

# How (Not) to Raise Money

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## Abstract

We consider auctions used to raise money for a public good. We show that winner-pay auctions are inept fund-raising mechanisms because of the positive externality bidders forgo if they top another's high bid. Revenues are suppressed as a result and remain finite even when bidders value a dollar donated to the public good the same as a dollar kept. This problem does not occur in "all-pay" auctions where bidders have to pay irrespective of whether they win or lose. We prove that the  $(k + 1)^{th}$ -price all-pay auction revenue dominates the  $k^{th}$ -price all-pay auction, and that the amount raised increases when bidders derive more benefit from the public good. Keeping the auction format fixed, an increase in the number of bidders may *decrease* revenues as low bids start resembling voluntary contributions, causing low-value bidders to "free ride." Fund-raisers may therefore benefit from restricting access to "a happy few." Finally, we investigate the fund-raising properties of the dynamic  $n$ -player "war of attrition."

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## 1. Introduction

It is well known that mechanisms used to finance public goods may yield disappointing revenues because they suffer from a free-rider problem. For example, simply asking for voluntary contributions generally results in underprovision of the public good (see Bergstrom, Blume, and Varian, 1986, and Andreoni, 1988).<sup>1</sup> From a purely theoretical viewpoint, Groves and Ledyard (1977) solved the decentralized public goods provision problem by identifying an optimal tax mechanism that overcomes the free-rider problem. This mechanism, however, is mainly of theoretical interest. In contrast, lotteries and auctions are frequently employed as practical means to raise money for a public good. Even the voluntary contribution method is commonly observed in practice, despite its inferior theoretical properties. The co-existence of these alternative formats raises the obvious question: “which method is superior at raising money?”

Morgan’s (2000) work constitutes an important first step in answering this question. He studies the fund-raising properties of lotteries and makes the point that the public good free-rider problem is mitigated by the negative externality present in lotteries. This negative externality occurs because an increase in the number of lottery tickets that one person buys lowers others’ chances. As a result, lotteries have a net positive effect on the amount of money raised vis-a-vis voluntary contributions. A similar negative externality emerges in auctions, where a bidder’s probability of winning is negatively affected by more aggressive bidding behavior of others.

A priori, most economists would probably expect that auctions are superior to lotteries in terms of raising money. Unlike lotteries, auctions are efficient; in equilibrium, the bidder with the highest value for the object places the highest bid and wins. This efficiency property promotes aggressive bidding and therefore has a positive effect on the amount of money raised. Ostensibly this suggests that lotteries are suboptimal and should be disregarded in favor of auctions. However, fund-raisers that use lotteries, or “raffles,” are quite prevalent, which casts doubt on the empirical validity of this conclusion.

The flaw in the above argument stems from a separate, antagonistic problem that emerges in auctions where only the winner pays. When a bidder tops the highest bid of others, she wins the object but concurrently eliminates the benefit she would have derived from free-riding off that (previously highest) bid. The possible elimination of positive

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<sup>1</sup>See Ledyard (1995) for a survey of experimental results.

externalities associated with others' high bids exerts downward pressure on equilibrium bids in winner-pay auctions. Notice that this feature does not occur in lotteries where all non-winning tickets are paid.

In this paper we determine the extent to which bids are suppressed in winner-pay auctions and find that these formats yield dramatically low revenues. Even when bidders value \$1 given to the public good the same as \$1 for themselves, revenues are finite. In contrast, lotteries generate infinite revenue in this case, notwithstanding their inefficiency. Though extreme, this example suggests that it may make sense to use lotteries instead of winner-pay auctions to raise money.

The main virtue of lotteries in the above example, i.e. that all tickets are paid, can be incorporated into an efficient mechanism. "All-pay" auctions, where everyone pays irrespective of whether they win or lose, avoid the problems inherent in winner-pay auctions. Since they are also efficient, they are prime candidates for superior fund-raising mechanisms. In this paper, we prove this intuition correct. We introduce a general class of all-pay auctions, rank their revenues, and demonstrate how they dominate winner-pay auctions. Furthermore, we show the optimal fund-raising mechanism is among the all-pay formats we consider.

Adding an all-pay element to fund-raisers seems very natural. Indeed, the popularity of lotteries as means to finance public goods indicates that people are willing to accept the obligation to pay even though they may lose. Presumably, the costs of losing the lottery are softened because they benefit a good cause. In some cases, it may even be awkward to not collect all bids. Suppose, for instance, that a group of parents submit sealed bids for a set of prizes that are auctioned, knowing that the proceeds benefit their children's school. Some parents may be offended when told they contributed nothing because they lost the auction, or, in other words, because their contributions were not high enough.<sup>2</sup>

This paper is organized as follows. In the next section, we consider winner-pay auctions where bidders derive utility from the revenue they generate. We build on the work of Engelbrecht-Wiggans (1994), who first studied such auctions for the two-bidder case. We extend his finding that second-price auctions revenue dominate first-price auctions by showing that both are dominated by a third-price auction. The main point of section 2,

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<sup>2</sup>In some cases it may be more natural that only the winner pays. Consider, for example, a fundraiser organized by the University of Virginia to renovate their Economics building on the lawn. If the prize reward consists of renaming Rouss Hall after the highest contributor (provided a minimum level of contribution is met), it seems natural to assume that only the winner would be willing to pay.

however, is punctuated by a novel revenue equivalence result for the case when people are indifferent between a dollar donated and a dollar kept. We show that the amount of money generated in this case is surprisingly low.

In section 3 we introduce a general class of all-pay auctions. We show how these formats avoid the shortcomings of winner-pay auctions and rank their revenues.<sup>3</sup> In section 4 we investigate how the amount raised changes as bidders derive greater benefit from the public good and when there is more competition for the auction's prize. We find that an increase in the number of bidders may decrease revenues as low bids start resembling voluntary contributions. Fund-raisers can therefore benefit from limiting the number of contestants.<sup>4</sup> In section 5 we consider a dynamic variant of the all-pay auction, i.e. the generalized "war of attrition." Our approach follows that of Bulow and Klemperer (1999a) who introduced this auction format and first determined its equilibrium properties.

Our work is related to several papers that consider auctions in which losing bidders gain by driving up the winner's price. In takeover situations, for example, losing bidders who own some of the target's shares ("toeholds") receive payoffs proportional to the sales price (e.g. Singh, 1998; Bulow, Huang, and Klemperer, 1999). A related topic is the dissolution of a partnership, as analyzed by Cramton, Gibbons, and Klemperer (1987). Graham and Marshall (1987) and McAfee and McMillan (1992) study "knockout auctions" where every member of a bidding ring receives a payment proportional to the winning bid. Other examples include creditors bidding in bankruptcy auctions (Burkart, 1995), and heirs bidding for a family estate (Engelbrecht-Wiggans, 1994). These papers restrict attention to standard winner-pay auctions, i.e. the first-price, second-price, and English auction, while our analysis mainly focuses on a class of all-pay formats. Another important difference is that in our paper, one bidder's benefit from the auction's revenue does not diminish its value to others.

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<sup>3</sup>A related paper is that of Krishna and Morgan (1997a) who study first-price and second-price all-pay auctions. They show that when bidders' values are affiliated, revenue equivalence does not hold. Baye, Kovenock, and de Vries (1998, 2000) also study these all-pay formats with affiliated values and consider their applications in a wide variety of two-person contests, including patent races, lobbying, and litigation.

<sup>4</sup>Bulow and Klemperer (1999b) show that in common-value auctions the "Law of Demand," i.e. that greater demand results in higher prices, may fail due to the "winner's curse." The intuition is that winning is more informative when there are more rivals, which warrants more conservative bidding. This effect can outweigh the increase in competition, and thus yield lower prices. Krishna and Morgan (1997b) demonstrate how a decrease in competition due to a merger can raise the auction's revenue because the merged party possesses better information that helps them to avoid the winner's curse. Winner's curse considerations do not play a role in the private-value auctions studied in this paper. Here the anti-competitive effects from an increase in the number of bidders are due to a "free-rider" problem.

The paper most closely related to ours is Engers and McManus (2001), who consider “charity auctions.”<sup>5</sup> They consider first-price and second-price auctions, and extend Engelbrecht-Wiggans’ (1994) ranking to the  $n$ -bidder case. Our results, however, demonstrate that (i) there exist other winner-pay formats that revenue-dominate the second-price auction, and (ii) all winner-pay formats are poor fund-raisers. Engers and McManus (2001) also find that a first-price all-pay auction yields a higher revenue than a first-price auction, but that its revenue may be more or less than that of a second-price auction. Our paper provides a framework to explain these results. We introduce a general class of all-pay auctions and rank their revenues. In particular, we show that the first-price all-pay auction is dominated by all other all-pay formats. An easy corollary to our analysis is that the lowest-price all-pay auction is the optimal fund-raising mechanism (see also Maasland and Onderstal, 2001, who derive a similar result in a different context). Figures 1 and 2 clearly demonstrate the difference in fund-raising efficacy between the formats studied here and those considered by Engers and McManus (2001).

Finally, our work is related to that of Jehiel, Moldovanu, and Stacchetti (1996) who consider auctions in which the winning bidder imposes an individual-specific negative externality on the losers. In this case, it may be in a bidder’s interest not to participate in the auction (Jehiel and Moldovanu, 1996). The reason is that participation can affect the identity of the final winner in a way that is unprofitable to the bidder. The positive externalities present in fund-raisers may also cause bidders not to participate. For instance, in the equilibrium of the war of attrition discussed in section 5, all but two bidders drop out of the auction at zero prices. One difference is that the magnitudes of the externalities that occur in fund-raisers are endogenously determined, while those considered by Jehiel, Moldovanu, and Stacchetti (1996) are fixed.

## 2. Winner-Pay Auctions

In this section we consider “standard” auctions in which only the winner has to pay. We start with a simple three-bidder example to illustrate and extend previous results in the literature and, more importantly, to demonstrate that winner-pay auctions are poor at raising money. We underscore our point by proving a novel revenue equivalence result:

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<sup>5</sup>Engers and McManus (2001) mention the annual wine sale organized by the Hospices de Beaune as an example. This wine sale benefits several local charities in the Burgundy region.

when bidders value \$1 given to the public good the same as \$1 for themselves, the revenue generated is identical for all winner-pay auctions. Most importantly, however, revenue in this case is only the expected value of the highest order statistic.

Consider three bidders who compete for a single indivisible object. Suppose bidders' values are independently and uniformly distributed on  $[0, 1]$  and the auction's proceeds accrue to a public good that benefits the bidders. We assume a particularly simple linear "production technology" where every bidder receives  $\alpha R$  from \$1 spent on the public good. Hence, bidders in the auction receive  $\alpha R$  in addition to their usual payoffs, where  $R$  is the auction's revenue. Engelbrecht-Wiggans (1994) first studied auctions where bidders benefit from the auction's revenue. He derived the optimal bids for the first-price and second-price auctions when there are two bidders. His answers, however, can easily be extrapolated to our three-bidder example. In the first-price auction the optimal bids are given by<sup>6</sup>

$$B_{1,3}(v) = \frac{2v}{3 - \alpha}, \quad (2.1)$$

where the first subscript indicates the auction format and the second the number of bidders. Similarly, the optimal bids of the second-price auction are<sup>7</sup>

$$B_{2,3}(v) = \frac{v + \alpha}{1 + \alpha}. \quad (2.2)$$

Since the bidding functions are linear, revenues follow by evaluating (2.1) and (2.2) at the expected value of the highest and second-highest of three draws:  $R_{1,3} = 3/(6 - 2\alpha)$  and  $R_{2,3} = (1 + 2\alpha)/(2 + 2\alpha)$ . Note that  $R_{1,3} = R_{2,3} = 1/2$  when  $\alpha = 0$ , which is the usual revenue equivalence result, and  $R_{1,3} = R_{2,3} = 3/4$  when  $\alpha = 1$ . For intermediate values of  $\alpha$  we have  $R_{2,3} > R_{1,3}$ , a result first shown by Engelbrecht-Wiggans (1994) for the case of two bidders.

This suggests that the second-price auction should be preferred for fund-raising. The result is of limited interest, however, as it is easy to find other formats that revenue

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<sup>6</sup>Consider a bidder with value  $v$  who bids as if he has value  $w$  and who faces rivals that bid according to  $B_{1,3}(\cdot)$ . The expected payoff is:  $\pi^e(B_{1,3}(w)|v) = (v - (1 - \alpha)B_{1,3}(w))w^2 + \alpha \int_w^1 B_{1,3}(y)dy^2$ . It is easy to verify that the first-order condition for profit maximization is:  $\partial_w \pi^e(B_{1,3}(w)|v) = 2(v - w)w$ , so it is optimal for a bidder with value  $v$  to bid  $B_{1,3}(v)$ .

<sup>7</sup>Consider a bidder with value  $v$  who bids as if he has value  $w$  and who faces rivals that bid according to  $B_{2,3}(\cdot)$ . The expected payoff is:  $\pi^e(B_{2,3}(w)|v) = \int_0^w (v - (1 - \alpha)B_{2,3}(z)) dz^2 + 2\alpha B_{2,3}(w)(1 - w)w + 2\alpha \int_w^1 B_{2,3}(z)(1 - z) dz$ . The first-order condition for profit maximization is:  $\partial_w \pi^e(B_{2,3}(w)|v) = 2(v - w)w$ , so it is optimal for a bidder with value  $v$  to bid  $B_{2,3}(v)$ .

dominate the second-price auction. Consider, for instance, a third-price auction in which the winner has to pay the third-highest price. The optimal bids for this format are given by<sup>8</sup>

$$B_{3,3}(v) = \frac{2(v - \alpha)}{1 - \alpha} + \frac{\alpha}{2(1 - \alpha)} \left(1 + \sqrt{1 + 8/\alpha}\right) (1 - v)^{\frac{1}{2}(\sqrt{1+8/\alpha}-1)} \quad (2.3)$$

with corresponding revenue

$$R_{3,3} = \frac{1 - \alpha + 3\alpha^2 \left(3 - \sqrt{1 + 8/\alpha}\right)}{2(1 - \alpha)(1 - 3\alpha)}. \quad (2.4)$$

Also the third-price auction yields revenue  $1/2$  when  $\alpha = 0$  as dictated by the Revenue Equivalence Theorem, and  $3/4$  when  $\alpha = 1$ . For intermediate values of  $\alpha$ , the third-price auction results in higher revenues than the other two formats, as shown in Figure 1.

The revenue-equivalence result for  $\alpha = 1$  holds quite generally. Consider a more general setting with  $n$  bidders whose values are identically and independently distributed on  $[0,1]$  according to some distribution  $F(\cdot)$ . To derive the amount of money raised when  $\alpha = 1$ , we focus on the first-price auction, for which it is a weakly dominant strategy to bid one's value. To verify this claim, consider bidder 1 and let  $b_{-1} = \max_{i=2,\dots,n} \{b_i\}$  denote the highest of the others' bids. When  $v_1 \geq b_{-1}$ , bidder 1's expected payoff when she bids her value is  $v_1$ , and she gets the same payoff for all bids with which she wins. When she bids too low and loses the auction, however, her expected payoff becomes  $b_{-1} < v_1$ . In other words, bidder 1 never gains but may lose when choosing a bid different from her value. Similarly, when  $v_1 < b_{-1}$ , bidder 1's expected payoff when she bids her value is  $b_{-1}$ . This payoff is the same for all bids with which she loses, but a bid that would lead her to win the auction yields a lower expected payoff equal to  $v_1$ . So it is optimal to bid one's value and the auction's revenue is simply the expected value of the highest order statistic. We next show that other winner-pay formats yield the same revenue (see the Appendix for a proof). Let  $Y_k^n$  denote the  $k^{th}$  highest order statistic from  $n$  value draws.

**Proposition 1.** *Any winner-pay auction yields revenue  $E(Y_2^n)$  for  $\alpha = 0$  and  $E(Y_1^n)$  for  $\alpha = 1$ .*

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<sup>8</sup>Consider a bidder with value  $v$  who bids as if he has value  $w$  and who faces rivals that bid according to  $B_{3,3}(\cdot)$ . The expected payoff is:  $\pi^e(B_{3,3}(w)|v) = vw^2 - 2(1 - \alpha) \int_0^w B_{3,3}(z)(w - z) dz + 2\alpha(1 - w) \int_0^w B_{3,3}(z) dz + \alpha B_{3,3}(w)(1 - w)^2$ . The first-order condition for profit maximization is:  $\partial_w \pi^e(B_{3,3}(w)|v) = 2(v - w)w$ , so it is optimal for a bidder with value  $v$  to bid  $B_{3,3}(v)$ .

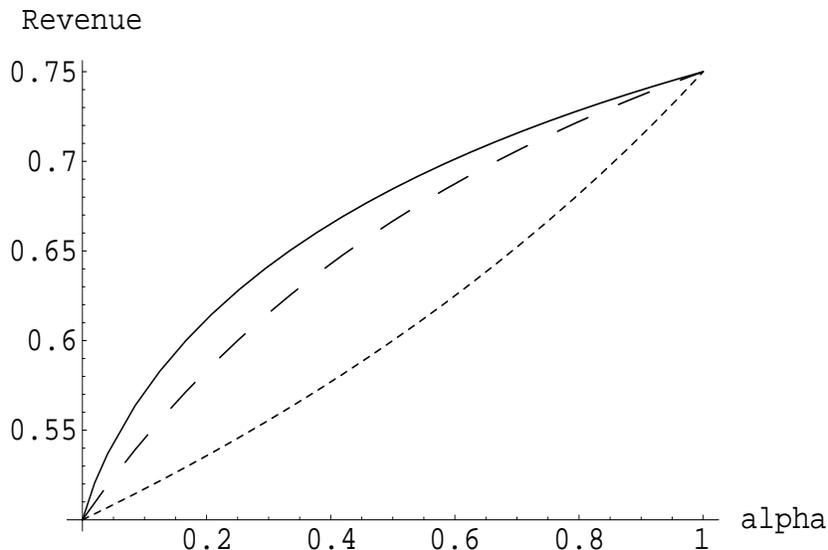


Figure 1: *Revenues from a first-price (short dashes), second-price (long dashes), and third-price (solid line) auction with three bidders for  $0 \leq \alpha \leq 1$ .*

This revenue equivalence result is somewhat interesting in its own right,<sup>9</sup> but the main point is that winner-pay auctions are ineffective at for raising money. Revenues are increasing with  $\alpha$  (see Figure 1) so the highest revenue should be expected for  $\alpha = 1$ . In this extreme case bidders are indifferent between keeping a \$1 for themselves or giving it to the public good, yet revenues are only  $E(Y_1^n)$ . Clearly, one should be able to raise more money when the benefit from the public good is this high. In a lottery, for instance, bidders have every incentive to buy an infinite number of tickets since the price they pay is effectively zero when  $\alpha$  tends to 1. In fact, even a voluntary contribution mechanism may yield infinite revenues in this limit case.

### 3. All-Pay Auctions

The problem with winner-pay auctions is one of opportunity costs. A high bid by one

<sup>9</sup>Klemperer (1999, 2000) provides a survey of revenue equivalence and demonstrates its use and applications in various areas of economics.

bidder imposes a positive externality on all others, who forgo this positive externality if they top the high bid. Bids are suppressed as a result, and so are revenues. This would not occur in situations where every bidder pays, regardless of whether they win or lose. In this section, we consider the class of  $k^{\text{th}}$ -price all-pay auctions where the highest bidder wins, the  $n - k$  lowest bidders pay their own bid, and the  $k$  highest bidders pay the  $k^{\text{th}}$  highest bid.<sup>10</sup>

The easiest way to derive the bidding functions is to consider the marginal benefits and costs of increasing one's bid, which have to be equal in equilibrium. The positive effects of increasing one's bid from  $B(v)$  to  $B(v + \epsilon) \approx B(v) + \epsilon B'(v)$  are twofold. First, it might lead one to win the auction that otherwise would have been lost. This occurs when the highest of the others' values falls between  $v$  and  $v + \epsilon$ , which happens with probability  $(n - 1)\epsilon f(v)F(v)^{n-2}$ . Second, an increase in one's bid raises revenue by  $\epsilon B'(v)$  if there are at least  $k - 1$  higher bids and by an additional  $\epsilon(k - 1)B'(v)$  if there are exactly  $k - 1$  higher bids. Let  $F_{Y_{k-1}^{n-1}}$  denote the distribution function of the  $(k - 1)^{\text{th}}$  order statistic from  $n - 1$  draws with the convention  $F_{Y_0^{n-1}}(v) = 0$  and  $F_{Y_n^{n-1}}(v) = 1$ . The probability that there are at least  $k - 1$  bidders with values higher than  $v$  is  $1 - F_{Y_{k-1}^{n-1}}(v)$ . Similarly, the probability that there are exactly  $k - 1$  such bidders is  $(1 - F_{Y_{k-1}^{n-1}}(v)) - (1 - F_{Y_k^{n-1}}(v)) = F_{Y_k^{n-1}}(v) - F_{Y_{k-1}^{n-1}}(v)$ . Combining the different terms, the expected marginal benefit can be written as  $\epsilon$  times

$$(n - 1)vf(v)F(v)^{n-2} + \alpha B'(v)\{(1 - F_{Y_{k-1}^{n-1}}(v)) + (k - 1)(F_{Y_k^{n-1}}(v) - F_{Y_{k-1}^{n-1}}(v))\}.$$

Likewise, the marginal cost is  $\epsilon B'(v)$  when there are at least  $k - 1$  higher bids, and the

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<sup>10</sup>Morgan (2000) considers lotteries as ways to fund public goods. Lotteries have an “all-pay” element in that losing tickets are not reimbursed. A major difference, of course, is that lotteries are not, in general, efficient, i.e. they do not necessarily assign the object for sale to the bidder that values it the most. This is not an issue for the symmetric complete information case Morgan studies, but does play a role when bidders' values are private information as in our model. Although no closed form solutions exist for such a “private value lottery,” it seems intuitive that its inefficiency would cause revenues to be lower than those of a first-price all-pay auction for example. Even in the complete information case, lotteries tend to generate less revenues because the highest bidder is not necessarily the winner. To make this more precise, consider Morgan's setup where the prize is worth  $V$  to all bidders. In the lottery the optimal number of tickets to buy is  $(n - 1)V/(n^2(1 - \alpha))$ , resulting in a revenue of  $(n - 1)V/(n(1 - \alpha))$ . In the first-price all-pay auction, the symmetric Nash equilibrium is in mixed-strategies. The equilibrium distribution of bids is  $F(b) = (b/((1 - \alpha)V))^{1/(n-1)}$ , and the resulting revenue is  $V/(1 - \alpha)$ , which exceeds that of a lottery for all  $n$ . Note, however, that the revenue of a lottery diverges when  $\alpha \rightarrow 1$  unlike that of a first-price auction, for instance, where the unique symmetric equilibrium entails bidding  $V$ , and hence revenue is  $V$ , for all  $\alpha \leq 1$ .

expected marginal cost is therefore  $\epsilon$  times

$$B'(v)(1 - F_{Y_{k-1}^{n-1}}(v)).$$

The optimal bids can be derived by equating marginal cost to marginal benefits. The resulting differential equation for the bidding function has a well-defined solution when  $\alpha < 1/k$ , a case we study first.

**Proposition 2.** *When  $\alpha < 1/k$ , the optimal bids of the  $k^{\text{th}}$ -price all-pay auction are*

$$B_{k,n}^{AP}(v) = \int_0^v \frac{(n-1)zf(z)F(z)^{n-2}}{(1-k\alpha)(1-F_{Y_{k-1}^{n-1}}(z)) + \alpha(k-1)(1-F_{Y_k^{n-1}}(z))} dz, \quad (3.1)$$

and revenues are

$$R_{k,n}^{AP} = \int_0^1 \frac{z(1-F_{Y_{k-1}^{n-1}}(z))}{(1-k\alpha)(1-F_{Y_{k-1}^{n-1}}(z)) + \alpha(k-1)(1-F_{Y_k^{n-1}}(z))} dF_{Y_2^n}(z). \quad (3.2)$$

The first-price and second-price all-pay auctions have been studied by Krishna and Morgan (1997a) for the case  $\alpha = 0$ . They consider a model with affiliated values (see Milgrom and Weber, 1982) and derive the following ranking of revenues:

$$\begin{array}{ccc} R_{2,n}^{AP} & > & R_{1,n}^{AP} \\ \vee & & \vee \\ R_{2,n} & > & R_{1,n} \end{array}$$

where the bottom line was first proved by Milgrom and Weber (1982). Revenue equivalence is violated in their model because bidders' values are not independent. In contrast, when bidders care about the auction's revenue, the amount raised varies across formats because of the different incentives these formats impose. In section 4 we prove a general ranking of all-pay formats when bidders' private values are independent and  $\alpha > 0$ .

Note from (3.2) that the first-price all-pay auction's revenue,  $R_{1,n}^{AP} = E(Y_2^n)/(1-\alpha)$ , diverges in the limit when  $\alpha$  tends to 1 (like that of a lottery). So for high values of  $\alpha$ , the first-price all-pay auction revenue dominates all winner-pay formats. Similarly, the revenue of the  $n^{\text{th}}$ -price all-pay auction,  $R_{n,n}^{AP} = E(Y_2^n)/(1-n\alpha)$ , diverges when  $\alpha \rightarrow 1/n$ .

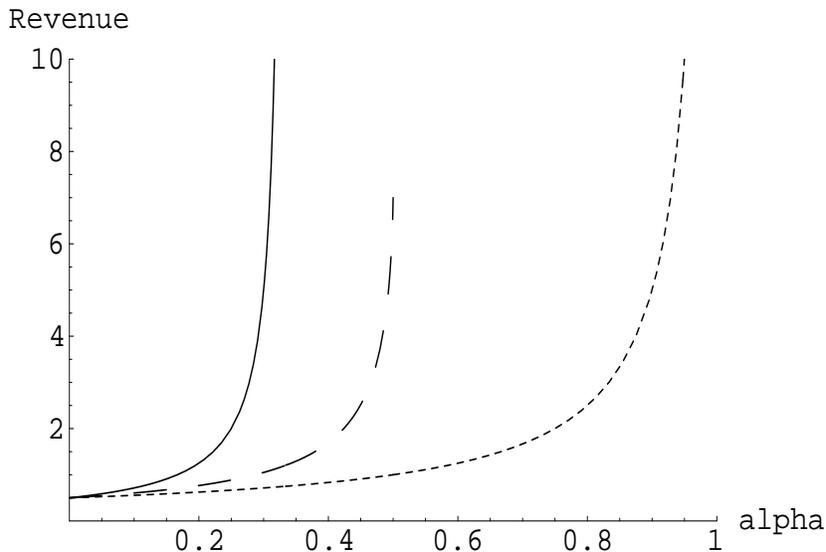


Figure 2: Revenues from a first-price (short dashes), second-price (long dashes), and third-price (solid line) all-pay auction with three bidders for  $0 \leq \alpha \leq 1$ .

For the intermediate cases  $k = 2, \dots, n - 1$ , however, revenues are finite when  $\alpha = 1/k$  and diverge for  $\alpha > 1/k$ .<sup>11</sup>

**Proposition 3.** *The  $k^{\text{th}}$ -price all-pay auction raises infinite revenues when  $\alpha > 1/k$ .*

This divergence is, of course, a consequence of our assumption of a linear production technology for the public good. If the marginal benefit of the public good is sufficiently decreasing (instead of being constant), revenues would be finite. We keep the constant marginal benefit assumption not because we claim to have found the perfect “money pump.” Rather it is the best way to demonstrate how much worse winner-pay auctions are in terms of raising money compared to all-pay formats.

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<sup>11</sup>The limit values are given by

$$\lim_{\alpha \rightarrow 1/k} R_{k,n}^{AP} = \frac{k}{k-1} \int_0^1 z \left( \frac{1 - F_{Y_{k-1}^{n-1}}(z)}{1 - F_{Y_k^{n-1}}(z)} \right) dF_{Y_2^n}(z), \quad k = 2, \dots, n-1.$$

The main results of this section are illustrated in Figure 2, which shows the revenues of a first-price, second-price, and third-price all-pay auction when there are three bidders whose values are uniformly distributed. The revenues of the first-price and third-price all-pay auction diverge when  $\alpha$  tends to 1 and  $1/3$  respectively, and the limit value of the revenue of a second-price all-pay auction when  $\alpha \rightarrow 1/2$  is 7.<sup>12</sup>

Comparing Figures 1 and 2 illustrates the extent to which revenues are suppressed in winner-pay auctions, and yields the following ranking of revenues for  $0 < \alpha < 1$ :

$$\begin{array}{ccccc} R_{3,3}^{AP} & > & R_{2,3}^{AP} & > & R_{1,3}^{AP} \\ \vee & & \vee & & \vee \\ R_{3,3} & > & R_{2,3} & > & R_{1,3} \end{array}$$

The above ranking is strict in the sense that the first-price all-pay auction revenue dominates the first-price auction for all  $0 < \alpha < 1$  but not, for instance, the third-price auction.

## 4. Comparative Statics

In this section we study how revenues of the all-pay auctions depend on the level of altruism,  $\alpha$ , the number of bidders,  $n$ , and the choice of format. The corresponding results for the standard first-price and second-price auctions are intuitive: revenues are increasing in  $\alpha$  and  $n$ .<sup>13</sup> Moreover, the second-price auction generates more revenues than

<sup>12</sup>This can be verified by considering the extreme value draws of 0 and 1. In the proof of Proposition 1 in the Appendix we show that revenue can be written as  $R = (E(Y_2^n) - n\pi^*(0))/(1 - n\alpha)$ , where  $\pi^*(0)$  is the expected profit of a bidder with value 0. This bidder loses for sure and only derives utility from the amount contributed to the public good. Both others pay  $E(B_{2,3}^{AP}(Y_2^3) | Y_3^3 = 0) = E(B_{2,3}^{AP}(Y_2^2))$  so  $\pi^*(0) = 2\alpha E(B_{2,3}^{AP}(Y_2^2)) = E(B_{2,3}^{AP}(Y_2^2))$  since  $\alpha = 1/2$ . Similarly, a bidder with a value of 1 wins for sure and receives 1. Moreover, a bidder with a value of 1 also receives  $\alpha E(B_{2,3}^{AP}(Y_3^3) | Y_1^3 = 1) + 2\alpha E(B_{2,3}^{AP}(Y_2^3) | Y_1^3 = 1) = \frac{1}{2}E(B_{2,3}^{AP}(Y_2^2)) + E(B_{2,3}^{AP}(Y_1^2))$  and pays  $E(B_{2,3}^{AP}(Y_1^2))$ . Hence,  $\pi^*(1) = 1 + \frac{1}{2}E(B_{2,3}^{AP}(Y_2^2))$ , or  $\pi^*(1) = 1 + \frac{1}{2}\pi^*(0)$ . A simple Envelope Theorem argument establishes an alternative link between the expected profits of the lowest and highest-value bidders:  $\pi^*(1) = \pi^*(0) + 1/3$ , see the Appendix. Taken together, these imply  $\pi^*(0) = 4/3$  and a resulting revenue of  $R = (1/2 - 4)/(1 - 3/2) = 7$ .

<sup>13</sup>The optimal bids in the first-price auction are

$$B_{1,n}(v) = \int_0^v z dF_\alpha^n(z|v),$$

where  $F_\alpha^n(z|v) \equiv (F(z)/F(v))^{\frac{n-1}{1-\alpha}}$ , see Goeree and Turner (2000). Note that  $F_\alpha^n$  first-order stochastically

the first-price auction. This result was first derived by Engelbrecht-Wiggans (1994) for the case of two bidders. Bulow, Huang, and Klemperer (1999) extend Engelbrecht-Wiggans' finding to situations with both private and common values and Engers and McManus (2001) allow for an arbitrary number of bidders.<sup>14</sup> Based on the example in section 2, we believe these results hold more generally for  $k^{\text{th}}$ -price winner-pay auctions. In particular, we conjecture that the  $(k + 1)^{\text{th}}$ -price auction generates more revenue than the  $k^{\text{th}}$ -price auction, for  $k = 1, \dots, n - 1$ .

We do not pursue this issue further since standard winner-pay auctions are poor fundraisers (see Proposition 1). Instead we focus on all-pay auctions for which the comparative statics are easy to determine.

**Proposition 4.** *The revenue of the  $k^{\text{th}}$ -price all-pay auction is increasing in  $\alpha$  but may be decreasing in  $n$ .*

The first result follows by differentiating (3.2) with respect to  $\alpha$ . The numbers effect could have been anticipated. For instance, we know that the revenue of the second-price auction diverges in the limit  $\alpha \rightarrow 1/2$  when there are two bidders while it remains finite with more bidders.

Figure 3 illustrates the numbers effect for the second-price all-pay auction. From this figure it is clear that more competition raises revenues when the marginal return from the public good,  $\alpha$ , is low. As bidders derive more utility from the public good, however, additional bidders may lower revenues. The intuition behind this result can be made clear by considering the second-price all-pay auction. With two bidders, the loser knows her bid determines the price paid by the winner, which provides the loser with an incentive to drive up the price. This is not true with three or more bidders, however, in which case

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dominates  $F_{\alpha'}^n$  for all  $\alpha \geq \alpha'$ , and  $F_{\alpha}^n$  first-order stochastically dominates  $F_{\alpha}^{n'}$  for all  $n \geq n'$ . An increase in  $\alpha$  or  $n$  thus yields higher bids and higher revenues. The optimal bids for the second-price auction are

$$B_{2,n}(v) = \int_v^1 z dG_{\alpha}(z|v),$$

where  $G_{\alpha}(z|v) \equiv 1 - \left(\frac{1-F(z)}{1-F(v)}\right)^{\frac{1}{\alpha}}$ , independent of  $n$ .  $G_{\alpha}$  first-order stochastically dominates  $G_{\alpha'}$  for all  $\alpha \geq \alpha'$ , and an increase in  $\alpha$  results in higher bids and higher revenues. Bids in the second-price auction are independent of the number of bidders, but the expected value of the second-highest order statistic increases with  $n$  and so does the auction's revenue.

<sup>14</sup>See also Goeree and Turner (2000) who consider a setting with  $n$  bidders and private and common values. The proof in Engers and McManus (2001) is similar to that in Goeree and Turner (2000).

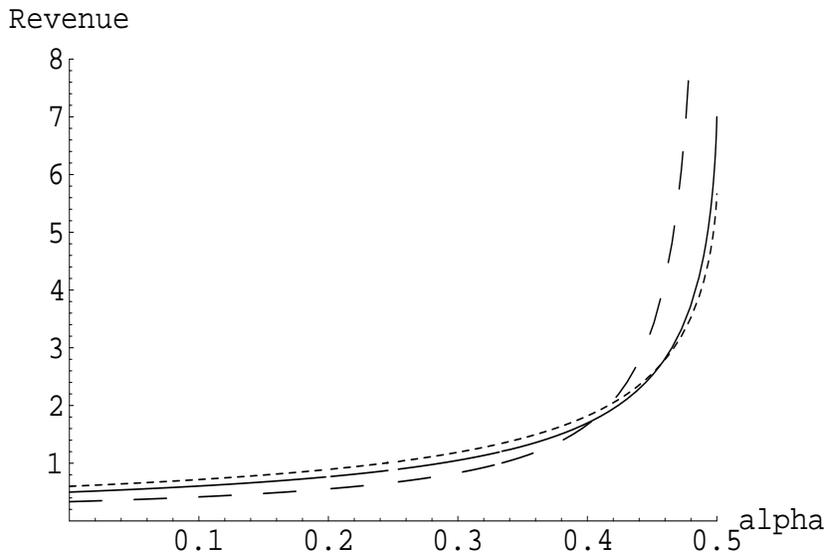


Figure 3: *Revenues from a second-price all-pay auction with two (long dashes), three (solid line), and four (short dashes) bidders for  $0 \leq \alpha \leq 1/2$ .*

the  $n - 2$  lowest bids are paid only by the losers. Hence there are no positive externalities associated with such bids, which become like “voluntary contributions” to the public good. This suppresses bids of low-value bidders, who “free ride” on the revenues generated by the bidders with higher values. Fund-raisers may thus benefit from “limiting competition” and restricting access to “a happy few.”

Even though the revenue of the  $k^{\text{th}}$ -price all-pay auction may decrease with the number of bidders, it exceeds that of any winner-pay auction even when the number of bidders becomes arbitrarily large. To see this, note that the revenue of the first-price all-pay auction,  $E(Y_2^n)/(1 - \alpha)$ , tends to  $1/(1 - \alpha)$  as  $n$  grows large. Likewise, the revenue of an  $n^{\text{th}}$ -price all-pay auction tends to  $1/(1 - n\alpha)$  for large  $n$ . In Proposition 5 below we show that the revenue for the  $k^{\text{th}}$ -price all-pay auction is increasing in  $k$ . Hence the amount of money raised by any of the all-pay formats lies between  $1/(1 - \alpha)$  and  $1/(1 - n\alpha)$  for large  $n$ . In contrast, the revenue of a winner-pay format lies between  $E(Y_2^n)$  and  $E(Y_1^n)$  (see Proposition 1), and thus limits to 1 as  $n$  grows large for all values of  $\alpha$ .

Next, we study how the choice of format affects revenues. Recall that for the standard

winner-pay auctions we conjectured that the  $(k+1)^{th}$ -price auction generates more revenue than the  $k^{th}$ -price auction. The corresponding result for the all-pay formats is proven in the Appendix.

**Proposition 5.** *The  $(k+1)^{th}$ -price all-pay auction raises more money than the  $k^{th}$ -price all-pay auction, for  $k = 1, \dots, n-1$ .*

The proof follows by showing that

$$\frac{F_{Y_k^{n-1}}(z) - F_{Y_{k-1}^{n-1}}(z)}{1 - F_{Y_{k-1}^{n-1}}(z)},$$

increases with  $k$ . In other words, the probability that exactly  $k-1$  bidders have values greater than  $z$  given that at least  $k-1$  bidders have values greater than  $z$ , increases with  $k$ .

With  $n$  bidders, the  $n^{th}$ -price all-pay auction not only revenue dominates other all-pay formats, but it is also optimal in the sense that the bidder with the lowest possible value of zero has zero expected payoffs.<sup>15,16</sup> This follows since the zero-value bidder loses for sure and also determines the price paid in the auction. Therefore, the zero-value bidder's expected payoff is  $n\alpha B_{n,n}^{AP}(0)$ , which is zero by (3.1) for all  $\alpha < 1/n$ . The intuition is that a strictly positive bid by the zero-value bidder implies that the expected value of the lowest bid is strictly positive. Since the zero-value bidder's profit is  $n\alpha - 1 < 0$  times the lowest bid, this bidder is better off bidding zero.

## 5. The War of Attrition

Like lotteries, the sealed-bid formats analyzed above are “one-shot” in nature. In this section, we consider a dynamic variant of the all-pay auction where bidders can condition their choices on the price levels at which others drop out: the  $n$ -bidder “war of attrition.” Bidders' exit strategies in this auction depend on their own value and that of the last bidder that exited. Let  $2 \leq m \leq n$  denote the number of bidders that are still active and

<sup>15</sup>A necessary condition for optimality, see Myerson (1981).

<sup>16</sup>See also Maasland and Onderstal (2001) who derive this result in a different context.

let  $v_m$  be the lowest possible value of an active bidder conditional on all other bidders thus far having followed equilibrium strategies. We write  $B_m(v|v_m)$  for the additional amount an active bidder with value  $v$  is willing to pay before dropping out given that none of the other active bidders drop out beforehand.

We first consider the situation when there are two active bidders left, in which case the war of attrition is equivalent to a sealed-bid second-price all-pay auction. Suppose the lowest possible value of an active bidder is  $v_2$ . We derive the optimal bid  $B_2(v|v_2)$  by equating the marginal cost and benefit of increasing one's bid to  $B_2(v + \epsilon|v_2)$ . Paying an extra  $\epsilon B'_2(v|v_m)$  results in winning the auction only when the rival's value lies between  $v$  and  $v + \epsilon$ , which occurs with probability  $\epsilon f(v)/(1 - F(v))$ . An additional benefit is that revenue increases by  $2\epsilon B'_2(v|v_m)$ , which pays back an extra  $2\alpha\epsilon B'_2(v|v_2)$  to the bidders. The marginal cost of staying in the auction longer is simply  $\epsilon B'_2(v|v_2)$ . Equating costs and benefits we have  $(1 - 2\alpha)B'_2(v|v_2) = f(v)/(1 - F(v))$ , so:

$$B_2(v|v_2) = \frac{1}{1 - 2\alpha} \int_{v_2}^v \frac{zf(z)}{1 - F(z)} dz, \quad (5.1)$$

which reproduces our earlier result for the second-price all-pay auction in (3.1) when  $v_2 = 0$ .

The case of more than two active bidders is less obvious. Bulow and Klemperer (1999a) study the  $n$ -player war of attrition when  $\alpha = 0$ , and show that the equilibrium involves the  $n - 2$  lowest-value bidders dropping out immediately after which the two highest-value bidders bid according to  $B_2(v|v_2)$ . Not surprisingly, a similar result holds for positive but small  $\alpha$ . The intuition is that with  $m > 2$  active bidders, dropping out  $\epsilon B'_m(v|v_m)$  later would not, to first order, affect the probability of winning. It does, however, extend the length of the bidding game with  $m$  active bidders, the marginal benefit of which is  $m\alpha B'_m(v|v_m)$ . But it also shortens the subsequent bidding game with  $m - 1$  bidders, which results in a loss of  $(m - 1)\alpha B'_{m-1}(v|v_{m-1})$ . Finally, the marginal cost of dropping out later equals  $\epsilon B'_m(v|v_m)$  so the net effect is  $\epsilon$  times

$$-(1 - m\alpha)B'_m(v|v_m) - (m - 1)\alpha B'_{m-1}(v|v_{m-1}),$$

which is strictly negative when  $\alpha \leq 1/m$  and a bidder is therefore better off dropping out  $\epsilon$  earlier. So all bidders prefer to drop out  $\epsilon$  earlier, and in fact would like to quit without delay until only two bidders remain active.

Bulow and Klemperer (1999a) point out that, strictly speaking, the game has no symmetric equilibrium. The above argument shows that there is no separating equilibrium in which all but the value-zero bidder remain in the auction at prices slightly above zero. But there is also no equilibrium in which all bidders drop out immediately with some positive probability, since then it would pay to wait. Bulow and Klemperer show how the outcome in which  $n - 2$  bidders drop out immediately can arise as the limiting case when drop out levels are ranked according to values, i.e. a separating equilibrium exists, but tend to zero for the  $n - 2$  lowest-value bidders.

In actual fund-raisers that employ the war of attrition this result could be motivated by modeling bidders' entry decisions as sequential. After two bidders have indicated they wish to compete for the prize (or have started bidding already), others will wish to refrain from bidding. Of course, if the selection of contestants is random (e.g. based on the first two bidders to raise their hands), the active bidders will not necessarily be the ones with the highest values. This would slightly change the revenue result in Proposition 6 below.

When  $\alpha > 1/n$ , the  $n$ -player war of attrition yields infinite revenues. Consider, for instance, the case of three bidders with  $1/3 < \alpha < 1/2$ . We claim that it is a symmetric equilibrium to "never drop out" while three bidders are active (i.e.  $B_3(v|0) = \infty$ ), and to bid according to (5.1) after one rival has dropped out. To verify this is an equilibrium suppose bidders 2 and 3 use this strategy. If bidder 1 also follows this strategy she earns  $3\alpha - 1 > 0$  per unit of time, and her total payoff blows up. If she drops out at price level  $X$ , her profits are at best  $(3\alpha - 1)X + \alpha R_{2,2}^{AP}$  which is finite for all finite  $X$ . Hence, bidder 1 is better off never dropping out. Above we already established it is optimal to bid according to (5.1), if one of the rivals drops out.

**Proposition 6.** *The war of attrition raises*

$$R_{2,n}^{WA} = \frac{E(Y_2^n)}{1 - 2\alpha} \quad (5.2)$$

*when  $\alpha < 1/n$ , while revenues are infinite for  $\alpha > 1/n$ .*

This revenue result sharply contrasts with that of an English (button) auction. Engers and McManus (2001) show that this dynamic winner-pay format yields the same revenues as a standard sealed-bid second-price auction. As shown in section 2, this revenue is low and remains finite even when  $\alpha = 1$ . In fact, Goeree and Turner (2000) show that

revenues of a second-price auction are finite even when  $\alpha > 1$ , i.e. when bidders value \$1 donated to the public good more than \$1 for themselves. (This situation may occur when the auction's revenue is matched by an outside party. Mathematically, such a matching is equivalent to a doubling of  $\alpha$ .) The intuition is that when  $\alpha > 1$ , the highest-value bidder does not gain from further increasing her bid (as she would in a first-price auction) because she does not determine the price she pays herself. In fact, in the limit  $\alpha \rightarrow \infty$  the second-price auction yields a disappointing revenue of 1.<sup>17</sup> Clearly, it would be better just to ask for voluntary contributions to the public good in this case.

## 6. Conclusion

Large voluntary contributions such as the recent \$24 billion committed by Bill Gates to the *Bill and Melinda Gates Foundation*, make up a substantial part of total fund-raising revenue today.<sup>18</sup> Not surprisingly, such gifts garner significant attention in the popular media.<sup>19</sup> The vast majority of fund-raising organizations, however, seek small contributions from a large number of donors. These organizations frequently prefer lotteries and auctions over the solicitation of voluntary contributions.<sup>20</sup>

Moreover, as electronic commerce on the Internet has grown, web sites offering charity auctions have proliferated. Electronic auction leaders such as *Ebay* and *Yahoo!* have specific sites for charity auctions where dozens of items are sold each day. The established fund-raising community has taken notice of these developments. In a recent report for the *W.K. Kellogg Foundation*, Reis and Clohesy (2000) identified auctions as one of the most important, and fastest growing, options that fund-raisers use to leverage the power of the Internet. Given these trends, it is clear that professional fund-raisers can profit from an improved auction design.

Currently, most fund-raisers employ standard auctions where only the winner pays. These familiar formats have long been applied in the sales of a variety of goods and their

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<sup>17</sup>Recall from footnote 13 that the optimal bids for the second-price auction are  $B_{2,n}(v) = \int_v^1 z dG_\alpha(z|v)$  where  $G_\alpha(z|v) \equiv 1 - \left(\frac{1-F(z)}{1-F(v)}\right)^{\frac{1}{\alpha}}$ . Note that  $G_\alpha(z|v)$  is well defined for all positive  $\alpha$  and that bids, and hence revenues, remain finite even when  $\alpha > 1$ . In the limit when  $\alpha$  tends to  $\infty$ , the distribution  $G_\alpha(z|v)$  becomes degenerate and puts all probability mass at  $z = 1$ , so  $B_2(v) = 1$  for all  $v$ .

<sup>18</sup>Total giving was an estimated \$190 billion in 1999, according to *Giving USA*.

<sup>19</sup>"Bill's Biggest Bet Yet," *Newsweek*, February 4, 2002, p. 46.

<sup>20</sup>For example, in the year 2000, *Ducks Unlimited* raised a total of \$75 million from special events organized by its 3,300 local chapters, with over 50% coming from auctions.

revenue-generating virtues are well established, both in theory and practice. We show, however, that they are ill suited for fund-raising. The problem with winner-pay auctions in this context is one of opportunity costs. A high bid by one bidder imposes a positive externality on all others, which they forgo if they top the high bid. Bids are suppressed as a result, and so are revenues. We show that the amount raised by winner-pay auctions is surprisingly low even when people are indifferent between a dollar donated and a dollar kept.

The elimination of positive externalities associated with others' bids does not occur when bidders have to pay irrespective of whether they win or lose. Many fund-raisers employ lotteries, for example, where losing tickets are not reimbursed (see Morgan, 2000). Lotteries are generally not efficient, however, which negatively affects revenues. We introduce a novel class of all-pay auctions, which are efficient while avoiding the shortcomings of winner-pay formats. We rank the different all-pay formats and demonstrate their superiority in terms of raising money (see Figures 1 and 2). An easy corollary to our analysis is that the lowest-price all-pay auction is the optimal fund-raising mechanism.

Our findings are not just of theoretical interest. The frequent use of lotteries as fund-raisers indicates that people are willing to accept an obligation to pay even though they may lose. The all-pay formats studied here may be characterized as incorporating "voluntary contributions" into an efficient mechanism. They are easy to implement and may revolutionize the way in which money is raised.

## A. Appendix

**Proof of Proposition 1.** Consider a standard auction format in which the highest bidder wins and only the winner pays. The surplus generated by the auction is  $S = E(Y_1^n) + n\alpha R$ , where  $R$  is the auction's revenue. This surplus is divided between the seller and the bidders:  $S = R + \pi_{bidders}$ . Solving for  $R$  we derive

$$R = \frac{E(Y_1^n) - \pi_{bidders}}{1 - n\alpha}. \quad (\text{A.1})$$

The revenue equivalence result for  $\alpha = 0$  is standard. When  $\alpha = 1$ , the winning bidder's *net* payment is zero. A bidder with a value of 1, who wins for sure, therefore has an expected payoff of 1. A simple Envelope Theorem argument shows

$$\pi^*(v) = \pi^*(0) + \int_0^v F^{n-1}(z) dz,$$

from which we derive

$$\begin{aligned} \pi^*(0) &= 1 - \int_0^1 F(z)^{n-1} dz \\ &= (n-1) \int_0^1 z f(z) F(z)^{n-2} dz \\ &= (n-1) \int_0^1 z f(z) \left( F(z)^{n-1} + F(z)^{n-2}(1 - F(x)) \right) dz \\ &= \frac{1}{n} \left( (n-1) E(Y_1^n) + E(Y_2^n) \right). \end{aligned}$$

The *ex ante* expected payoffs for the bidders are given by  $n \int_0^1 \pi^*(v) dF(v)$ , or:

$$\begin{aligned} \pi_{bidders} &= n\pi^*(0) + n \int_0^1 \int_0^v F(z)^{n-1} dz dF(v) \\ &= n\pi^*(0) + n \int_0^1 \int_z^1 dF(v) F(z)^{n-1} dz, \\ &= n\pi^*(0) + E(Y_1^n) - E(Y_2^n), \\ &= nE(Y_1^n). \end{aligned}$$

Using the next-to-last line, (A.1) can be rewritten as

$$R = \frac{E(Y_2^n) - n\pi^*(0)}{1 - n\alpha},$$

an expression used in footnote 12. Moreover, from the last line and (A.1) we derive

$$R = \frac{E(Y_1^n) - nE(Y_1)}{1 - n} = E(Y_1^n),$$

which completes the proof. *Q.E.D.*

**Proof of Proposition 2.** Let  $B(\cdot)$  denote the bidding function given in (3.1). Since the denominator in (3.1) is bounded away from 0 for all  $v < 1$  when  $\alpha < 1/k$ , the bidding function is well defined for all  $v < 1$  and possibly diverges in the limit  $v \rightarrow 1$ . The derivative of the expected profit of a bidder with value  $v$  who bids as if of type  $w$  and who faces rivals bidding according to  $B(\cdot)$  is

$$\begin{aligned} \partial_w \pi^e(B(w)|v) &= (n-1)v f(w) F(w)^{n-2} - (1-\alpha) B'(w) (1 - F_{Y_{k-1}^{n-1}}(w)) \\ &\quad + \alpha(k-1) B'(w) (F_{Y_k^{n-1}}(w) - F_{Y_{k-1}^{n-1}}(w)). \end{aligned}$$

Using the expression for  $B(\cdot)$  given by (3.1), the marginal expected profits can be rewritten as  $\partial_w \pi^e(B(w)|v) = (n-1)(v-w)f(w)F(w)^{n-2}$  and it is therefore optimal for a bidder with value  $v$  to bid  $B(v)$ . The revenue of the  $k^{\text{th}}$ -price all-pay auction equals

$$R = \sum_{i=k+1}^n \int_0^1 B(v) dF_{Y_i^n}(v) + k \int_0^1 B(v) dF_{Y_k^n}(v) = n \int_0^1 B(v) dG(v),$$

where  $G(v) \equiv \frac{1}{n} \sum_{i=k+1}^n F_{Y_i^n}(v) + \frac{k}{n} F_{Y_k^n}(v)$ . Note that  $G(0) = 0$  and  $G(1) = 1$  and that  $G(\cdot)$  is everywhere increasing. Using  $\frac{1}{n} \sum_{i=1}^n F_{Y_i^n} = F$ , the distribution  $G(\cdot)$  can be rewritten as

$$\begin{aligned} G(v) &= F(v) + \frac{1}{n} \sum_{i=1}^{k-1} (F_{Y_k^n}(v) - F_{Y_i^n}(v)) \\ &= F(v) + \frac{1}{n} \sum_{i=1}^{k-1} \left\{ \sum_{j=n+1-k}^n \binom{n}{j} F(v)^j (1-F(v))^{n-j} - \sum_{j=n+1-i}^n \binom{n}{j} F(v)^j (1-F(v))^{n-j} \right\} \\ &= F(v) + \frac{1}{n} \sum_{i=1}^{k-1} \sum_{j=n+1-k}^{n-i} \binom{n}{j} F(v)^j (1-F(v))^{n-j} \\ &= F(v) + \frac{1}{n} \sum_{j=n+1-k}^{n-1} \sum_{i=1}^{n-j} \binom{n}{j} F(v)^j (1-F(v))^{n-j} \\ &= F(v) + (1-F(v)) F_{Y_{k-1}^{n-1}}(v), \end{aligned}$$

where we used some basic properties of order statistics, see Mood, Graybill, and Boes (1962). The revenue of the  $k^{\text{th}}$ -price all-pay auction thus becomes

$$\begin{aligned}
R &= n \int_0^1 \int_0^v \frac{(n-1)zf(z)F(z)^{n-2}}{(1-k\alpha)(1-F_{Y_{k-1}^{n-1}}(z)) + \alpha(k-1)(1-F_{Y_k^{n-1}}(z))} dz dG(v) \\
&= n \int_0^1 \left( \int_z^1 dG(v) \right) \frac{(n-1)zf(z)F(z)^{n-2}}{(1-k\alpha)(1-F_{Y_{k-1}^{n-1}}(z)) + \alpha(k-1)(1-F_{Y_k^{n-1}}(z))} dz \\
&= \int_0^1 \frac{z(1-F_{Y_{k-1}^{n-1}}(z))}{(1-k\alpha)(1-F_{Y_{k-1}^{n-1}}(z)) + \alpha(k-1)(1-F_{Y_k^{n-1}}(z))} dF_{Y_k^{n-1}}(z),
\end{aligned}$$

where we used  $G(1) - G(z) = (1 - F(z))(1 - F_{Y_{k-1}^{n-1}}(z))$ .

*Q.E.D.*

**Proof of Proposition 3.** To show that revenues diverge when  $\alpha > 1/k$ , first consider the intermediate cases  $k = 2, \dots, n-1$ . Suppose the seller imposes an upper-bound on bids,  $M$ , where  $M$  is much larger than 1. There exists a mixed-strategy equilibrium of the  $k^{\text{th}}$ -price all-pay auction where bidders bid  $M$  with probability  $p$  and 0 with probability  $1-p$ . The expected profit of bidding 0 is

$$\pi_i^e(0) = \frac{v_i}{n} (1-p)^{n-1} + M \sum_{i=k}^{n-1} \binom{n-1}{i} \alpha i p^i (1-p)^{n-i-1}$$

and the expected profit of bidding  $M$  is

$$\pi_i^e(M) = \sum_{i=0}^{n-1} \frac{v_i}{i+1} p^i (1-p)^{n-i-1} + M \sum_{i=k-1}^{n-1} \binom{n-1}{i} (\alpha(i+1) - 1) p^i (1-p)^{n-i-1}$$

The equilibrium value of  $p$  follows by equating these expected profits. Ignoring terms proportional to  $v_i/M$  (since  $v_i$  is much smaller than  $M$ ) yields:

$$\binom{n-1}{k-1} \frac{k\alpha - 1}{1 - \alpha} = \sum_{i=k}^{n-1} \binom{n-1}{i} \left( \frac{p}{1-p} \right)^{i-k+1} \quad (\text{A.2})$$

This equation has a unique solution when  $\alpha > 1/k$  since the right side is strictly increasing in  $p$  and ranges from 0 (when  $p = 0$ ) to  $\infty$  (when  $p = 1$ ). The expected revenue generated in this mixed-strategy equilibrium is linear in  $M$ , and the seller can thus guarantee an arbitrarily high revenue by increasing  $M$ .

Finally, when  $k = 1$  and  $\alpha > 1/k$ , any contribution to the public good returns more than it costs and it is optimal to bid  $M$  for sure. Likewise, when  $k = n$  and  $\alpha > 1/k$ , the expected payoff of bidding  $M$  is linear in  $M$  while the expected payoff of bidding zero is at most  $v_i/n$ . Also in this case it is therefore optimal to bid  $M$  for sure. The seller can guarantee an arbitrarily high revenue by increasing  $M$ . *Q.E.D.*

**Proof of Proposition 4.** The derivative of (3.2) with respect to  $\alpha$  is the integral of a strictly positive function times

$$\begin{aligned} & k(1 - F_{Y_{k-1}^{n-1}}(z)) - (k-1)(1 - F_{Y_k^{n-1}}(z)) \\ &= (1 - F_{Y_{k-1}^{n-1}}(z)) + (k-1)(F_{Y_k^{n-1}}(z) - F_{Y_{k-1}^{n-1}}(z)) > 0, \end{aligned}$$

for all  $z < 1$ . Hence revenues are increasing in  $\alpha$ . Figure 3 shows an example where revenues are increasing with the number of bidders for low  $\alpha$  but decreasing with the number of bidders for high  $\alpha$ . *Q.E.D.*

**Proof of Proposition 5.** Note that the revenue of the  $k^{\text{th}}$ -price all-pay auction (3.2) can be written as

$$R_{k,n}^{AP} = \int_0^1 z \left( (1 - \alpha) - (k-1)\alpha \left\{ \frac{F_{Y_k^{n-1}}(z) - F_{Y_{k-1}^{n-1}}(z)}{1 - F_{Y_{k-1}^{n-1}}(z)} \right\} \right)^{-1} dF_{Y_2^n}(z).$$

A sufficient condition for revenues to be increasing in  $k$  is that the term between the curly brackets is increasing in  $k$  for all  $z \neq 0, 1$ . We first make this condition somewhat more intuitive. Consider an urn filled with red and blue balls and let  $p = 1 - F(z)$  be the chance of drawing a blue ball, where  $0 < p < 1$ . Suppose we draw  $n - 1$  times with replacement. The above condition can then be rephrased as: the chance of drawing exactly  $k - 1$  blue balls given that at least  $k - 1$  blue balls were drawn, is increasing in  $k$ . Hence, for all  $k$  is has to be true that

$$\frac{\binom{n-1}{k-1} p^{k-1} (1-p)^{n-k}}{\sum_{j=k-1}^{n-1} \binom{n-1}{j} p^j (1-p)^{n-j-1}} < \frac{\binom{n-1}{k} p^k (1-p)^{n-k-1}}{\sum_{j=k}^{n-1} \binom{n-1}{j} p^j (1-p)^{n-j-1}}.$$

Introducing  $x \equiv p/(1-p) > 0$ , the above inequality can be rearranged as:

$$\left(1 - \frac{\binom{n-1}{k}}{\binom{n-1}{k-1}} x\right) \left(1 + \sum_{j=k+1}^{n-1} \frac{\binom{n-1}{j}}{\binom{n-1}{k}} x^{j-k}\right) < 1.$$

The left side of this inequality can be expanded as  $1 + \sum_{i=1}^{n-k} a_i x^i$  where

$$a_i = \frac{\binom{n-1}{k+i}}{\binom{n-1}{k}} - \frac{\binom{n-1}{k+i-1}}{\binom{n-1}{k-1}} = -\frac{\binom{n-1}{k+i-1}}{\binom{n-1}{k-1}} \frac{ni}{(n-k)(k+i)} < 0,$$

which completes the proof. Q.E.D.

**Proof of Proposition 6.** For  $n = 2$ , the war of attrition is equivalent to the second-price all-pay auction and the proof follows from that of Propositions 2 and 3. We therefore focus on the case  $n \geq 3$ . First, consider the case  $\alpha < 1/n$ . When the  $n - 2$  lowest-value bidders drop out immediately and only the two highest-value bidders are left, revenues are

$$R = \frac{2}{1-2\alpha} E\left(\int_{Y_3^n}^{Y_2^n} \frac{zf(z)}{1-F(z)} dz\right).$$

Using  $f_{Y_3^n, Y_2^n}(x, y) = (n!/(n-3)!)F(x)^{n-3}(1-F(y))f(x)f(y)$  for  $y \geq x$  (see, for instance, Mood, Graybill, and Boes, 1962), this can be worked out as

$$\begin{aligned} R &= \frac{2n!}{(1-2\alpha)(n-3)!} \int_0^1 \int_x^1 \int_x^y F(x)^{n-3}(1-F(y))f(x)f(y) \frac{zf(z)}{1-F(z)} dz dy dx \\ &= \frac{2n!}{(1-2\alpha)(n-3)!} \int_0^1 \int_x^1 \left(\int_z^1 (1-F(y))f(y) dy\right) F(x)^{n-3} f(x) \frac{zf(z)}{1-F(z)} dz dx \\ &= \frac{n!}{(1-2\alpha)(n-3)!} \int_0^1 \int_x^1 F(x)^{n-3} f(x) zf(z)(1-F(z)) dz dx \\ &= \frac{n!}{(1-2\alpha)(n-3)!} \int_0^1 \int_0^z F(x)^{n-3} f(x) zf(z)(1-F(z)) dx dz \\ &= \frac{n(n-1)}{1-2\alpha} \int_0^1 zf(z)F(z)^{n-2}(1-F(z)) dz \\ &= \frac{E(Y_2^n)}{1-2\alpha}. \end{aligned}$$

Next, when  $1/n < \alpha \leq 1/(n-1)$ , it is a symmetric equilibrium for all bidders to “never drop out” while  $n$  bidders are active, i.e.  $B_n(v|0) = \infty$ . Note that this strategy would pay  $n\alpha - 1 > 0$  per unit of time to each bidder, and total payoffs would diverge (as would revenues). If one bidder drops out, the net effect for an active bidder of staying in the auction  $\epsilon$  longer is  $\epsilon$  times

$$-(1 - (n-1)\alpha)B'_{n-1}(v|v_{n-1}) - (n-2)\alpha B'_{n-2}(v|v_{n-2}) < 0,$$

which implies that  $n-3$  others will drop out right away, leaving only two active bidders (see the discussion in section 5). The expected payoffs from dropping out at a finite price level  $X$  are thus at best  $(n\alpha - 1)X + \alpha R_{2,2}^{AP}$  which is finite. Hence, a bidder is better off never dropping out of the auction. Finally, when  $1/m < \alpha \leq 1/(m-1)$  for some  $m = 2, \dots, n-1$ , it is an equilibrium for  $n-m$  bidders to drop out right away and for the remaining  $m$  active bidders to never drop out of the auction. *Q.E.D.*

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