire space, or the real line. Then the following strict inclusions hold for each $1 \leq n < \omega$:

$$\begin{tikzcd}
 & \mathbf{\Sigma}^1_n \arrow[dr, phantom, sloped, "\subsetneq"] & \\
\mathbf{\Delta}^1_n \arrow[ur, phantom, sloped, "\subsetneq"] \arrow[dr, phantom, sloped, "\subsetneq"] & & \mathbf{\Delta}^1_{n+1} \\
 & \mathbf{\Pi}^1_n \arrow[ur, phantom, sloped, "\subsetneq"] &
\end{tikzcd}$$

In particular, every Borel set is a $\mathbf{\Sigma}^1_1$ set and there is a $\mathbf{\Sigma}^1_1$ set which is not a Borel set.[1]

*Proof.* See, e.g., [45, Theorem 37.7]. $\square$

Definable sets in the second-order arithmetic are related to $\mathbf{\Sigma}^0_n$ sets, $\mathbf{\Pi}^0_n$ sets, $\mathbf{\Sigma}^1_n$ sets, and $\mathbf{\Pi}^1_n$ sets in the Baire space. By the *second-order structure*, we mean the two-sorted structure $\mathcal{A}^2 = (\omega, {}^{\omega}\omega, \mathrm{app}, +, \cdot, =, 0, 1)$, where app is the function from ${}^{\omega}\omega \times \omega$ to $\omega$ such that $\mathrm{app}(x, n) = x(n)$ and $+, \cdot, =$ are summation, multiplication, and equality on the natural numbers. By $\Sigma^0_n$*-formulas*, we mean the formulas in the language of the second-order structure of the form

$$(\exists^0 x_1)\ (\forall^0 x_2)\ \dots (Q_n x_n)\ \phi,$$

where $\exists^0, \forall^0$ are the existential quantifier and the universal quantifier for natural numbers respectively, $Q_n$ is $\forall^0$ if $n$ is even and $\exists^0$ if $n$ is odd, $x_i$ $(1 \leq i \leq n)$ are variables for natural numbers, and $\phi$ is a quantifier-free formula. By $\Pi^0_n$*-formulas*, we mean the formulas in the language of the second-order structure of the form

$$(\forall^0 x_1)\ (\exists^0 x_2)\ \dots (Q_n x_n)\ \phi,$$

where $Q_n$ is $\exists^0$ if $n$ is even and $\forall^0$ if $n$ is odd, $x_i$ $(1 \leq i \leq n)$ are variables for natural numbers, and $\phi$ is a quantifier-free formula. By *arithmetical formulas*, we mean $\Sigma^0_n$-formulas or $\Pi^0_n$-formulas for some natural number $n$. By $\Sigma^1_n$*-formulas*, we mean the formulas in the language of the second-order structure of the form

$$(\exists^1 x_1)\ (\forall^1 x_2)\ \dots (Q_n x_n)\ \phi,$$

where $\exists^1, \forall^1$ are the universal quantifier and the existential quantifier for elements in the Baire space respectively, $Q_n$ is $\forall^1$ if $n$ is even and $\exists^1$ if $n$ is odd, $x_i$ $(1 \leq i \leq n)$ are variables for elements in the Baire space, and $\phi$ is an arithmetical formula. By $\Pi^1_n$*-formulas*, we mean the formulas in the language of the second-order structure of the form

$$(\forall^1 x_1)\ (\exists^1 x_2)\ \dots (Q_n x_n)\ \phi,$$

where $Q_n$ is $\exists^1$ if $n$ is even and $\forall^1$ if $n$ is odd, $x_i$ $(1 \leq i \leq n)$ are variables for elements in the Baire space, and $\phi$ is an arithmetical formula. Let $n$ be a natural number with $n \geq 1$, $A$ be a subset of the Baire space and $a$ be an element of the Baire space. We say $A$ is a $\Sigma^0_n(a)$ *set* if there is a $\Sigma^0_n$-formula $\phi$ such that $A = \{x \mid \mathcal{A}^2 \vDash \phi(x, a)\}$. One can define $\Pi^0_n(a)$ *sets*, $\Sigma^1_n(a)$ *sets*, and $\Pi^1_n(a)$ *sets* in the same way. We also use $\Sigma^0_n(a), \Pi^0_n(a), \Sigma^1_n(a)$, and $\Pi^1_n(a)$ to denote the set of all $\Sigma^0_n(a)$ sets, $\Pi^0_n(a)$ sets, $\Sigma^1_n(a)$ sets, and $\Pi^1_n(a)$ sets respectively.

[1] The last statement is due to Suslin [82].

**Theorem 1.5.6.** Let $n$ be a natural number with $n \geq 1$. Then

$$\mathbf{\Sigma}^0_n = \bigcup_{a \in {}^\omega\omega} \Sigma^0_n(a), \quad \mathbf{\Pi}^0_n = \bigcup_{a \in {}^\omega\omega} \Pi^0_n(a),$$
$$\mathbf{\Sigma}^1_n = \bigcup_{a \in {}^\omega\omega} \Sigma^1_n(a), \quad \mathbf{\Pi}^1_n = \bigcup_{a \in {}^\omega\omega} \Pi^1_n(a).$$

*Proof.* See, e.g., [66, 8B.5 & 8B.15]. □

## 1.6 Gale-Stewart games

In this section, we introduce *Gale-Stewart games*, which are infinite games with perfect information.

In 1913, Ernst Zermelo [93] investigated finite games with perfect information as a formalization of the game of chess and proved the determinacy of these games. In 1953, Gale and Stewart [27] developed the general theory of infinite games, so-called *Gale-Stewart games*, which are two-player zero-sum infinite games with perfect information. The theory of Gale-Stewart games has been investigated by many logicians and now it is one of the main topics in set theory and it has connections with other topics in set theory as well as model theory and computer science.

Let us start with the definition of Gale-Stewart games.

**Definition 1.6.1** (Gale-Stewart games)**.** Let $X$ be a nonempty set and $A$ be a subset of ${}^\omega X$. The *Gale-Stewart game* $G_X(A)$ is played by two players, player I and player II. They play elements of $X$ $\omega$-many times in turn, i.e., player I starts with choosing an element $x_0$ of $X$, then player II responds with $x_1 \in X$, then player I moves with $x_2 \in X$ and player II chooses $x_3$ and so on. After $\omega$ moves, they have produced an $\omega$-sequence $x = \langle x_n \mid n \in \omega \rangle \in {}^\omega X$. Player I wins if $x$ is in $A$ and player II wins if $x$ is not in $A$.

This game is an infinite zero-sum game with perfect information because one of the players always wins and when one player wins, the other loses, and because both players know what they have previously played and they can decide the next move considering their previous moves.

We are interested in whether one of the players has a winning strategy in the game $G_X(A)$, i.e., whether one of the players has a way to play this game such that no matter her opponent moves, she will always win this game. Let us formulate the notion of winning strategies.

**Definition 1.6.2.** A *strategy for player I* is a function $\sigma\colon X^{\mathrm{Even}} \to X$, where $X^{\mathrm{Even}}$ is the set of finite sequences of elements in $X$ with even length. A *strategy for player II* is a function $\tau\colon X^{\mathrm{Odd}} \to X$, where $X^{\mathrm{Odd}}$ is the set of finite sequences of elements in $X$ with odd length. Given a strategy $\sigma$ for player I and a strategy

$\tau$ for player II, one can produce the run $\sigma * \tau$ of the game $G_X(A)$ according to $\sigma$ and $\tau$ by letting player I follow $\sigma$ and player II follow $\tau$, more precisely, the run $\sigma * \tau$ of the game $G_X(A)$ is a unique $\omega$-sequence of elements in $X$ with the following property: For any natural number $n$,

$$(\sigma * \tau)(n) = \nu_{\sigma,\tau}\big((\sigma * \tau)\restriction n\big),$$

where for a finite sequence $s$ of elements in $X$, $\nu_{\sigma,\tau}(s) = \sigma(s)$ if the length of $s$ is even and $\nu_{\sigma,\tau}(s) = \tau(s)$ if the length of $s$ is odd. A strategy $\sigma$ for player I is *winning in the game* $G_X(A)$ if for any strategy $\tau$ for player II, $\sigma * \tau$ is in $A$. A strategy $\tau$ for player II is *winning in the game* $G_X(A)$ if for any strategy $\sigma$ for player I, $\sigma * \tau$ is not in $A$. A subset $A$ of ${}^{\omega}X$ is *determined* if one of the players has a winning strategy in the game $G_X(A)$.

Hence we are interested in what kind of sets $A$ are determined. Let us list some results concerning this question. Recall from § 1.4 that the topology of ${}^{\omega}X$ is given by the product topology where each coordinate (i.e., $X$) is seen as the discrete space.

**Theorem 1.6.3** (Gale and Stewart)**.** (AC) Let $X$ be a nonempty set.

1. Any closed subset of ${}^{\omega}X$ and any open subset of ${}^{\omega}X$ are determined. If $X$ is well-ordered, one does not need AC.

2. There is a subset of ${}^{\omega}\omega$ which is not determined.

*Proof.* See, e.g., [37, Lemma 33.1, Lemma 33.17]. □

**Theorem 1.6.4** (Martin)**.** (AC) Let $X$ be a nonempty set. Then every Borel subset of ${}^{\omega}X$ is determined.

*Proof.* See, e.g., [45, Theorem 20.5]. □

**Theorem 1.6.5** (Davis; Gödel and Addison)**.** ZFC cannot prove that every $\mathbf{\Sigma}^1_1$ subset of the Baire space is determined.

*Proof.* The statement follows from the combination of the following two results: The first is that if every $\mathbf{\Sigma}^1_1$ subset of the Baire space is determined, then every $\mathbf{\Pi}^1_1$ subset of the Baire space has the perfect set property and the second one is that ZFC cannot prove that every $\mathbf{\Pi}^1_1$ subset of the Baire space has the perfect set property. The first result is due to Davis [23] and the second result was announced by Gödel [28] and the details of the proof given by Addison [1]. For the proofs, see, e.g., [66, p. 224 & 225] and [37, Corollary 25.37]. □

Gale-Stewart games are general enough that they can be used to simulate several kinds of infinite games in mathematics (e.g., Banach-Mazur games; for the definition of Banach-Mazur games, see § 1.8). In particular, the determinacy of Gale-Stewart games implies that of several other kinds of games. From this, one can prove several properties of sets of reals assuming the determinacy of Gale-Stewart games such as Lebesgue measurability, the Baire property (for the definition, see § 1.8), and the perfect set property (for the definition, see § 1.4).

Mycielski and Steinhaus [68] introduced the *Axiom of Determinacy* (AD), which states that every subset of the Baire space is determined, and investigated the consequences of this axiom. They proved that AD implies that every set of reals is Lebesgue measurable and that every subset of the Baire space has the Baire property and the perfect set property where each of these statements contradicts the Axiom of Choice. Beside such properties for sets of reals, AD supplies a beautiful structural theory. Moreover, models of AD have been investigated for a long time and they are essential for the research on inner models with large cardinals (for inner models, see § 1.11). In this way, the study of AD has been one of the central topics in set theory despite the fact that AD contradicts AC.

One can define $\mathrm{AD}_X$ for a nonempty set $X$ as follows: Every subset of ${}^{\omega}X$ is determined. Let us list some known observations on $\mathrm{AD}_X$:

**Proposition 1.6.6.**

1. Let $X, Y$ be nonempty sets and assume that there is an injection from $X$ to $Y$. Then $\mathrm{AD}_Y$ implies $\mathrm{AD}_X$. In particular, $\mathrm{AD}_{\mathbb{R}}$ implies $\mathrm{AD}_{\omega} = \mathrm{AD}$.

2. The axioms $\mathrm{AD}_{\omega_1}$ and $\mathrm{AD}_{\mathcal{P}(\mathbb{R})}$ are inconsistent.

*Proof.* The first statement is a folklore and it is easy. For the second statement, the inconsistency of $\mathrm{AD}_{\omega_1}$ is due to Mycielski [67] and that of $\mathrm{AD}_{\mathcal{P}(\mathbb{R})}$ follows from the inconsistency of $\mathrm{AD}_{\omega_1}$, the fact that there is an injection from $\omega_1$ into $\mathcal{P}(\mathbb{R})$, and the first item of this proposition. (One can send a countable ordinal $\alpha$ to the set of all reals $x$ such that $(\omega, x)$ is isomorphic to $(\alpha, \in)$ and this is an injection from $\omega_1$ into $\mathcal{P}(\mathbb{R})$.) $\square$

We investigate AD and $\mathrm{AD}_{\mathbb{R}}$ further in Chapter 3.

## 1.7 Pointclasses, parametrization, and Recursion Theorem

As with Borel sets, one often looks at the properties of a class of sets of reals rather those of a set of reals. Such classes are called *pointclasses*. In this section, we introduce basic properties for pointclasses. When we are talking about "reals", we mean elements of the Cantor space ${}^{\omega}2$ and we use $\mathbb{R}$ to denote the Cantor space.

A *pointclass* is the union of sets of subsets of $\omega^m \times \mathbb{R}^n$ for natural numbers $m \geq 0, n \geq 1$. If $\boldsymbol{\Gamma}$ is a pointclass, $\boldsymbol{\Gamma}$ is called a *boldface pointclass* if it is closed under continuous preimages, i.e., for natural numbers $m_1, m_2 \geq 0$ and $n_1, n_2 \geq 1$, a continuous function $f\colon \omega^{m_1} \times \mathbb{R}^{n_1} \to \omega^{m_2} \times \mathbb{R}^{n_2}$, and a subset $A \in \boldsymbol{\Gamma}$ of $\omega^{m_2} \times \mathbb{R}^{n_2}$, $f^{-1}(A)$ is also in $\boldsymbol{\Gamma}$. Closure under recursive preimages is similarly defined with recursive functions.

A pointclass $\Gamma$ is *$\omega$-parametrized* if for all natural numbers $m \geq 0$ and $n \geq 1$ there is a subset $G^{m,n}$ of $\omega^{m+1} \times \mathbb{R}^n$ in $\Gamma$ such that for any subset $A$ of $\omega^m \times \mathbb{R}^n$ in $\Gamma$, there is a natural number $e$ such that $A = G^{m,n}_e = \{(x, y) \mid (e, x, y) \in G^{m,n}\}$. The following lemma is useful: Let $\Gamma$ be a pointclass and $x$ be a real. Then the pointclass $\Gamma(x)$ is the set of all sets $A$ such that there is a set $C \in \Gamma$ such that $A = C_x$ where $C_x = \{y \in \mathbb{R} \mid (y, x) \in C\}$. Set $\boldsymbol{\Gamma} = \bigcup_{x \in \mathbb{R}} \Gamma(x)$.

**Lemma 1.7.1.** Suppose $\Gamma$ is an $\omega$-parametrized pointclass which is closed under recursive preimages. Then for each natural number $n \geq 1$, there is a set $G^n \subseteq \mathbb{R} \times \mathbb{R}^n$ in $\Gamma$ such that the following hold:

1. For each $n \geq 1$, $G^n$ is universal for subsets of $\mathbb{R}^n$ in $\boldsymbol{\Gamma}$, i.e., for any subset $A \in \boldsymbol{\Gamma}$, there is a real $x$ such that $A = G^n_x$,
2. For $A \subseteq \mathbb{R}^n$ in $\Gamma$, there is a recursive real $x$ such that $A = G^n_x$, and
3. For all natural numbers $n, m \geq 1$, there is a recursive function $S^{n,m}\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ such that for any real $a$, $x \in \mathbb{R}^n$, and $y \in \mathbb{R}^m$, $G^{m+n}(a, x, y) \iff G^m(S^{n,m}(a, x), y)$.

*Proof.* See [66, 3H.1]. □

We fix some notions for projections. For natural numbers $m \geq 0$ and $n \geq 1$ and a subset $A$ of $\omega \times \omega^m \times \mathbb{R}^n$, let $\exists^\omega A = \{(x, y) \in \omega^m \times \mathbb{R}^n \mid (\exists e \in \omega)\ (e, x, y) \in A\}$ and $\forall^\omega A = \{(x, y) \in \omega^m \times \mathbb{R}^n \mid (\forall e \in \omega)\ (e, x, y) \in A\}$. The sets $\exists^{\mathbb{R}} A$ and $\forall^{\mathbb{R}} A$ are defined in the similar way. A pointclass $\Gamma$ is *closed under* $\exists^\omega$ if for any $A$ in $\Gamma$, $\exists^\omega A$ is in $\Gamma$. Closure under $\forall^\omega$, $\exists^{\mathbb{R}}$, and $\forall^{\mathbb{R}}$ is defined in the similar way.

**Definition 1.7.2.** A pointclass $\Gamma$ is a *Spector pointclass* if it satisfies the following:

1. It contains all the $\Sigma^0_1$ sets and it is closed under recursive substitutions, finite intersections and unions, $\exists^\omega$, and $\forall^\omega$,
2. It is $\omega$-parametrized,
3. It has the substitution property, and
4. It has the prewellordering property.

For the definition the substitution property and the basic theory of $\Gamma$-recursive functions, see [66, 3D & 3G]. For the definition of prewellordering property, see [66, 4B]. Typical examples of Spector pointclasses are $\Pi^1_1$ and $\Sigma^1_2$. Assuming the determinacy of all the projective sets, one can prove that $\Pi^1_{2n+1}$ and $\Sigma^1_{2n+2}$ are also Spector pointclasses for each natural number $n$.

We use the following general form of Kleene's Recursion Theorem for Spector pointclasses in Chapter 3:

**Theorem 1.7.3** (Recursion Theorem)**.** (Kleene) Let $\Gamma$ be a Spector pointclass and suppose $f\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is $\mathbf{\Gamma}$-recursive on its domain. Then there exists a fixed real $a^*$ such that for all reals $x$, if $f(a^*, x)$ is defined, then $f(a^*, x) = \{a^*\}(x)$, where $\{a^*\}$ is the $\mathbf{\Gamma}$-recursive function on its domain coded by $a^*$.

*Proof.* See [66, 7A.2]. □

## 1.8 The Baire property and Banach-Mazur games

In this section, we introduce the *Baire property* and *Banach-Mazur games* and discuss the connection between them. In the Scottish Café "Kawiarnia Szzkocka" in Lwów, Polish mathematicians in the Lwów School of Mathematics would often meet and spend their afternoons discussing mathematical problems in 1920s and 1930s. Their discussions produced the famous book so-called "the Scottish book of problems". In this book (see [63]), Mazur described infinite games nowadays called *Banach-Mazur games* and conjectured their connection to the Baire property. The conjecture was confirmed by Banach in 1935 and the statement was generalized to arbitrary topological space by Oxtoby [69] in 1957.

We start with the definition of the Baire property:

**Definition 1.8.1.** Let $X$ be a topological space and $A$ be a subset of $X$.

1. We say $A$ is *nowhere dense* if the interior of the closure of $A$ is empty.
2. We say $A$ is *meager* if it is a countable union of nowhere dense sets.
3. We say $A$ is *comeager* if the complement of $A$ is meager.
4. We say $A$ has the *Baire property* if there is an open subset $U$ of $X$ such that the symmetric difference between $A$ and $U$ (i.e., $\big((A\setminus U)\cup(U\setminus A)\big)$, denoted by $A\triangle U$) is meager.

Nowhere dense sets and meager sets are small in the sense of topology, e.g., on the Baire space, the Cantor space and the real line, any singleton is nowhere dense and any countable set is meager. Sets with the Baire property can be approximated by open sets modulo such small sets. But if some nonempty open set was meager, this property would not make sense. To avoid that problem, we introduce a property for topological spaces: A topological space $X$ is called a *Baire space* if any nonempty open subset of $X$ is not meager.[2] All the topological spaces that appear in this thesis will be Baire spaces.

If $X$ is a topological space, many subsets of $X$ have the Baire property in $X$: Trivially every open set has the Baire property, also every closed set has the Baire property (if we take $U$ to be the interior of the given closed set $A$, then symmetric difference between $A$ and $U$ is $A\setminus U$ and it is nowhere dense by the

[2] Note that being *a* Baire space is different from being *the* Baire space ${}^\omega\omega$. Being a Baire space is a property for topological spaces while the Baire space is one particular topological space.

definition of interior, hence meager). From this, we can conclude that the set of subsets of $X$ with the Baire property is closed under complements. Moreover, since the set of meager sets is closed under countable unions, the set of subsets with Baire property is also closed under countable unions and hence every Borel subset of $X$ has the Baire property.

It is natural to ask whether the converse is true for the Baire space, i.e., if a subset of the Baire space has the Baire property, then is it Borel? The answer is 'No'. In 1923, Lusin and Sierpinski [57] proved that every $\mathbf{\Sigma}^1_1$ set of reals has the Baire property and there is a $\mathbf{\Sigma}^1_1$ set of reals which is not Borel by Theorem 1.5.5. So one could ask, "How far can we go?" Actually, in the constructible universe L, there is a $\mathbf{\Delta}^1_2$ set of reals without the Baire property.[3] On the other hand, starting with a model of ZFC, one can construct a model of ZFC extending the given model such that every $\mathbf{\Delta}^1_2$ set has the Baire property. Hence the statement that every $\mathbf{\Delta}^1_2$ set of reals has the Baire property is *independent from* ZFC. Then one could naturally ask the following: When is it true and when is it not? We discuss this question in Chapter 2. Next, we introduce Banach-Mazur games, which characterize meagerness of topological spaces:

**Definition 1.8.2** (Banach-Mazur games)**.** Let $X$ be a topological space and $A$ be a subset of $X$. The *Banach-Mazur game* of $A$, denoted by $G^{**}(A)$ (or $G^{**}(A, X)$), is defined as follows: Players I and II choose alternatively nonempty open sets $V_n$ ($n \in \omega$) with $V_0 \supseteq V_1 \supseteq V_2 \supseteq V_3 \supseteq \ldots$ in $\omega$ moves,

$$\begin{array}{lccccc} \mathrm{I} & V_0 & & V_2 & & \ldots \\ \mathrm{II} & & V_1 & & V_3 & \ldots \end{array}$$

Player II wins this run of the game if $\bigcap_{n\in\omega} V_n \cap A = \emptyset$.

The notions of strategies and winning strategies are defined in the same way as for Gale-Stewart games in § 1.6.

**Theorem 1.8.3** (Banach and Mazur, Oxtoby)**.** Let $X$ be a topological space and $A$ be a subset of $X$. Then $A$ is meager if and only if player II has a winning strategy in the game $G^{**}(A)$.

*Proof.* See, e.g., [45, Theorem 8.33]. □

One can characterize when a subset $A$ of $X$ has the Baire property in $X$ in terms of Banach-Mazur games: Let $U_A$ be the union of all open sets $U$ in $X$ such that $U \setminus A$ is meager in $X$. Then $A$ has the Baire property if and only if the

[3] Although Gödel [28] announced the similar result for Lebesgue measurability in 1938 and seemed to know about this result at that time, it seems to have been first made explicit in [67, p. 216] (cf. [44, p. 169]).

set $A \setminus U_A$ is meager, hence if and only if player II has a winning strategy in the Banach-Mazur game $G^{**}(A \setminus U_A)$.

It is natural to ask whether one could characterize when player I has a winning strategy in Banach-Mazur games in terms of topology. The answer is: If $X$ is a completely metrizable topological space, then player I has a winning strategy in $G^{**}(A)$ if and only if there is a nonempty open subset $U$ of $X$ such that $U \setminus A$ is meager in $U$, where $U$ is equipped with the relative topology of $X$ in this case. (But this characterization is not true if $X$ is a general topological space. For the proof, see, e.g., [45, Theorem 8.33].) It follows from this result that player I cannot have a winning strategy in the Banach-Mazur game $G^{**}(A \setminus U_A)$. Hence we can conclude that a subset $A$ of $X$ has the Baire property if and only if the Banach-Mazur game $G^{**}(A \setminus U_A)$ is *determined*, i.e., either player I or II has a winning strategy in this game. Now we have reduced the problem of the Baire property of a given set to the problem of determinacy of Banach-Mazur games. This is how the Polish school of mathematics found out the following: Assume every Banach-Mazur game in the Baire space is determined, then every set of reals has the Baire property.

We also use a variant of Banach-Mazur games so-called the *unfolded Banach-Mazur games*:

**Definition 1.8.4** (The unfolded Banach-Mazur games)**.** Let $X$ be a topological space and $F$ be a subset of $X \times {}^{\omega}\omega$. Define the *unfolded Banach-Mazur game* $G^{**}_u(F)$ (or $G^{**}_u(F, X)$) as follows:

$$\begin{array}{llllll} \mathrm{I} & V_0, y_0 & & V_2, y_1 & & \dots \\ \mathrm{II} & & V_1 & & V_3 & \dots \end{array}$$

Players I and II choose $V_0, V_1, \dots$ as in the Banach-Mazur game, but additionally I plays a natural number $y_n$ in her $n$th move. Let $y = \langle y_n \mid n \in \omega \rangle$. Player II wins if $\left(\bigcap_{n\in\omega} V_n \times \{y\}\right) \cap F = \emptyset$.

We have the same kind of characterization theorem as Banach-Mazur games:

**Theorem 1.8.5** (Folklore)**.** Let $X$ be a topological space and $F$ be a subset of $X \times {}^{\omega}\omega$. Let $A = \exists^{\mathbb{R}} F$.

1. If $A$ is meager in $X$, then player II has a winning strategy in the game $G^{**}_u(F)$.

2. Suppose that $F$ is of the form $(f \times \mathrm{id})^{-1}(C)$, where $f \colon X \to {}^{\omega}\omega$ is a continuous function, $f \times \mathrm{id} \colon X \times {}^{\omega}\omega \to {}^{\omega}\omega \times {}^{\omega}\omega$ is defined by $(f \times \mathrm{id})\,(x, y) = (f(x), y)$, and $C$ is a subset of ${}^{\omega}\omega \times {}^{\omega}\omega$. Then if player II has a winning strategy in the game $G^{**}_u(F)$, then $A$ is meager in $X$.

*Proof.* We show the first item. By Theorem 1.8.3, if $A$ is meager, then player II has a winning strategy $\tau$ in the game $G^{**}(A, X)$. But $\tau$ can be viewed as a winning strategy for player II in the game $G^{**}_u(F)$ by ignoring I's moves of $y_n$s.

Next we show the second item. The point is that given a winning strategy $\tau$ for player II in the game $G^{**}_u(F)$, she can modify $\tau$ so that in her $n$th move, she can decide the $n$th digit of $f(x)$ by the continuity of $f$. The rest of the argument is the same as in [45, Theorem 21.5]. $\square$

Using Theorem 1.8.5, one can characterize when player I has a winning strategy in the game $G^{**}_u(F)$ as well: Player I has a winning strategy in the game $G^{**}_u(F)$ if and only if there is a nonempty open set $U$ in $X$ such that $U \setminus A$ is meager in $U$. As before, it follows from this fact that a subset $A$ of $X$ has the Baire property if and only if the game $G^{**}_u(F')$ is determined, where $F'$ is a subset of $X \times {}^\omega\omega$ with $\exists^{\mathbb{R}} F' = A \setminus U_A$ and $U_A$ is the same as in the paragraphs after Theorem 1.8.3.

The advantage of the unfolded Banach-Mazur games over Banach-Mazur games is that one can reduce the complexity of the payoff sets (from $A$ to $F$ in the above definition). If $A$ is a $\mathbf{\Sigma}^1_1$ set in the Baire space, then $A \setminus U_A$ is also $\mathbf{\Sigma}^1_1$, hence there is a closed subset $F$ of ${}^\omega\omega \times {}^\omega\omega$ such that $\exists^{\mathbb{R}} F = A \setminus U_A$. Since there is no difference between playing basic open sets and playing open sets for Banach-Mazur games and the unfolded ones and basic open sets in the Baire space are easily coded by natural numbers, one can simulate the unfolded Banach-Mazur games by Gale-Stewart games in a simple way. By the first item of Theorem 1.6.3, all the closed Banach-Mazur games and the unfolded ones are determined. Hence we can conclude that every $\mathbf{\Sigma}^1_1$ set of reals has the Baire property.[4]

## 1.9 Forcing

While Zermelo-Fraenkel set theory with the axiom of choice (ZFC), which is a set-theoretic axiomatization for the foundation of mathematics, is a very good basis for most of mathematical practice, some mathematical questions remain undetermined by ZFC and one such typical question is whether the Continuum Hypothesis (CH) is true or not. In 1963, Cohen introduced *forcing* to prove that CH does not follow from ZFC and since then, forcing has been one of the most important basic tools in set theory. Starting from a model of ZFC (called the “ground model”), Cohen produced an extension of the given model (called a “generic extension”) which is a model of ZFC and the negation of CH. This technique is so general that one can define a generic extension for each partial order in the given ground model, and one can change the truth-value of many mathematical statements between ground models and their generic extensions which yield the consistency and the independence of those statements from ZFC.

[4] This is not the original proof of Lusin. It is due to Solovay (cf. [44, Exercise 27.14]).

In Chapter 2 and Chapter 3, we assume the basic theory of forcing which can be found in, e.g., [52, § 7, 8]. Let us fix the notation concerning forcing and list the partial orders we will use in this thesis.

The *Universe* is the class of all sets and it is denoted by $V$. Let $M$ be a model of ZF, $\mathbb{P}$ be a partial order belonging to $M$, and $G$ be a $\mathbb{P}$-generic filter over $M$. By $M[G]$, we mean the generic extension of $M$ via $G$. For a $\mathbb{P}$-name $\tau$ in $M$, $\tau^G$ denotes the interpretation of $\tau$ via $G$. For a set $x$, $\check{x}$ denotes the standard $\mathbb{P}$-name for $x$, i.e., $\check{x}^G = x$ for any filter $G$.

The following is the list of partial orders we will use:

**Cohen forcing.** The partial order is $({}^{<\omega}\omega, \supseteq)$ denoted by $\mathbb{C}$ where $\supseteq$ is reverse inclusion on finite sequences of natural numbers. Given a model $M$ of ZF and a $\mathbb{C}$-generic filter $G$ over $M$, set $x_G = \bigcup\{p \in \mathbb{C} \mid p \in G\}$. By the genericity of $G$, $x_G$ is a function from $\omega$ to itself (i.e., an element of the Baire space). Such objects are called *Cohen reals over $M$*. Also one can reconstruct $G$ from $x_G$ and $\mathbb{C}$ as follows: $G = \{p \in \mathbb{C} \mid p \subseteq x_G\}$. Hence there is a canonical one-to-one correspondence between $\mathbb{C}$-generic filters over $M$ and Cohen reals over $M$. We often identify these two objects.

**Random forcing** Elements of the partial order are Borel sets in the Baire space (or in the real line) with positive Lebesgue measure ordered by inclusion and it is denoted by $\mathbb{B}$. Given a model $M$ of ZF+AC$_\omega(\mathbb{R})$ and a $\mathbb{B}$-generic filter $G$ over $M$, the set $\bigcap\{B^{M[G]} \mid B \in G\}$ is a singleton $\{x_G\}$, where $B^{M[G]}$ is the interpretation of $B$ in $M[G]$ via Borel codes for $B$ in $M$.[5] Such reals $x_G$ are called *random reals over $M$*. As with Cohen reals, one can recover $G$ from $x_G$ and $M$ as follows: $G = \{B \in \mathbb{B} \mid x_G \in B^{M[G]}\}$. Hence there is a canonical one-to-one correspondence between $\mathbb{B}$-generic filters over $M$ and random reals over $M$. We often identify these two objects.

**Hechler forcing.** Elements of the partial order are pairs $(n, f)$ where $n$ is a natural number and $f$ is a function from $\omega$ to itself and it is denoted by $\mathbb{D}$. Given $(n, f)$ and $(m, g)$ in $\mathbb{D}$, $(n, f) \leq (m, g)$ if $n \geq m$, $f\restriction m = g\restriction m$ and $f(k) \geq g(k)$ for any $k \geq m$. Given a model $M$ of ZF and a $\mathbb{D}$-generic filter $G$ over $M$, $x_G = \bigcup\{f\restriction n \mid (n, f) \in G\}$ is a function from $\omega$ to itself by the genericity of $G$. Such reals $x_G$ are called *Hechler reals over $M$*. One can recover $G$ from $x_G$ and $M$ as follows: $G = \{(n, f) \in \mathbb{D} \mid x_G \supseteq f\restriction n \text{ and } (\forall k \geq n)\ f(k) \leq x_G(k)\}$. Hence there is a canonical one-to-one correspondence between $\mathbb{D}$-generic filters over $M$ and Hechler reals over $M$. We often identify these two objects.

[5] For the definition and the basic properties of Borel codes, see § 1.13.

**Mathias forcing.** Elements of the partial order are pairs $(s, A)$ where $s$ is a finite set of natural numbers and $A$ is an infinite set of natural numbers such that $\max(s) < \min(A)$ and the forcing is denoted $\mathbb{R}$.[6] Given $(s, A)$ and $(t, B)$ in $\mathbb{R}$, $(s, A) \leq (t, B)$ if $s \cap (n+1) = t$, $A \subseteq B$ and $s \setminus t \subseteq B$, where $n = \max t$. Given a model $M$ of ZF and a $\mathbb{R}$-generic filter over $M$, $x_G = \bigcup\{s \mid (\exists A)\ (s, A) \in G\}$ is an infinite set of natural numbers by the genericity of $G$. Such reals are called *Mathias reals over $M$*. One can reconstruct $G$ from $x_G$ and $M$ as follows: $G = \{(s, A) \in \mathbb{R} \mid s \subseteq x_G \text{ and } x_G \subseteq s \cup A\}$. Hence there is a canonical one-to-one correspondence between $\mathbb{R}$-generic filters over $M$ and Mathias reals over $M$. We often identify these two objects.

**Sacks forcing.** Elements of the partial order are perfect trees on 2 ordered by inclusion and it is denoted by $\mathbb{S}$. Given a model $M$ of ZF and an $\mathbb{S}$-generic filter $G$ over $M$, $x_G = \bigcup\{\text{stem}(T) \mid S \in G\}$ is a function from $\omega$ to 2 by the genericity of $G$. Such reals are called *Sacks reals over $M$*. One can recover $G$ from $x_G$ and $M$ as follows: $G = \{S \in \mathbb{S} \mid x_G \in [S]\}$. Hence there is a canonical one-to-one connection between $\mathbb{S}$-generic filters over $M$ and Sacks reals over $M$. We often identify these two objects.

**Silver forcing.** Elements of the partial order are uniform perfect trees on 2 ordered by inclusion and it is denoted by $\mathbb{V}$, where a perfect tree $T$ on 2 is *uniform* if for any $s$ and $t$ in $T$ with the same length and $i = 0, 1$, $s^\frown\langle i\rangle \in T$ if and only if $t^\frown\langle i\rangle \in T$. Given a model $M$ of ZF and a $\mathbb{V}$-generic filter $G$ over $M$, one can define $x_G$ in the same way as Sacks reals and such reals are called *Silver reals over $M$*. There is a canonical one-to-one correspondence between $\mathbb{V}$-generic filters over $M$ and Silver reals over $M$ as in Sacks forcing. We often identify these two objects.

**Miller forcing.** Elements of the partial order are superperfect trees on $\omega$ ordered by inclusion and it is denoted by $\mathbb{M}$, where a tree $T$ on $\omega$ is *superperfect* if for any node $t$ of $T$, there is an extension $u$ of $t$ in $T$ such that $\{n \in \omega \mid u^\frown\langle n\rangle \in T\}$ is infinite. Given a model $M$ of ZF and a $\mathbb{M}$-generic filter $G$ over $M$, one can define $x_G$ in the same way as Sacks reals and such reals are called *Miller reals over $M$*. There is a canonical one-to-one correspondence between $\mathbb{M}$-generic filters over $M$ and Miller reals over $M$ as in Sacks forcing. We often identify these two objects.

**Laver forcing.** Elements of the partial order are trees $T$ on $\omega$ such that for each node $t \supseteq \text{stem}(T)$ of $T$, the set $\{n \in \omega \mid t^\frown\langle n\rangle \in T\}$ is infinite and they are ordered by inclusion. The partial order is denoted by $\mathbb{L}$. Given a model $M$ of ZF and a $\mathbb{L}$-generic filter $G$ over $M$, one can define $x_G$ in the same way as

[6] We use this notation only in Chapter 2 where we do not use $\mathbb{R}$ either for the real line, the Baire space or the Cantor space. Hence there will be no confusion for this notation.

Sacks reals and such reals are called *Laver reals over $M$*. There is a canonical one-to-one correspondence between $\mathbb{L}$-generic filters over $M$ and Laver reals over $M$ as in Sacks forcing. We often identify these two objects.

**Eventually different forcing.** Elements of the partial order are pairs $(s, F)$ where $s$ is a finite sequence of natural numbers and $F$ is a finite set of functions from $\omega$ to itself and it is denoted by $\mathbb{E}$. Given $(s, F)$ and $(t, F')$ in $\mathbb{E}$, $(s, F) \leq (t, F')$ if $s \supseteq t$, $F' \subseteq F$ and $(\forall f \in F') \big(\forall n \in \mathrm{dom}(s \setminus t)\big)\ s(n) \neq f(n)$. Given a model $M$ of ZF and a $\mathbb{E}$-generic filter $G$ over $M$, $x_G = \bigcup\{s \mid (\exists F)\ (s, F) \in G\}$ is a function from $\omega$ to itself by the genericity of $G$. Such reals are called *$\mathbb{E}$-generic reals over $M$* and one can reconstruct $G$ from $x_G$ and $M$ as follows: $G = \{(s, F) \in \mathbb{E} \mid s \subseteq x_G \text{ and } (\forall f \in F) \big(\forall n \geq \mathrm{dom}(s)\big)\ x_G(n) \neq f(n)\}$. Hence there is a canonical one-to-one correspondence between $\mathbb{E}$-generic filters over $M$ and $\mathbb{E}$-generic reals over $M$. We often identify these two objects.

Next, we introduce useful classes of forcings that we use in Chapter 2. Let $\mathbb{P}$ be a partial order. For $p$ and $q$ in $\mathbb{P}$, $p$ and $q$ are *compatible* (denoted by $p||q$) if there is an $r$ in $\mathbb{P}$ such that $r \leq p$ and $r \leq q$. They are called *incompatible* (denoted by $p \perp q$) if they are not compatible. A subset $A$ of $\mathbb{P}$ is an *antichain* if any two different elements of $A$ are incompatible. A subset $D$ of $\mathbb{P}$ is *dense* if for any $p$ in $\mathbb{P}$ there is a $d$ in $D$ such that $d \leq p$. Let $D$ be a subset of $\mathbb{P}$ and $p$ be an element of $\mathbb{P}$. The set $D$ is *predense below* $p$ if for any $q \leq p$ in $\mathbb{P}$ there is a $d$ in $D$ such that $q$ and $d$ are compatible.

For a regular cardinal $\theta$, $\mathcal{H}_\theta$ denotes the set of all sets $a$ such that $|\mathrm{TC}(a)| < \theta$, where $\mathrm{TC}(a)$ denotes the transitive closure of $a$, i.e., the smallest set $b$ containing $a$ and which is transitive, i.e., $(\forall x \in b)\ x \subseteq b$.

**The countable chain condition (ccc).** A partial order $\mathbb{P}$ has the *countable chain condition* (or $\mathbb{P}$ is ccc) if every antichain of $\mathbb{P}$ is countable. Since the invention of forcing, ccc forcings have been fundamental partial orders and they enjoy many nice properties, e.g., they preserve cardinalities, i.e., given a ccc partial order $\mathbb{P}$ and a $\mathbb{P}$-generic filter $G$ over $V$, for any ordinal $\alpha$, $\alpha$ is a cardinal in $V$ if and only if it is a cardinal in $V[G]$. In particular, $\omega_1^V = \omega_1^{V[G]}$. Typical examples of ccc forcings are Cohen forcing, random forcing, Hechler forcing, and eventually different forcing. Mathias forcing, Sacks forcing, Silver forcing, Miller forcing, and Laver forcing are not ccc.

**Proper forcings.** A partial order $\mathbb{P}$ is *proper* if for any sufficiently large regular cardinal $\theta$ (e.g., $\theta \geq 2^{|\mathbb{P}|}$) and any countable elementary substructure $X$ of $\mathcal{H}_\theta$ with $\mathbb{P} \in X$, and any $p$ in $\mathbb{P} \cap X$, there is a $q \leq p$ in $\mathbb{P}$ such that $q$ is $(X, \mathbb{P})$-generic, i.e., for any dense set $D$ of $\mathbb{P}$ in $X$, $D \cap X$ is predense below $q$. Proper forcings were introduced by Shelah and they are also fundamental in modern set theory. They are a generalization of ccc forcings (i.e., every ccc forcing is proper) and they enjoy several properties ccc forcings satisfy, e.g., for a proper forcing $\mathbb{P}$, a $\mathbb{P}$-generic filter $G$ over $V$, and any countable set of ordinals $A$ in $V[G]$, there is a countable set of ordinals $B$ in $V$ such that $A \subseteq B$. In particular, $\omega_1^V = \omega_1^{V[G]}$. All the examples of forcings listed above are proper.

## 1.10 Large cardinals

*Large cardinals* are cardinals with certain transcendence properties over cardinals smaller than them. Many such properties are the analogies of the ones $\omega$ has over finite numbers. For the basics and background for large cardinals, we refer the reader to [44]. Let us list the large cardinals (or the large cardinal properties) we will use in this thesis:

**Inaccessible cardinals.** Inaccessible cardinals are the least and the oldest large cardinals. An uncountable cardinal $\kappa$ is *inaccessible* if it is regular, i.e., for any ordinal $\alpha < \kappa$ and any function $f\colon \alpha \to \kappa$, $f$ is bounded, i.e., there is a $\beta < \kappa$ such that $\mathrm{ran}(f) \subseteq \beta$, and it is strong limit, i.e., for any $\alpha < \kappa$, $2^\alpha < \kappa$. If $\kappa$ is inaccessible, then $V_\kappa$ is a model of ZFC. Hence the existence of an inaccessible cardinal implies the consistency of ZFC and by Gödel's Incompleteness Theorem, the consistency of ZFC+"There is an inaccessible cardinal" is strictly stronger than that of ZFC.

**Sharps.** Let $X$ be a set. By $X^\#$, we mean the complete theory of $\mathrm{L}(X)$ in the language $(\in, \{c_n\}_{n\in\omega}, \{d_a\}_{a\in\mathrm{TC}(X)})$ with some special properties, where $c_n$ is the constant for the $n$-th indiscernible for $\mathrm{L}(X)$ and $d_a$ is the constant for $a \in \mathrm{TC}(X)$. For the details, see, e.g., [22]. The existence of $X^\#$ is equivalent to the existence of a closed unbounded proper class of indiscernibles for $\mathrm{L}(X)$ with some properties. Also it is equivalent to the existence of an elementary embedding $j$ from $\mathrm{L}(X)$ to itself whose critical point is above the rank of $X$. (Here the critical point of $j$ is the least ordinal $\kappa$ such that $j(\kappa) > \kappa$.) We say *every real has a sharp* if for any real $x$, $x^\#$ exists. We say *every set has a sharp* if for any set $X$, $X^\#$ exists.

**Measurable cardinals.** Measurable cardinals are one of the most fundamental large cardinals. An uncountable cardinal $\kappa$ is a *measurable cardinal* if there is an elementary embedding from $V$ to a transitive proper class whose critical point is $\kappa$. There is a first-order characterization of measurable cardinals: An uncountable cardinal $\kappa$ is measurable if and only if there is a non-trivial $\kappa$-complete ultrafilter on $\kappa$; here a filter is *non-trivial* if it is not principal and it is *$\kappa$-complete* if it is closed under intersections with $<\kappa$ many sets. It is easy to see that if $\kappa$ is a measurable cardinal, then for any set $X \in V_\kappa$, $X^\#$ exists.

**Strong cardinals.** Most large cardinals stronger than measurable cardinals assert the existence of elementary embeddings from $V$ to a transitive class $M$ with certain properties. The more $M$ is close to $V$, the stronger the large cardinal property is. Strong cardinals are one of the natural strengthening of measurable cardinals in this sense. Let $\alpha$ be an ordinal. An uncountable cardinal $\kappa$ is *$\alpha$-strong* if there is an elementary embedding $j$ from $V$ to $M$ such that $M$ is transitive, the critical point of $j$ is $\kappa$, and $V_\alpha \subseteq M$. An uncountable cardinal $\kappa$ is *strong* if it is $\alpha$-strong for any ordinal $\alpha$. It is immediate that any $\alpha$-strong cardinal is measurable. If $\kappa$ is $(\kappa+2)$-strong, then there are unboundedly many measurable cardinals below $\kappa$.

**Woodin cardinals.** Woodin cardinals were introduced when Shelah and Woodin tried to decide the optimal upper bound for the consistency strength of the saturation of the nonstationary ideal on $\omega_1$ and they are tightly connected to the determinacy of projective sets in Gale-Stewart games. Let $\alpha < \delta$ be ordinals and $A$ be a subset of $V_\delta$. An uncountable cardinal $\kappa < \delta$ is *$\alpha$-$A$-strong* if there is an elementary embedding $j$ from $V$ to a transitive class $M$ such that $\kappa$ is the critical point of $j$, $V_\alpha \subseteq M$, and $A \cap V_\alpha = j(A) \cap V_\alpha$. An uncountable cardinal $\kappa$ is *$<\delta$-$A$-strong* if it is $\alpha$-$A$-strong for every $\alpha < \delta$. An inaccessible cardinal $\delta$ is *Woodin* if it is a limit of $<\delta$-$A$-strong cardinals for any subset $A$ of $V_\delta$. If $\delta$ is Woodin, then $V_\delta$ satisfies "There is a proper class of strong cardinals".

## 1.11 Inner models and inner model theory

*Inner models* are transitive proper class models of ZF. The study of *inner model theory* is about canonical inner models with large cardinals. The *Gödel's Constructible Universe* L is the most basic canonical inner model. It always exists in ZF and it is the least inner model of ZFC. Gödel introduced L to prove the consistency of AC, CH, and moreover the Generalized Continuum Hypothesis (GCH) with ZF. Beside this fact, L has many interesting properties, e.g., in L, there is a $\Delta^1_2$ set of reals without the Baire property and which is not Lebesgue measurable, and there is a $\Pi^1_1$ set of reals without the perfect set property. As at the end of § 1.8, every $\boldsymbol{\Sigma}^1_1$ set of reals has the Baire property. Also every $\boldsymbol{\Sigma}^1_1$ set of reals is Lebesgue measurable and has the perfect set property. Hence the above facts about L show that $\boldsymbol{\Sigma}^1_1$ sets of reals are the limit for proving the above regularity properties in ZFC.

One can relativize the construction of L to any set in the following two ways: For a set $A$, $\mathrm{L}[A]$ denotes the least inner model such that $A \cap \mathrm{L}[A] \in \mathrm{L}[A]$ and $\mathrm{L}(A)$ denotes the least inner model containing $A$ as an element. The model $\mathrm{L}[A]$ is always a model of ZFC and $A$ might not belong to $\mathrm{L}[A]$ in general (e.g., $\mathrm{L}[\mathbb{R}] = \mathrm{L}$ and $\mathbb{R}$ does not belong to L in general) while $\mathrm{L}(A)$ might not be a model of AC (e.g., if there are $\omega$-many Woodin cardinals and a measurable cardinal above all of them, then AC fails in $\mathrm{L}(\mathbb{R})$). For a set of ordinals $A$, $\mathrm{L}[A] = \mathrm{L}(A)$.

Let us list the basic properties of L we use later:

**Lemma 1.11.1** (Gödel)**.**

1. The relation $\{(x, a) \in {}^{\omega}\omega \times {}^{\omega}\omega \mid x \in \mathrm{L}[a]\}$ is a $\Sigma^1_2$ set of reals.
2. For any real $a$, $\mathrm{L}[a] \models$ "There is a $\Delta^1_2(a)$ wellordering of the reals".

*Proof.* See, e.g., [66, Theorem 8F.7, 8F.23, 8F.24]. □

*Core models* are canonical inner models with the following special properties: first they are fine structural (constructed with Jensen's $J_\alpha$ Hierarchy), second, they are forcing invariant (they are absolute between ground models and their forcing extensions), and lastly they are close to $V$, e.g., they have covering properties or weak covering. If $0^\#$ does not exist, L is the basic core model. Unlike many canonical inner models, one needs to assume some anti-large cardinal hypothesis to prove the existence of core models. The following is a general result for the existence of the core model:

**Theorem 1.11.2** (Dodd and Jensen [24]; Koepke [50]; Jensen [38]; Mitchell [64]; Jensen [39]; Steel [79]; Jensen and Steel [41, 40])**.** Suppose every real has a sharp. If there is no inner model of ZFC with a Woodin cardinal, then the core model K exists. More generally, if $\boldsymbol{\Delta}^1_2$-determinacy fails, then there is a real $a_0$ such that for any $a \geq_{\mathrm{T}} a_0$, the $a$-relativized version of the core model $\mathrm{K}_a$ exists, where $\leq_{\mathrm{T}}$ is the Turing order.[7] Moreover, the core models have the following properties:

1. the relation $\{(x, a) \in {}^{\omega}\omega \times {}^{\omega}\omega \mid x \in \mathrm{K}_a\}$ is a $\Sigma^1_3$ set of reals, and
2. for any real $a$, $\mathrm{K}_a \models$ "There is a $\Delta^1_3(a)$ wellordering of the reals".

*Proof.* When there is a real $a$ such that $a^\dagger$ does not exist, see [24]. In the other case, see [79]. Note that in [79], Steel assumed the existence of a measurable cardinal to construct K. But Jensen and Steel [41, 40] omitted this assumption. □

To build core models, one needs to study fragments of core models or more general objects, which are called *mice*. Standard examples of mice are L and the core model K. For a set $a$, there are $a$-relativized version of mice called *$a$-mice*. Basic examples are $\mathrm{L}[a]$ and $\mathrm{K}_a$. The following two theorems are essential to study mice:

**Theorem 1.11.3** (Comparison Lemma)**.** Let $M, N$ be mice and $\theta = \max\{|M|^+, |N|^+\}$. After $<\theta$ steps of coiterations, one of them is an initial segment of the other.

[7] Note that $\Delta^1_2$-determinacy (lightface) is equivalent to the existence of an inner model of ZFC with a Woodin cardinal. This is why we said "More generally,".

*Proof.* See, e.g., [92, Lemma 9.1.8]. □

**Theorem 1.11.4** (Dodd-Jensen Lemma)**.** Let $M$ be a mouse and $i\colon M \to M'$ be an iteration map according to the unique iteration strategy of $M$. Suppose there is a $\Sigma^*$ preserving map $\sigma\colon M \to M'$. Then

1. there is no drop in the iteration tree for $i$, and
2. for any ordinal $\xi$ in $M$, $i(\xi) \leq \sigma(\xi)$.

In particular, any two iteration maps without drops from a mouse to a mouse are the same.

*Proof.* See, e.g., [92, Lemma 9.2.10]. □

## 1.12 Absoluteness

We speak of *absoluteness* if a sentence or a class of sentences does not change truth values of mathematical statements between models of set theory and it is one of the basic and central notions in set theory. Given models of set theory $M$ and $N$ with $M \subseteq N$ and a formula $\phi$, $\phi$ is *absolute between $M$ and $N$* if for any finite sequence of elements $\vec{x}$ in $M$, $M \vDash \phi(\vec{x})$ if and only if $N \vDash \phi(\vec{x})$. For example, the formula "$x$ is $\omega$" is absolute between any two transitive models of ZF. The first nontrivial and important absolute notion is *wellfoundedness*. A relation $R$ on a set $A$ is *wellfounded* if for any nonempty subset $B$ of $A$, there is an $R$-minimal element of $B$, i.e., there is a $b \in B$ such that for any element $a$ of $B$, $(a, b) \notin R$.

**Lemma 1.12.1.** The formula "$R$ is a wellfounded relation on $A$" is absolute between any two transitive models of ZF.

*Proof.* See, e.g., [37, Lemma 13.11] and the two paragraphs preceding it. □

Given a $\Pi^1_1$ formula $\phi$, one can recursively compute a tree $T$ on $\omega \times \omega$ such that $\{x \mid \mathcal{A}^2 \vDash \phi(x)\} = \{x \mid [T_x] = \emptyset\}$, where $T_x = \{t \in {}^{<\omega}\omega \mid \big(x{\restriction}\mathrm{dom}(t), t\big) \in T\}$ in ZF+AC$_\omega(\mathbb{R})$. But $[T_x] = \emptyset$ if and only if $(T_x, \supseteq)$ is wellfounded. Hence $\mathcal{A}^2 \vDash \phi(x)$ if and only if $(T_x, \supseteq)$ is wellfounded. Hence the problem of membership for a $\Pi^1_1$ set is reduced to the one for the wellfoundedness of certain trees. Combining with Lemma 1.12.1,

**Theorem 1.12.2** (Mostowski)**.** Every $\Pi^1_1$ formula is absolute between transitive models of ZF+AC$_\omega(\mathbb{R})$. Hence every $\Sigma^1_1$ formula is also absolute between transitive models of ZF+AC$_\omega(\mathbb{R})$.

*Proof.* See, e.g., [37, Theorem 25.4]. □

In general, a $\Pi^1_2$ formula is not absolute between transitive models of ZF. Shoenfield proved that any $\Pi^1_2$ formula is absolute between inner models of ZF+ $\mathrm{AC}_\omega(\mathbb{R})$:

**Theorem 1.12.3** (Shoenfield)**.** For any $\Pi^1_2$ formula $\phi$ and real $a$, there is a tree $T$ on $\omega\times\omega_1$ in $\mathrm{L}[a]$ such that for any real $x$, $\mathcal{A}^2 \vDash \phi(x,a)$ if and only $T_x$ is wellfounded. This tree is called a *Shoenfield tree* and one can construct a Shoenfield tree in any inner model of ZF+$\mathrm{AC}_\omega(\mathbb{R})$ and the construction depends only on $\phi$, $a$, and a fixed uncountable ordinal (in this case, $\omega_1^V$).

Hence Shoenfield trees are absolute and thus every $\Pi^1_2$ formula (and $\Sigma^1_2$ formula) is absolute between between inner models of ZF+$\mathrm{AC}_\omega(\mathbb{R})$, especially between L and $V$.

*Proof.* See, e.g., [66, 8F.8, 8F.9, 8F.10]. □

In general, a $\Pi^1_3$ formula is not absolute between L and $V$, e.g., the statement "Every real is in L" is equivalent to a $\Pi^1_3$ formula and one can add nonconstructible real (e.g., a Cohen real over L) via forcing starting from L. Using sharps for reals, Martin and Solovay constructed a tree called *Martin-Solovay tree* for a $\Pi^1_3$ formula which is like a Shoenfield tree for a $\Pi^1_2$ formula. We will give a sufficient condition for the absoluteness of Martin-Solovay trees. Assume every real has a sharp. For a real $a$, let $I_a$ be the closed unbounded class of indiscernibles derived from $a^\#$ and set $I = \bigcap_{a\in{}^\omega\omega} I_a$. The class $I$ is called the *class of uniform indiscernibles* and $u_2$ denotes the second element of $I$ and is called the *second uniform indiscernible.*

**Theorem 1.12.4** (Martin and Solovay)**.** Let $M$, $N$ be inner models of ZFC+ "Every real has a sharp". If $u_2^M = u_2^N$ with $M \subseteq N$, then Martin-Solovay trees are absolute between $M$ and $N$ and hence every $\Pi^1_3$ formula (and $\Sigma^1_3$ formula) is absolute between $M$ and $N$.

*Proof.* See, e.g., [33, Theorem 2.1]. □

Every $\Sigma^1_3$ formula is absolute between the core model K and $V$ when K exists:

**Theorem 1.12.5** (Dodd and Jensen; Steel)**.** Assume every real has a sharp. If $\boldsymbol{\Delta}^1_2$-determinacy fails, then there is a real $a_0$ such that for any $a \geq_{\mathrm{T}} a_0$, the $a$-relativized version of the core model $\mathrm{K}_a$ exists and every $\Sigma^1_3$ formula is absolute between $\mathrm{K}_a$ and $V$.

*Proof.* In case there is a real $a$ such that $a^\dagger$ does not exist, this is due to Dodd and Jensen [24]. If every real has a dagger, then this is due to Steel [79, Theorem 7.9].[8] □

[8]In [79, Theorem 7.9], he assumed two measurable cardinals. But one can replace this assumption with daggers for reals. See [71, Theorem 0.1].

Before closing this section, we discuss the absoluteness of being a winning strategy for Gale-Stewart games with closed payoff sets:

**Theorem 1.12.6** (Folklore)**.** Let $X$ be a nonempty set and $M$ be a transitive model of ZF with $X \in M$. For any closed subset $A$ of ${}^{\omega}X$, given a strategy $\sigma$ for player I in $M$, $M \vDash$ "$\sigma$ is winning in $A$" if and only if $V \vDash$ "$\sigma$ is winning in $A$". The same holds for player II.

*Proof.* As described in [45, 20.B], if there is a winning strategy for player I in the game $G_X(A)$ for a closed set $A$, then there is a canonical winning quasistrategy $\Sigma_A$ for player I and a strategy $\sigma$ for I is winning for the game $G_X(A)$ if and only if $\sigma \subseteq \Sigma_A$. Since the construction of $\Sigma_A$ is absolute between transitive models of ZF, the statement "$\sigma$ is winning in $A$" is absolute between transitive models of ZF, as desired. $\square$

## 1.13 Borel codes and $\infty$-Borel codes

If $X$ is the Baire space, the Cantor space, or the real line, it is easy to show that there is a surjection from the Cantor space to the set of all Borel subsets of $X$. (By induction on $1 \leq \xi < \omega_1$, one can construct surjections from the Cantor space to $\mathbf{\Sigma}^0_\xi$ subsets of $X$ and one can amalgamate them into one surjection.) *Borel codes* are effective realizations of such surjections introduced by Solovay. To introduce them, we first fix some notions and notations. Let $Y$ be a set. A tree $T$ on $Y$ is *wellfounded* if $(T, \supseteq)$ is wellfounded. A node $s$ of $T$ is *terminal* if there is no node $t$ in $T$ extending $s$. Let $\mathrm{Term}(T)$ denote the set of all terminal nodes of $T$. Let $s, t$ be nodes of $T$. The node $t$ is a *successor of $s$ in $T$* if $t$ extends $s$ and $\mathrm{lh}(t) = \mathrm{lh}(s) + 1$. For a node $s$ of $T$, $\mathrm{Succ}_T(s)$ denotes the set of successors of $s$ in $T$.

We introduce Borel codes for Borel subsets of the Cantor space. One can introduce Borel codes for the Baire space and the real line in the same way. *Borel codes* are pairs $(T, f)$ where $T$ is a wellfounded tree on $\omega$ and $f$ is a function from $\mathrm{Term}(T)$ to ${}^{<\omega}2$. One can simply regard Borel codes as elements of the Cantor space by identifying trees on $\omega$ with a map from ${}^{<\omega}\omega$ to $\{0, 1\}$ and fixing a simple bijection between ${}^{<\omega}\omega$ and $\omega$. With this identification, we regard Borel codes as elements of the Cantor space. Given a Borel code $c = (T, f)$, the *decode* $B_c$ is defined as follows: For each node $t$ of $T$,

$$B_t = \begin{cases} [f(t)] & \text{if } t \in \mathrm{Term}(T) \\ {}^{\omega}2 \setminus B_s & \text{if } (\exists s \in T)\ \{s\} = \mathrm{Succ}_T(t) \\ \bigcup_{s \in \mathrm{Succ}_T(t)} B_s & \text{otherwise.} \end{cases}$$

We set $B_c = B_\emptyset$. This is well-defined because $T$ is wellfounded. One can easily check any Borel set is of the form $B_c$ for some Borel code $c$. The following are basic observations on Borel codes:

**Lemma 1.13.1** (Solovay)**.** The set of Borel codes and the relations $x \in B_c$, $x \notin B_c$ are $\Pi^1_1$ sets and hence they are absolute between transitive models of ZF+$\mathrm{AC}_\omega(\mathbb{R})$.

*Proof.* See, e.g., [37, Lemma 25.44 & Lemma 25.55]. □

Infinitary Borel codes ($\infty$-Borel codes) are a transfinite generalization of Borel codes: Let $\mathcal{L}_{\infty,0}(\{\mathbf{a}_n\}_{n\in\omega})$ be the language allowing arbitrary many conjunctions and disjunctions and no quantifiers with atomic sentences $\mathbf{a}_n$ for each $n \in \omega$. The $\infty$-*Borel codes* are the sentences in $\mathcal{L}_{\infty,0}(\{\mathbf{a}_n\}_{n\in\omega})$ belonging to any $\Gamma$ such that

- the atomic sentence $\mathbf{a}_n$ is in $\Gamma$ for each $n \in \omega$,
- if $\phi$ is in $\Gamma$, then so is $\neg\phi$, and
- if $\alpha$ is an ordinal and $\langle \phi_\beta \mid \beta < \alpha \rangle$ is a sequence of sentences each of which is in $\Gamma$, then $\bigvee_{\beta<\alpha} \phi_\beta$ is also in $\Gamma$.

To each $\infty$-Borel code $\phi$, we assign a set of reals $B_\phi$ in the same way as decoding Borel codes:

- if $\phi = \mathbf{a}_n$, then $B_\phi = \{x \in {}^\omega 2 \mid x(n) = 1\}$,
- if $\phi = \neg\psi$, then $B_\phi = {}^\omega 2 \setminus B_\psi$, and
- if $\phi = \bigvee_{\beta<\alpha} \psi_\beta$, then $B_\phi = \bigcup_{\beta<\alpha} B_{\psi_\beta}$.

A set of reals $A$ is called $\infty$-*Borel* if there is an $\infty$-Borel code $\phi$ such that $A = B_\phi$.

As Borel codes, one can regard $\infty$-Borel codes as wellfounded trees with atomic sentences $\mathbf{a}_n$ on terminal nodes and decode them by assigning sets of reals on each node recursively from terminal nodes. (If a node has only one successor, then it means "negation" and if a node has more than one successors, then it means "disjunction".) The only difference between Borel codes and $\infty$-Borel codes is that trees are on $\omega$ for Borel codes while trees are on ordinals for $\infty$-Borel codes. From this visualization, it is easy to see that the statement "$\phi$ is an $\infty$-Borel code" is absolute between any transitive models of ZF by Lemma 1.12.1.

Given an $\infty$-Borel code $\phi$ and a real $x$, the problem whether $x$ is in $B_\phi$ can be easily translated into the following kind of satisfaction game using the above visualization of $\infty$-Borel codes via wellfounded trees: Let us regard $\phi$ as a wellfounded tree $T_\phi$ on ordinals with terminal nodes labeled by atomic sentences. In the game $G_c(T_\phi)$, there are two players, Spoiler and Duplicator, and a counter designating which player should move next. We start with the top node (the empty sequence) with the counter designating Duplicator. If the node has only one successor, no player is supposed to decide anything and they move to the unique successor and exchange the name in the counter. (This is for the negation.) If the node has more than one successors, then the player designated by the counter chooses one of the successors and keeps the name of the counter. (This is for the disjunction.) If the node is a terminal node, then look at the atomic sentence labeled at the node, say $\mathbf{a}_n$. If the real $x$ satisfies that $x(n) = 1$, then the player designated by the counter wins, otherwise the other player wins. It is fairly easy to see that a real $x$ is in $B_\phi$ if and only if Duplicator has a winning strategy in the game $G_c(T_\phi)$. By the fact that the payoff set of this game is a clopen subset of ${}^\omega\gamma$ for some ordinal $\gamma$, being a winning strategy in this game is absolute in any transitive model of ZF by Theorem 1.12.6. Hence the statement "a real $x$ is in $B_\phi$" is absolute between transitive models of ZF.

The following characterization of $\infty$-Borel sets is very useful:

**Fact 1.13.2** (Folklore)**.** Let $A$ be a set of reals. Then the following are equivalent:

1. $A$ is $\infty$-Borel, and
2. There is a formula $\phi$ in the language of set theory and a set $S$ of ordinals such that for each real $x$,

$$x \in A \iff \mathrm{L}[S, x] \vDash \phi(x).$$

*Proof.* See [80]. □

Standard examples of $\infty$-Borel sets are Suslin sets. A set of reals $A$ is *Suslin* if there are an ordinal $\gamma$ and a tree $T$ on $2 \times \gamma$ such that $A = \mathrm{p}[T]$, where $\mathrm{p}[T]$ is the projection of $[T]$ to the first coordinate, i.e.,

$$\mathrm{p}[T] = \{x \in {}^\omega 2 \mid (\exists f \in {}^\omega\gamma)\ (x, f) \in [T]\}.$$

By the above fact, every Suslin set is $\infty$-Borel. Assuming the Axiom of Choice, it is easy to see that every set of reals is Suslin, in particular $\infty$-Borel. Hence the property $\infty$-Borelness is trivial in the ZFC context while it is nontrivial and powerful in a determinacy world, as we will see in Chapter 3.

## 1.14 Blackwell games

In this section, we introduce *Blackwell games*, which are infinite games with imperfect information and compare them with Gale-Stewart games.

In 1928, John von Neumann proved his famous *minimax theorem* which is about finite games with imperfect information. Infinite versions of von Neumann's games were introduced by David Blackwell [15] where he proved the analogue of von Neumann's theorem for $\mathrm{G}_\delta$ sets of reals (i.e., $\mathbf{\Pi}^0_2$ sets of reals). The games he introduced are called *Blackwell games* and they were called by him "games with slightly imperfect information" in his paper [16].

We start with the definition of Blackwell games.[9] Let $X$ be a nonempty set and assume $\mathrm{AC}_\omega({}^\omega\mathbb{R})$. Recall from § 1.4 that the topology of ${}^\omega X$ is given by the product topology where each coordinate (i.e., $X$) is seen as the discrete space. In Blackwell games, players choose probabilities on $X$ instead of elements of $X$ and with those probabilities, one can deduce a Borel probability on ${}^\omega X$, i.e., a measure assigning probability to each Borel subset of ${}^\omega X$. Player I wins if the probability of a given payoff set is 1 and player II wins if the probability of the payoff set is 0. Let us formulate this in detail.

**Definition 1.14.1.** A *mixed strategy for player I* is a function $\sigma\colon X^{\mathrm{Even}} \to \mathrm{Prob}_\omega(X)$, where $\mathrm{Prob}_\omega(X)$ is the set of functions $\mu\colon X \to [0,1]$ with $\sum_{x\in X}\mu(x) = 1$.[10] A *mixed strategy for player II* is a function $\tau\colon X^{\mathrm{Odd}} \to \mathrm{Prob}_\omega(X)$.

Given mixed strategies $\sigma$, $\tau$ for player I and II respectively, let $\nu(\sigma,\tau)\colon {}^{<\omega}X \to \mathrm{Prob}_\omega(X)$ be as follows: For each finite sequence $s$ of elements of $X$,

$$\nu(\sigma,\tau)(s) = \begin{cases} \sigma(s) & \text{if } s \in X^{\mathrm{Even}}, \\ \tau(s) & \text{if } s \in X^{\mathrm{Odd}}. \end{cases}$$

For each finite sequence $s$ of elements of $X$, define

$$\mu_{\sigma,\tau}([s]) = \prod_{i=0}^{\mathrm{lh}(s)-1} \nu(\sigma,\tau)(s\restriction i)\ \big(s(i)\big).$$

Recall that $[s]$ denotes the set of $x \in {}^\omega X$ such that $x \supseteq s$ and these sets are basic open sets in the space ${}^\omega X$. With the help of $\mathrm{AC}_\omega({}^\omega X)$, we can uniquely extend $\mu_{\sigma,\tau}$ to a Borel probability on ${}^\omega X$, i.e., the probability whose domain is the set of all Borel sets in the space ${}^\omega X$. Let us also use $\mu_{\sigma,\tau}$ for denoting this Borel probability.

Let $A$ be a subset of ${}^\omega X$. A mixed strategy $\sigma$ for player I is *optimal in $A$* if for any mixed strategy $\tau$ for player II, $A$ is $\mu_{\sigma,\tau}$-measurable and $\mu_{\sigma,\tau}(A) = 1$. A mixed strategy $\tau$ for player II is *optimal in $A$* if for any mixed strategy $\sigma$ for player I, $A$ is $\mu_{\sigma,\tau}$-measurable and $\mu_{\sigma,\tau}(A) = 0$. A set $A$ is *Blackwell-determined* if one of the players has an optimal strategy in $A$. The axiom $\mathrm{Bl\text{-}AD}_X$ states that every subset of ${}^\omega X$ is Blackwell-determined. We write Bl-AD for $\mathrm{Bl\text{-}AD}_\omega$.

Note that since there is a bijection between $\mathbb{R}$ and ${}^\omega\mathbb{R}$, by Remark 1.2.1, $\mathrm{AC}_\omega(\mathbb{R})$ implies $\mathrm{AC}_\omega({}^\omega\mathbb{R})$ and hence one can formulate Blackwell games in ${}^\omega\mathbb{R}$ and $\mathrm{Bl\text{-}AD}_\mathbb{R}$ within $\mathrm{ZF}+\mathrm{AC}_\omega(\mathbb{R})$. The following is an analogy with Proposition 1.6.6:

[9] Our definitions of Blackwell games and Blackwell determinacy are different from the original ones given by Blackwell [16] where Blackwell determinacy is formulated as an extension of von Neumann's minimax theorem, but our formulation is equivalent to the original one when it is about the Cantor space (i.e., when $X = 2$). For the original formulation of Blackwell games and Blackwell determinacy, see, e.g., [56, § 3 & § 5].

[10] We use $\mathrm{Prob}_\omega(X)$ to denote such functions because they are the same as Borel probabilities $\mu$ on $X$ with countable support, i.e., there is a countable subset $A$ of $X$ with $\mu(A) = 1$.

**Proposition 1.14.2.**

1. Let $X, Y$ be nonempty sets and suppose that there is an injection from $X$ to $Y$ and assume $\mathrm{AC}_\omega({}^\omega Y)$. Then $\mathrm{Bl\text{-}AD}_Y$ implies $\mathrm{Bl\text{-}AD}_X$. In particular, $\mathrm{Bl\text{-}AD}_{\mathbb{R}}$ implies Bl-AD.

2. The axioms Bl-AD and $\mathrm{Bl\text{-}AD}_2$ are equivalent.

*Proof.* The first item is easy to see. For the second item, see [55, Corollary 4.4]. □

As for Gale-Stewart games, one could ask what kind of subsets of ${}^\omega X$ are Blackwell-determined for a nonempty set $X$. After proving that every $\mathrm{G}_\delta$ subset of the Cantor space is Blackwell-determined, Blackwell asked whether every Borel subset of the Cantor space is determined. It was Donald Martin who found a general connection between the determinacy of Gale-Stewart games and Blackwell determinacy.[11]

**Theorem 1.14.3** (Martin)**.** Let $X$ be a set and assume $\mathrm{AC}_\omega({}^\omega X)$. If there is a winning strategy for player I (resp., II) in a subset $A$ of ${}^\omega X$, then there is an optimal strategy for player I (resp., II) in $A$. In particular, AD implies that Bl-AD and $\mathrm{AD}_{\mathbb{R}}$ implies that $\mathrm{Bl\text{-}AD}_{\mathbb{R}}$.

*Proof.* Given a strategy $\sigma$ for player I (resp., II), one can naturally translate $\sigma$ into a mixed strategy $\hat{\sigma}$ for player I (resp., II) by setting $\hat{\sigma}(s)$ to be the Dirac measure concentrating on $\sigma(s)$. It is easy to see that if $\sigma$ is winning in $A$, then $\hat{\sigma}$ is optimal in $A$. □

By Theorem 1.6.4, every Borel subset of the Cantor space is Blackwell-determined in ZFC and this answers the question of Blackwell. After proving Theorem 1.14.3, Martin conjectured the following:

**Conjecture 1.14.4** (Martin)**.** Bl-AD implies AD.

This conjecture is still not known to be true. The best known result toward AD from Bl-AD is as follows: Recall the notion of Suslinness from § 1.13. A set of reals is *co-Suslin* if its complement is Suslin.

**Theorem 1.14.5** (Martin, Neeman, and Vervoort)**.** Assume Bl-AD. Then every Suslin and co-Suslin set of reals is determined.

*Proof.* See [59, Lemma 4.1].[12] □

[11] In [58], Martin proved the Blackwell determinacy in the original formulation as mentioned in Footnote 9, not in our formulation.

[12] In [59, Lemma 4.1], they assume the Blackwell determinacy for sets of reals in a weakly scaled pointclass. But the argument shows the statement in Theorem 1.14.5.

Together with the following result, one can establish the equiconsistency between AD and Bl-AD:

**Theorem 1.14.6** (Kechris and Woodin)**.** Assume that every Suslin and co-Suslin set of reals is determined. Then $\mathrm{AD}^{\mathrm{L}(\mathbb{R})}$ holds.

*Proof.* See [46]. □

**Corollary 1.14.7** (Martin, Neeman, and Vervoort)**.** In $\mathrm{L}(\mathbb{R})$, AD and Bl-AD are equivalent. In particular, AD and Bl-AD are equiconsistent.

Also, Bl-AD has some consequence on regularity properties:

**Theorem 1.14.8** (Vervoort)**.** Assume Bl-AD. Then every set of reals is Lebesgue measurable.

*Proof.* See [86]. □

We discuss the connection between Blackwell determinacy and other regularity properties such as the Baire property in § 3.2.

It is not difficult to see that if finite games are Blackwell determined, then they are determined. As a corollary, one can obtain the following:

**Theorem 1.14.9** (Löwe)**.** Assume $\mathrm{Bl\text{-}AD}_{\mathbb{R}}$. Then every relation on the reals can be uniformized by a function.

*Proof.* See [56, Theorem 9.3]. □

Since there is a relation on the reals which cannot be uniformized by a function in $\mathrm{L}(\mathbb{R})$, $\mathrm{Bl\text{-}AD}_{\mathbb{R}}$ does not hold in $\mathrm{L}(\mathbb{R})$. Since $\mathrm{Bl\text{-}AD}_{\mathbb{R}}$ implies Bl-AD by the first item of Remark 1.14.2 and Bl-AD implies $\mathrm{AD}^{\mathrm{L}(\mathbb{R})}$ by Corollary 1.14.7, AD does not imply $\mathrm{Bl\text{-}AD}_{\mathbb{R}}$.

In Chapter 3, we discuss the connection between $\mathrm{AD}_{\mathbb{R}}$ and $\mathrm{Bl\text{-}AD}_{\mathbb{R}}$.

## 1.15 Wadge reducibility and Wadge games

When we study descriptive set theory, we often would like to compare given two sets of reals via some measure of complexity, i.e., we would like to ask the question "Which set of reals is more complex than the other?". In 1972, Wadge [88] introduced *Wadge reducibility* for sets of reals in the Baire space, which is an analogue of many-one reducibility in recursion theory: A set of reals $A$ is *Wadge reducible* to a set of reals $B$ if there is a continuous function $f$ from the Baire space to itself such that $A = f^{-1}(B)$. After its introduction, set theorists in California developed a beautiful theory of Wadge reducibility under the Axiom of Determinacy (AD) plus the principle of Dependent Choice (DC). Nowadays this theory is one of the basic tools in the research of determinacy and is essential to the study of descriptive set theory. The key tool of the analysis of Wadge reducibility is a type of infinite games called *Wadge games*, which characterize continuous functions from the Baire space to itself.

For a subset $A$ of a topological space $X$, $A^{\mathrm{c}}$ denotes the complement of $A$ and $\overline{A}$ denotes the closure of $A$ in $X$.

We start with the definition of Wadge reducibility for a general topological space. Let $X$ be a topological space and $A, B$ be subsets of $X$. The set $A$ is *Wadge reducible to* $B$ (write $A \leq^X_{\mathrm{W}} B$) if there is a continuous function $f\colon X \to X$ such that $A = f^{-1}(B)$. Hence the problem of the membership of $A$ can be reduced to that of the membership of $B$ via a continuous function, and in this sense $B$ is more complicated than (or as complicated as) $A$. This notion reminds us of the many-one reducibility for subsets of $\omega$ in recursion theory given by replacing continuous functions with recursive functions. We define three other notions of Wadge reducibility. A subset $A$ of $X$ is *Wadge equivalent to* a subset $B$ of $X$ ($A \equiv^X_{\mathrm{W}} B$) if $A \leq^X_{\mathrm{W}} B$ and $B \leq^X_{\mathrm{W}} A$. A subset $A$ of $X$ is *strictly Wadge reducible to* a subset $B$ of $X$ ($A <^X_{\mathrm{W}} B$) if $A \leq^X_{\mathrm{W}} B$ and $B \not\leq^X_{\mathrm{W}} A$. A subset $A$ of $X$ is *Wadge comparable to* a subset $B$ of $X$ if $A \leq^X_{\mathrm{W}} B$ or $B \leq^X_{\mathrm{W}} A$ holds. It is easy to see that the Wadge order $\leq^X_{\mathrm{W}}$ is a preorder (i.e., reflexive and transitive) and that the Wadge equivalence $\equiv^X_{\mathrm{W}}$ is an equivalence relation on subsets of $X$. An equivalence class of this equivalence relation is called a *Wadge degree*.

When $X$ is the Baire space, the study of Wadge degrees is interesting to descriptive set theorists in the way that Turing degrees are interesting to recursion theorists. Since each boldface pointclass is closed under continuous preimages, it consists of an initial segment of all the subsets of reals via Wadge reducibility and hence the study of Wadge degrees gives us a finer analysis of boldface pointclasses such as Borel classes $\mathbf{\Sigma}^0_\xi$ and projective classes $\mathbf{\Sigma}^1_n$. Wadge introduced Wadge games to analyze Wadge reducibility for the Baire space. Given two set of reals $A, B$ in the Baire space, the *Wadge game* $G_{\mathrm{W}}(A, B)$ is played by two players I and II in the following way: I plays a natural number $x_0$, then II plays a natural number $y_0$ or she can pass, then I plays again a natural number $x_1$ and II plays a natural number or she can pass. After $\omega$ rounds of this process, they will produce sequences $x = \langle x_n \mid n \in \omega\rangle$ and $y = \langle y_n \mid n < i\rangle$ where $i \leq \omega$. Player II wins if $i = \omega$ (i.e., player II plays natural numbers infinitely often) and $x \in A \iff y \in B$. Otherwise player I wins. It is easy to see that $A \leq^{{}^\omega\omega}_{\mathrm{W}} B$ if and only if player II has a winning strategy in the Wadge game $G_{\mathrm{W}}(A, B)$. Since Wadge games can be easily simulated by Gale-Stewart games, under AD, we can conclude the following:

**Theorem 1.15.1** (Wadge's Lemma)**.** Assume AD and let $A, B$ be two sets of reals in the Baire space. Then either $A \leq^{{}^\omega\omega}_{\mathrm{W}} B$ or $B \leq^{{}^\omega\omega}_{\mathrm{W}} A^{\mathrm{c}}$ holds.

*Proof.* Suppose $A \not\leq^{{}^\omega\omega}_{\mathrm{W}} B$. Then by the above observation, player I has a winning strategy in the game $G_{\mathrm{W}}(A, B)$. But using this strategy, player II can win the game $G_{\mathrm{W}}(B, A^{\mathrm{c}})$ because the negation of $x \in A \iff y \in B$ is the same as

$y \in B \iff x \in A^{\mathrm{c}}$. Hence player II has a winning strategy in the game $G_{\mathrm{W}}(B, A^{\mathrm{c}})$ and $B \leq_{\mathrm{W}}^{\omega_\omega} A^{\mathrm{c}}$. □

By the above theorem, we can deduce that the Wadge order $\leq_{\mathrm{W}}^{\omega_\omega}$ is almost linear in the following sense: Let $X$ be a topological space and $A$ be a subset of $X$. We say $A$ is *selfdual* if $A \leq_{\mathrm{W}}^{X} A^{\mathrm{c}}$ (equivalently $A^{\mathrm{c}} \leq_{\mathrm{W}}^{X} A$) and *non-selfdual* if $A \not\leq_{\mathrm{W}}^{X} A^{\mathrm{c}}$ (equivalently $A^{\mathrm{c}} \not\leq_{\mathrm{W}}^{X} A$). Let $A$ be a selfdual set of reals and $B$ be a set of reals in the Baire space. Then either i) $B <_{\mathrm{W}}^{\omega_\omega} A$, ii) $B \equiv_{\mathrm{W}}^{\omega_\omega} A$, or iii) $A <_{\mathrm{W}}^{\omega_\omega} B$ holds. Let $A$ be a non-selfdual set of reals and $B$ be a set of reals in the Baire space. Then either i) $B <_{\mathrm{W}}^{\omega_\omega} A$ and $B <_{\mathrm{W}}^{\omega_\omega} A^{\mathrm{c}}$, ii) $B \equiv_{\mathrm{W}}^{\omega_\omega} A$, iii) $B \equiv_{\mathrm{W}}^{\omega_\omega} A^{\mathrm{c}}$, or iv) $A <_{\mathrm{W}}^{\omega_\omega} B$ and $A^{\mathrm{c}} <_{\mathrm{W}}^{\omega_\omega} B$ holds.

Donald Martin and Leonard Monk proved that the Wadge order $\leq_{\mathrm{W}}^{\omega_\omega}$ is wellfounded. Hence we can measure the complexity of sets of reals via ordinals by taking their rank in the Wadge order.

**Theorem 1.15.2** (Martin and Monk)**.** Assume AD+DC$_{\mathbb{R}}$. Then the Wadge order $\leq_{\mathrm{W}}^{\omega_\omega}$ is wellfounded.

*Proof.* See, e.g., [83, Theorem 2.2]. □

The above two theorems are essential parts of the basic theory of the Wadge order for the Baire space. In Chapter 5, we show that both theorems fail for the Wadge order for the real line.