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On Nonobtuse Simplicial Partitions*

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Dedicated to Ivan Hlaváček on his 75th birthday

Abstract. This paper surveys some results on acute and nonobtuse simplices and associated spatial partitions. These partitions are relevant in numerical mathematics, including piecewise polynomial approximation theory and the finite element method. Special attention is paid to a basic type of nonobtuse simplices called path-simplices, the generalization of right triangles to higher dimensions. In addition to applications in numerical mathematics, we give examples of the appearance of acute and nonobtuse simplices in other areas of mathematics.

Key words. ortho-simplices, path-simplices, Delaunay triangulation, refinements, Kuhn partition, Sommerville tetrahedron, polytope, simplicial finite elements, discrete maximum principle

AMS subject classifications. 52B11, 51M20, 65N30, 52B05

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1. Introduction.

A *simplex (d-simplex)* in the Euclidean space $\mathbb{R}^d$, $d \in \{1, 2, 3, \dots\}$, is a convex hull of $d+1$ points that do not all belong to the same hyperplane. Those points are said to be the *vertices* of the simplex. Opposite each vertex lies a $(d-1)$-dimensional *facet.* For $d = 1$ facets are just points. For $d \geq 1$ the *dihedral angle* α between two facets is defined by means of the inner product of their outward unit normals n_1 and n_2,

$$\cos\alpha = -n_1 \cdot n_2.$$

If $d = 1$, these normals necessarily make an angle of $180°$ and thus $\alpha = 0$. Each simplex in $\mathbb{R}^d$ has $\binom{d+1}{2}$ dihedral angles. If all dihedral angles of a given simplex are less than $90°$ (less than or equal to $90°$), we say that the simplex is *acute* (*nonobtuse*).

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Let $\Omega \subset \mathbb{R}^d$ be a domain. If the boundary of the closure $\partial\overline{\Omega}$ of Ω is contained in a finite number of $(d-1)$-dimensional hyperplanes, we say that $\overline{\Omega}$ is *polytopic.* Moreover, if $\overline{\Omega}$ is bounded, it is called a *polytope*; in particular, $\overline{\Omega}$ is called a *polygon* for $d = 2$ and a *polyhedron* for $d = 3$.

DEFINITION 1. *By a* partition *(or* triangulation*) of a closed polytopic domain $\overline{\Omega}$ we mean a set of simplices whose union is $\overline{\Omega}$, and for which any two simplices have disjoint interiors and each facet of a simplex is a facet of another simplex from the partition or belongs to the boundary $\partial\overline{\Omega}$. Moreover, we assume that the set of vertices of all simplices do not have an accumulation point in $\mathbb{R}^d$. A partition is called* acute *(*nonobtuse*) if all its simplices are acute (nonobtuse).*

This definition allows us to consider partitions (face-to-face tilings) of the whole space $\mathbb{R}^d$ as well.

2. Acute Partitions. First we present a basic property of acute simplices (see, e.g., [29], [31, p. 110], [11]):

THEOREM 2. *Let $d > 2$. If a d-simplex is acute, then each of its facets is an acute $(d-1)$-simplex.*

The converse implication does not hold. Indeed, the tetrahedron with vertices $A = (-1, 0, 0)$, $B = (1, 0, 0)$, $C = (0, -1, \frac{1}{2})$, and $D = (0, 1, \frac{1}{2})$ has congruent acute triangular faces, but the dihedral angles at the edges AB and CD are obtuse. Choosing $C = (0, -1, 1)$ and $D = (0, 1, 1)$, we get the so-called *Sommerville tetrahedron* (No. 1 according to the classification in [39]) with acute triangular faces and right angles at the edges AB and CD.

Figure 1 presents a triangulation of an obtuse triangle and a square into seven and eight acute triangles, respectively (cf. [37]). It was shown in [65] and [69] that these numbers are minimal. Later, Cassidy and Lord [17] proved that for any $n \geq 10$ there exists a triangulation of a square into n acute triangles. They also showed why such a triangulation does not exist for $n = 9$.

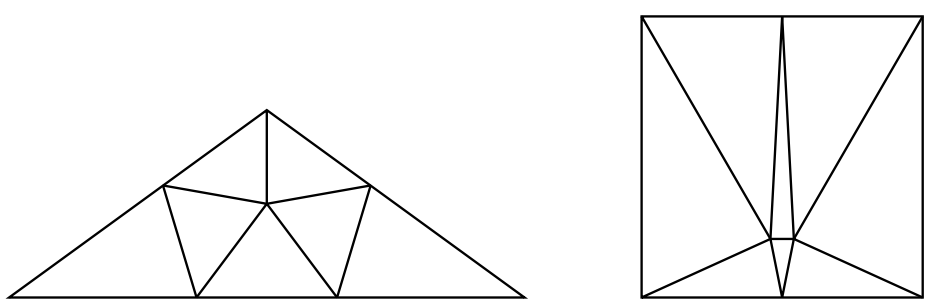

Fig. 1 *Partition of an obtuse triangle and square into acute triangles.*

Each triangle and also each (possibly nonconvex) quadrangle is a plane-filler (see [40]). Using this fact we can easily construct acute "periodic" triangulations of the plane. Dividing the Penrose rhombic tiles [72] into two isosceles triangles, we can generate acute "aperiodic" triangulations with maximal angle 72°. No such tiling is periodic [40]. Burago and Zalgaller in [13] give an algorithm to construct an acute triangulation for an arbitrary polygon. Gerver [38] presents an algorithm that enables us to decompose special polygons into "almost equilateral triangles" with maximal

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angle 72°. Maehara [68] proves that every n-gon can be triangulated into $\mathcal{O}(n)$ acute triangles (see also [94]). A short recent survey on acute triangulations was done by Zamfirescu in [95].

It is obvious that in acute (nonobtuse) triangulations of planar domains, each inner vertex is surrounded by at least five (four) triangles (see Figure 1). The ratio between the corresponding numbers in $\mathbb{R}^3$ is much bigger, namely, 20:8. In [60, p. 165] the following theorem was proved.

THEOREM 3. *In any acute (nonobtuse) partition of a polyhedral domain each inner vertex is surrounded by at least* 20 (8) *tetrahedra.*

To show that these numbers are attainable, consider the classical Platonic bodies, the regular tetrahedron, the cube, the regular octahedron, the regular dodecahedron, and the regular icosahedron (see Figure 17). The regular icosahedron and the regular octahedron (depicted in Figure 2) can be partitioned into 20 acute (8 nonobtuse) tetrahedra by taking the 20 (8) convex hulls of the center of gravity with each of the triangular faces.

For $d = 4$, the above ratio seems to be 600:16 (see Conjecture 4). In fact, in around 1852, Schläfli [76] studied regular polytopes in $\mathbb{R}^4$, in particular the regular 600-cell and 16-cell (also called 4-orthoplex). Their three-dimensional surfaces are formed by congruent regular tetrahedra (see, for instance, [79, 83]), whose convex hulls with the center of gravity G of the regular polytope form 600 acute and 16 nonobtuse 4-simplices surrounding G, respectively.

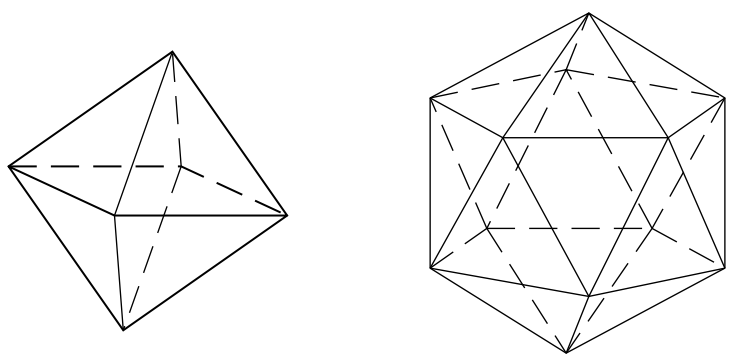

Fig. 2 *The regular octahedron and icosahedron.*

It is easy to verify that a vertex in $\mathbb{R}^4$ cannot be surrounded by less than 16 nonobtuse simplices, but the situation with acute simplices is much more difficult.

CONJECTURE 4. *Each vertex in an acute partition of* $\mathbb{R}^4$ *is surrounded by at least* 600 *simplices.*

For $d \geq 5$ the situation is very different, due to Theorem 11 later on.

Generating acute partitions in $\mathbb{R}^3$ is much harder than in $\mathbb{R}^2$, as can be understood from the following open problem.

CONJECTURE 5. *Each tetrahedron (or cube) can be partitioned into acute tetrahedra.*

Aristotle in his treatise *On the Heaven* (350 B.C.) incorrectly conjectured that the regular tetrahedron is a space-filler [2, Vol. 3, Chap. 8]. This would require the dihedral angle between its faces to be equal to 72°. Since Aristotle was a recognized

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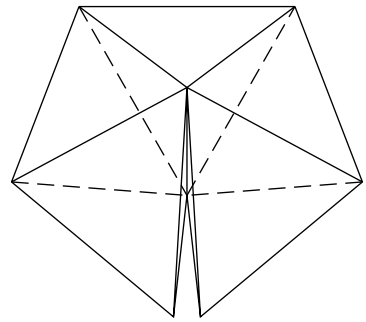

Fig. 3 *The regular tetrahedron is not a space-filler. To a given face of the regular tetrahedron we may join face-to-face another regular tetrahedron in a unique way. Repeating this process, five regular tetrahedra may surround a common edge, but a small gap will appear, since all dihedral angles are approximately* 71° *only.*

person, nobody doubted his statement. Only in the Middle Ages was it realized that he was mistaken (see Figure 3). All dihedral angles of the regular tetrahedron are equal to $\arccos\frac{1}{3}$ which, rounded to entire degrees, gives 71°. Also, Averroës (1126–1198) calculated [85, p. 127] that the length of each edge of the regular icosahedron, inscribed in the unit ball, is

$$\frac{1}{5}\sqrt{10(5-\sqrt{5})} \;\approx\; 1.05,$$

which does not equal 1, as would follow from the Aristotle conjecture.

An algorithm for partitioning $\mathbb{R}^3$ into acute tetrahedra was discovered only very recently. In 2001, Üngör [89] published the following result.

THEOREM 6. *There exists an acute partition of* $\mathbb{R}^3$.

The proof of this theorem is constructive and based on the partition of the regular icosahedron (see Figure 2) into twenty acute tetrahedra, as discussed above. The projection of the regular icosahedron onto the plane on which it stands (on one of its triangular faces) is a regular hexagon, which is a plane-filler. Congruent regular icosahedra may thus be placed into a spatial regular lattice such that two neighboring isocahedra have a common edge, but not a common face. The remaining gaps can be partitioned by four different types of acute tetrahedra (see the sketch in Figure 4).

Eppstein, Sullivan, and Üngör [24] introduced four more algorithms for generating acute partitions of $\mathbb{R}^3$ which use information about the position of atoms in crystals of zeolites. The volume of these minerals increases with pressure due to a special dense arrangement of atoms, such as in silicon dioxide. Their chemical structure was studied in 1958 (see [32]), and half a century later they inspired the construction of acute partitions of $\mathbb{R}^3$.

The main idea is the following: Denote the centers of the particular atoms by $A_1, A_2, \dots$. For each $i \in \{1, 2, \dots\}$ define the corresponding *Voronoi cells* (also called *Dirichlet regions* [77]) in $\mathbb{R}^d$. Properties of these convex polytopes (see Figure 5) were studied by Voronoi (1868–1908) in [92], though they were defined earlier by Dirichlet (1805–1859) as

$$V_i = \{x \in \mathbb{R}^d \mid \|x - A_i\| \le \|x - A_j\| \text{ for all } j = 1, 2, \dots\},$$

where $\|\cdot\|$ stands for the Euclidean norm. The set V_i thus contains all points $x \in \mathbb{R}^d$ whose distance from A_i is less than or equal to the distances to each of the other points

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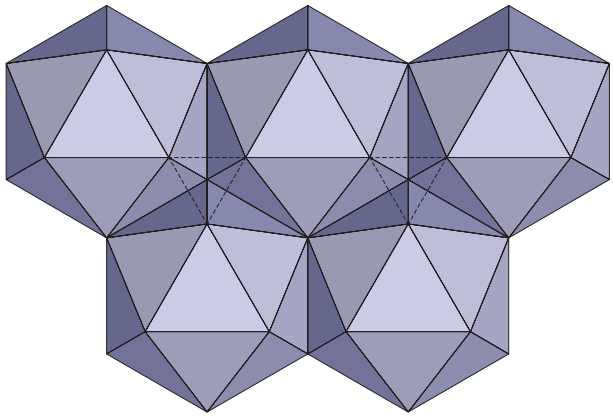

Fig. 4 *Üngör's construction of an acute partition of $\mathbb{R}^3$ by means of regular icosahedra. The adjacent layers of icosahedra are just mirror images of the sketched layer.*

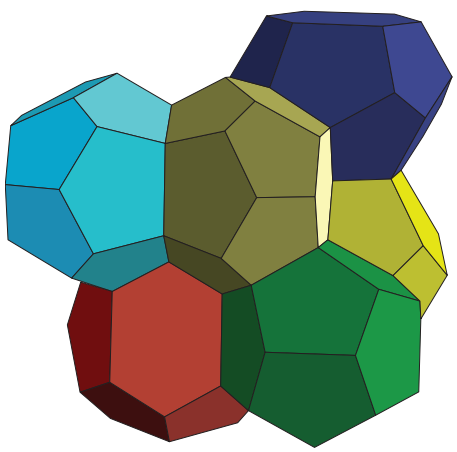

Fig. 5 *Voronoi cells in $\mathbb{R}^3$ corresponding to atoms of special chemical compounds.*

A_j. It is possible to associate a so-called Delaunay triangulation (see Definition 7 below) to the face-to-face division into Voronoi cells, which is the required acute triangulation. Its vertices are $\{A_1, A_2, \dots\}$ and each edge is surrounded by either five or six tetrahedra.

DEFINITION 7. *Let $D \subset \mathbb{R}^d$ be the set of all vertices of all simplices from a given triangulation. If the interiors of the circumscribed balls about any simplex from the triangulation do not contain points from D (see Figure 6), then the triangulation is said to be* Delaunay.

In the pioneering paper *Sur la sphère vide* (see [21]) by Delone[1] it was shown that for any finite number of points that do not all belong to the same hyperplane, such a triangulation exists. Moreover, if no $d+2$ points from D lie on the surface of

[1]Usually spelled (French) phonetically as Delaunay.

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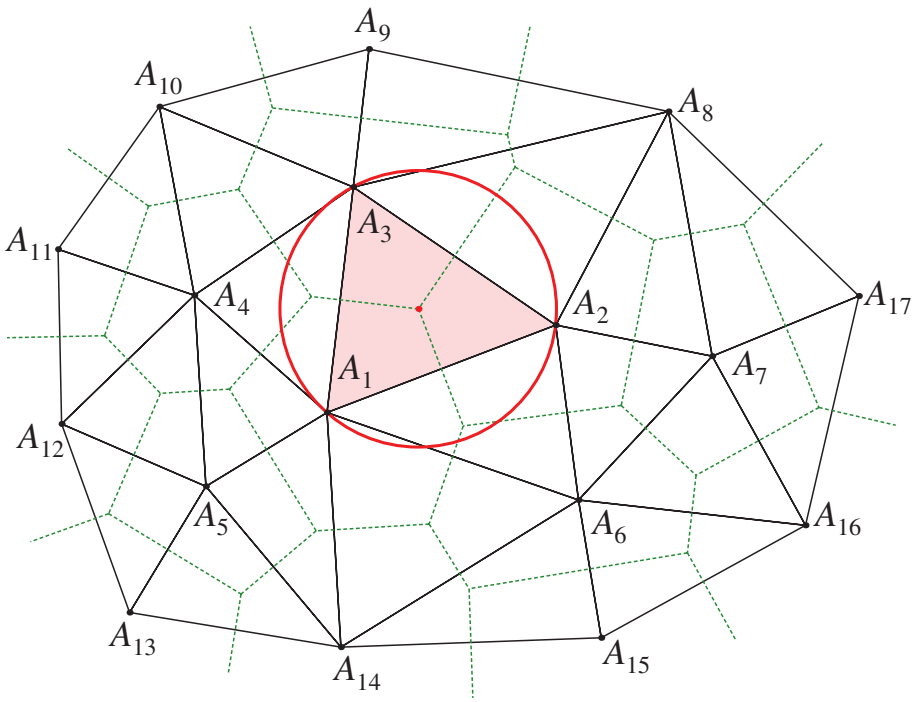

Fig. 6 *Voronoi cells in $\mathbb{R}^2$ are indicated by broken lines. Notice that the vertices of the Delaunay triangulation are not contained in the interiors of the circumscribed balls about the particular triangles. For $d = 2$, this triangulation maximizes the minimal angle among all triangulations having the same vertices.*

a d-dimensional ball, the Delaunay triangulation is determined uniquely. There also exist other constructive definitions of Delaunay triangulations in [70, 71, 84]. Some authors say that the Delaunay triangulation is dual to a division into Voronoi cells (see Figure 6).

It should be noted that one of the four above-mentioned algorithms (introduced in [24]), which uses the crystal lattice of zeolites, produces dihedral angles not greater than 74.2° (see [24]). It is not known whether there exists an acute partition of $\mathbb{R}^3$ with a smaller maximal dihedral angle. This angle cannot be less than 72°, which is due to the following theorem (see [56]).

THEOREM 8. *In any partition of $\mathbb{R}^3$ there exist an edge that is surrounded by at least six tetrahedra and an edge that is surrounded by at most five tetrahedra.*

COROLLARY 9. *In any partition of $\mathbb{R}^3$ there exist a dihedral angle less than or equal to 60° and a dihedral angle greater than or equal to 72°.*

From this we again see why the Aristotle conjecture cannot hold.

In [24] an algorithm is given that enables us to decompose an infinite slab of constant thickness into acute tetrahedra with maximal angle not greater than 87.7°.

Zhang [96] examines some problems arising during successive refinements of tetrahedral partitions, which is needed in numerical mathematics. In Figure 7 we see a standard red refinement technique applied to an acute or nonobtuse tetrahedron. The four "exterior" subtetrahedra are similar to the original tetrahedron. Since the remaining four "interior" subtetrahedra share a common edge, the associated four dihedral angles are right or at least one of them is obtuse.

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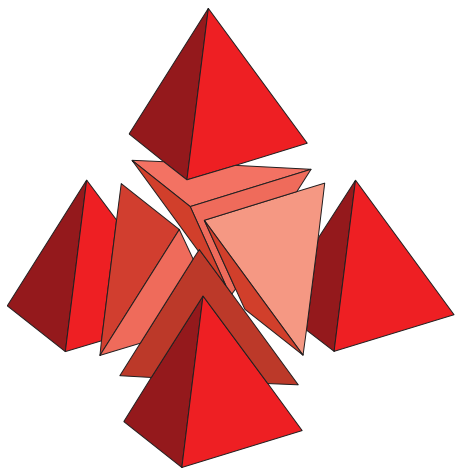

Fig. 7 *Refinement of a tetrahedron by plane cuts through midlines of its faces.*

Remark 10. It is not known how to refine acute partitions of polyhedra keeping all dihedral angles acute. The main obstacle here is the fact that a simple bisection of a half-space yields two dihedral angles that add up to 180°. In particular, edges of acute partitions that lie inside polygonal faces must be surrounded by at least three tetrahedra. Consequently, the acuteness condition seems considerably harder (especially at the boundary) than the nonobtuseness condition which admits orthogonal bisections (cf. Figures 9–11, 15–17). On the other hand, contrary to nonobtuse partitions, allowing an acute partition is a stable property: any small enough perturbation of the original geometry still allows an acute partition.

Let us point out that the dihedral angle of the regular simplex in $\mathbb{R}^d$ is

$$\alpha(d) = \arccos\frac{1}{d} < 90^\circ.$$

For $d \geq 4$ this angle is greater than 72°, since $\alpha(4) \approx 76^\circ$ and the sequence $\{\alpha(d)\}_{d=2}^{\infty}$ is increasing. Consequently, $k\alpha(d) \neq 360^\circ$ for any integer k and $d > 2$. This means that the regular simplex is not a space-filler in higher-dimensional spaces. For $d \geq 5$, options are even more limited:

THEOREM 11. *There is no acute partition of* $\mathbb{R}^d$ *for* $d \geq 5$.

The proof [57] resembles Fermat's method of infinite descent. It uses some results from combinatorial topology, in particular the Euler–Poincaré formula [67] which implies that in five-dimensional space a point cannot be surrounded by acute simplices. On the other hand, as mentioned after Theorem 3, a point in $\mathbb{R}^4$ can be surrounded by at least 600 acute simplices due to the existence of the 600-cell. In spite of that we believe that the following conjecture is true.

CONJECTURE 12. *There is no acute partition of* $\mathbb{R}^4$.

Our belief is based on the following heuristics. If Conjecture 4 is true, then each vertex is surrounded by at least 600 acute simplices. On the other hand, in an arbitrary partition this number should be "on average" much lower. For instance, in a uniform partition of $\mathbb{R}^4$ based on the Kuhn partition (see Theorem 21), each vertex is surrounded by $(4+1)! = 120$ simplices only. Moreover, we also have to take into

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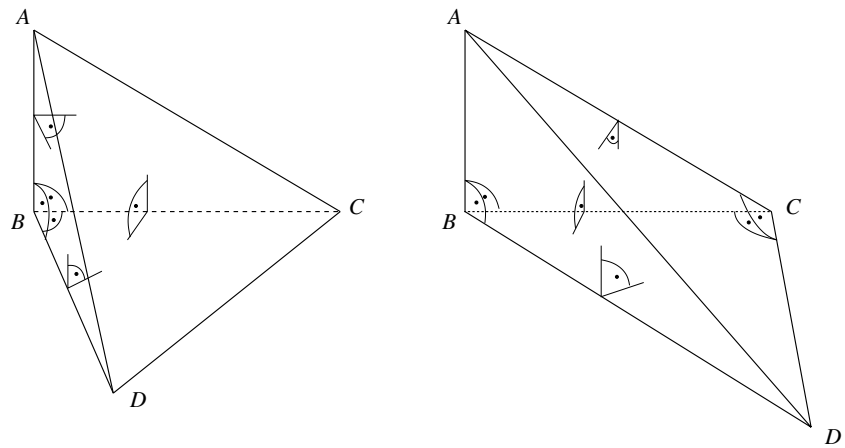

Fig. 8 *Two types of ortho-simplices in $\mathbb{R}^3$. The one on the right is a path-tetrahedron whose three mutually orthogonal edges AB, BC, and CD form a path. All its faces are right triangles.*

account the fact that the sum Σ of dihedral angles of a 4-simplex satisfies

$$720^\circ < \Sigma < 1080^\circ,$$

whereas, for acute simplices, the upper bound is likely to be 810°. The above ingredients seem to be mutually contradictory, but a formal proof has not yet been found. For bounds on the sum of the angles of d-simplices for $d \geq 2$, see [35, 36].

In the following section we will also allow right dihedral angles to show up in the partition. We shall see that construction and analysis of nonobtuse partitions will be much simpler than for the case of acute partitions.

3. Nonobtuse Partitions and Path-Simplices. Nonobtuse triangulations of polygons are very well studied; see, for instance, [4, 50, 51, 68, 94]. We start with a useful property of nonobtuse simplices in higher dimensions (see [11, 31]).

THEOREM 13. *Let $d > 2$. If a d-simplex is nonobtuse, then each of its facets is a nonobtuse $(d-1)$-simplex.*

The converse implication is not valid. To see that it is enough to consider the same tetrahedron as the one described just after Theorem 2. However, the converse implication does hold in a special case formulated in Theorem 17.

Now we introduce two notions that are not always consistently defined in the literature [76].

DEFINITION 14. *An* ortho-simplex *in $\mathbb{R}^d$, $d \in \{1, 2, \dots\}$, is a simplex having d mutually orthogonal edges. A* path-simplex *in $\mathbb{R}^d$ is an ortho-simplex whose d orthogonal edges form a path (in the sense of graph theory); in particular, for $d = 3$ we shall speak about a* path-tetrahedron.

A right triangle in $\mathbb{R}^2$ is an ortho-simplex and also a path-simplex. In Figure 8 we see both types of ortho-simplices in $\mathbb{R}^3$. The left one, called a *cube corner tetrahedron*, has three mutually orthogonal edges that share a common vertex B. Three of its faces are right triangles and the fourth face ACD is an acute triangle. The second type of ortho-simplex only has right triangular faces (see the right side of Figure 8). We observe that its three orthogonal edges form a path. Therefore, it is a path-tetrahedron.

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The following three theorems can be found in the monograph by Fiedler [31] (see also [5, 11, 50]).

THEOREM 15. *Each simplex has at least d acute dihedral angles.*

This theorem, originally published in [27, p. 315], is so nice that it was rediscovered and published fifty years later as [63], even though it can be found in Mathematical Reviews 0069507.

THEOREM 16. *Each ortho-simplex has $\binom{d+1}{2} - d$ right angles.*

Combining Theorems 15 and 16 shows that each ortho-simplex is nonobtuse.

THEOREM 17. *Let $d > 2$. A d-simplex is a path-simplex if and only if each of its facets is a path $(d-1)$-simplex.*

From this we inductively find that a d-simplex is a path simplex if and only if each of its two-dimensional faces is a right triangle (see [28, 31]).

Let us present some further results. It is easy to see (compare with Figure 12) that the cube corner tetrahedron on the left of Figure 8 does not contain its circumcenter. (A formula for the radius of the circumscribed ball of a simplex in $\mathbb{R}^d$ is derived in [27, p. 316].) On the other hand, the circumcenter of the path-tetrahedron on the right in Figure 8 lies at the midpoint of the longest edge AD. This is also true in $\mathbb{R}^d$ (see [5, p. 194] and [28, 31]).

THEOREM 18. *An ortho-simplex contains its circumcenter if and only if it is a path-simplex.*

Notice that the circumscribed ball about a path-simplex for $d = 2$ is, in fact, the Thales circle. In 1994 Rajan proved [74, p. 200] another assertion in this context.

THEOREM 19. *If each simplex in a given triangulation in $\mathbb{R}^d$ contains its circumcenter, then the triangulation is Delaunay.*

In particular, each nonobtuse triangulation in the plane is Delaunay (see Figure 6). The converse implication does not hold. Indeed, there exist Delaunay triangulations in $\mathbb{R}^2$ containing triangles with obtuse angles. For $d > 2$, neither the nonobtuseness nor the acuteness of simplices implies that the triangulation is Delaunay. For instance, Üngör in [89] (see also [24, p. 245]) presents a non-Delaunay partition consisting of two acute tetrahedra that do not contain their circumcenters. A combination of the previous two theorems results in the following theorem.

THEOREM 20. *A triangulation into path-simplices is Delaunay.*

In his paper [33] from 1942, Freudenthal shows the following.

THEOREM 21. *The unit cube $[0,1]^d$ can be partitioned into $d!$ path-simplices.*

Indeed, the path-simplices from the above theorem can be defined as

$$S_\sigma = \{x = (x_1, \dots, x_d) \in \mathbb{R}^d \mid 0 \le x_{\sigma(1)} \le \cdots \le x_{\sigma(d)} \le 1\},$$

where σ ranges over all permutation of the numbers $1, 2, \dots, d$. See Figure 9 for $d = 2, 3$.

Thus, $\mathbb{R}^d$ can be first partitioned into d-cubes and then also into path-simplices. The Freudenthal triangulation is also called the Kuhn partition due to the paper [61]. Another idea was published by Goldberg in [39]. He divided $\mathbb{R}^3$ into congruent infinite prisms having an equilateral triangle as their cross-section. Then each prism was partitioned into congruent tetrahedra (see Figure 10), which can be chosen to be nonobtuse. If the partition of each triangular prism is a mirror image of an adjacent prism, we get a nonobtuse partition of the three-dimensional space in the sense of Definition 1, that is, face-to-face.

A special tetrahedral space-filler was found in 1923 by Sommerville (see [80, 39]). The lengths of two of its opposite edges equal 2, while the other four edges have

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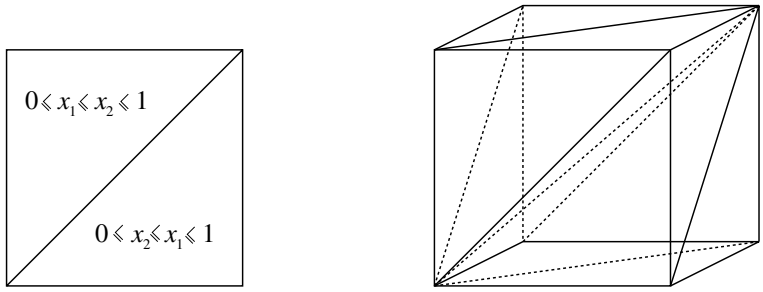

Fig. 9 *Triangulation of a square (cube) into two (six) path-simplices.*

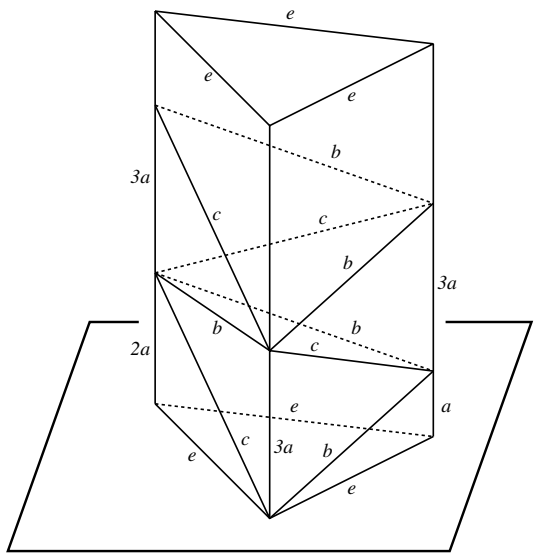

Fig. 10 *Goldberg's division of an infinite triangular prism into congruent tetrahedra. If $\sqrt{2}b \geq c$, then no dihedral angle exceeds $90°$ (see [39, p. 353]). For instance, the choice $a = 4$, $b = 11$, and $c = 13$ yields integer lengths of edges.*

length $\sqrt{3}$. Dihedral angles at the two longer edges are right, whereas the other four dihedral angles equal $60°$. This Sommerville tetrahedron can be partitioned into eight congruent tetrahedra that are all similar to the original one [53] (cf. Figure 7). It can also be partitioned into four path-tetrahedra or into two congruent ortho-simplices [81]. As mentioned in [23, p. 288], there does not exist an acute tetrahedral space-filler. Partitions of $\mathbb{R}^d$ with congruent nonobtuse simplices are described in [20].

Any right triangle can be bisected by the altitude onto the hypotenuse into two smaller right triangles. A partition of a path-tetrahedron into three smaller path-tetrahedra (see Figure 11) was described by Coxeter [19] in 1989. The same construction had been, in fact, used implicitly by Lenhard in 1960 to trisect tetrahedra of the so-called class $T1c$ (see [64]) to which path-simplices belong. The result was recently generalized to any space dimension in [11].

THEOREM 22. *Each path-simplex in $\mathbb{R}^d$ can be partitioned into d (and also $d+1$) smaller path-simplices.*

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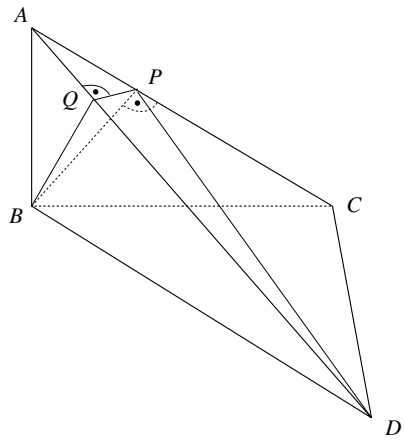

Fig. 11 *Coxeter's trisection of a path-tetrahedron $ABCD$ into smaller path-tetrahedra $AQPB$, $BPQD$, and $BPCD$. The points P and Q are the orthogonal projections of the points B and P on the segments AC and AD, respectively.*

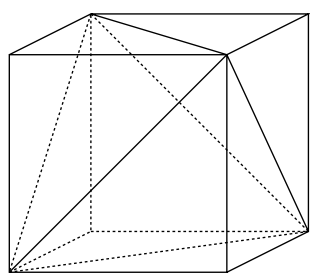

Fig. 12 *Partition of a cube into five tetrahedra having the same circumscribed ball.*

Induction is used to prove the statement for $d+1$, after which the result for d follows as a degenerate case. Geometrically, part of the proof resembles the Gram–Schmidt orthogonalization process.

By Theorem 21, each d-dimensional cube can be partitioned into $d!$ path-simplices (see Figure 9). However, the smallest number of simplices into which the d-dimensional cube can be partitioned is given by [7, 43]

$$1, 2, 5, 16, 67, 308, 1493$$

for $d \leq 7$. In Figure 12 we see five tetrahedra that form a cube. One of them is the regular tetrahedron, and the other four are cube corner tetrahedra, hence ortho-simplices, but not path-tetrahedra. The center of the cube is the center of the circumscribed ball of each of the five tetrahedra.

In 1957, Hadwiger [41] stated the following conjecture.

CONJECTURE 23. *Each simplex in $\mathbb{R}^d$ can be dissected into a finite number of path-simplices.*

The correctness of this statement has so far been proved only for $d \leq 5$. For example, each triangle can be bisected into two right triangles by means of the altitude onto the longest edge.

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In 1960, Lenhard [64] described a method to decompose an arbitrary tetrahedron into at most twelve path-tetrahedra, though possibly not face-to-face. Later, Böhm [8] showed that this number cannot be reduced.

In 1977, Harazišvili [42] showed that an arbitrary 4-simplex can be decomposed into at most 730 path-simplices. This number was reduced in 1986 by Kaiser [46] to 610 and in 1993 by Tschirpke [86] to 500. The minimal number is yet unknown.

Tschirpke also investigated the case $d = 5$. In [88], and based on the dissertation [87], it was shown that it is sufficient to use 12,598,800 path-simplices.

Remark 24. Each nonconvex polytope can be cut into convex polytopes by a finite number of bisecting planes whose union contains $\partial\overline{\Omega}$ (cf. [53] for $d = 3$). Each convex polytope can be easily decomposed into simplices. If the Hadwiger conjecture is valid, then each polytope can be decomposed into a finite number of path-simplices. Path-simplices in the geometry of polytopes would thus be basic building blocks like atoms in nature, even more elementary than simplices themselves.

4. Applications in Numerical Mathematics. Nonobtuse partitions are frequently employed in numerical mathematics, mainly because partitions give rise to function spaces containing piecewise polynomials relative to the partition, like, for instance, the space of globally continuous functions that are linear on each simplex. A sufficient condition for Lagrange or Hermite interpolation into such spaces to have optimal approximation properties is that the partition is nonobtuse (see, e.g., [45, 54, 55]). Also, the standard reference simplex known from approximation theory of the finite element method is nonobtuse.

The finite element method uses piecewise polynomials to approximate solutions of partial differential equations. If these solutions satisfy certain maximum principles [62, 73, 82], it is desirable that their finite element approximations satisfy certain of their discrete analogues. Nonobtuse and acute partitions indeed yield finite element approximations that satisfy so-called discrete maximum principles when solving (possibly nonlinear) elliptic [47, 48, 52, 58, 91] and parabolic [26, 34, 44, 75] problems, semiconductor equations [98, 99], and convection-diffusion problems [1] by means of globally continuous, piecewise linear functions.

A key observation in this context is that the gradient of a nonzero linear function on a simplex S that vanishes on a facet F_j of S is a constant nonzero normal to F_j. Hence, the sign of inner products between gradients of such functions is in one-to-one correspondence with the type of angle between the facets of S. To be more explicit, for $d \geq 1$ we have the following expression, which was derived in [12] directly from [11] and [58]:

$$(\nabla v_i)^\top \nabla v_j = -\frac{|F_i||F_j|}{h_i h_j} \cos \alpha_{ij}, \quad i, j = 1, \ldots, d+1, \quad i \neq j, \tag{1}$$

where α_{ij} is the dihedral angle between F_i and F_j of S. Moreover, $|F_\ell|$ denotes the volume of F_ℓ, and S has height h_ℓ above F_ℓ. Finally, v_ℓ is the linear function that vanishes on F_ℓ and has value 1 at the vertex B_ℓ opposite F_ℓ (see Figure 13). A similar expression was introduced by Xu and Zikatanov [93].

Basically, the discrete Laplacian that results from the standard finite element method has a nonnegative inverse if each of the above inner products in the partition is nonpositive for distinct i and j, which is the case for nonobtuse partitions. If the partition is in fact acute, the discrete Laplacian has a positive inverse and then both reaction and convection terms of small enough size can be handled using perturbation arguments. See, for instance, the paper [18] by Ciarlet and Raviart, in which the

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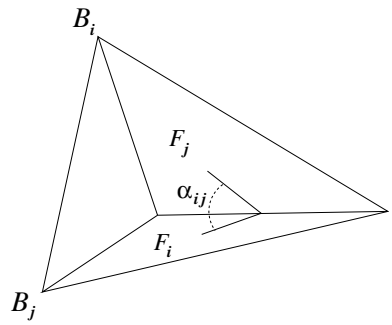

Fig. 13 *Illustration of the formula in* (1) *for* $d = 3$.

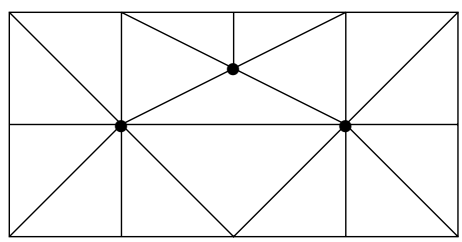

Fig. 14 *Triangulation with a single obtuse triangle.*

presence of a reaction term led to the condition that the partition should be acute and the diameters of the simplices should be small enough. In that paper, the discrete maximum principle resulted in uniform error bounds for the finite element method. The discrete maximum principle is also of interest to avoid negative numerical values of typical positive physical quantities like concentration, temperature (in Kelvins), density, and pressure.

A single obtuse triangle in the partition can destroy the discrete maximum principle [22, 78]. To show this, consider the Poisson equation $-\Delta u = f$ with homogeneous Dirichlet boundary conditions on the domain $[0,4]\times[0,2]$ triangulated as in Figure 14.

The space of continuous piecewise linear functions relative to this triangulation satisfying the boundary condition has dimension 3: values at the vertices indicated with bullets are the degrees of freedom. Assume that they have coordinates $v_1 = (1,1), v_2 = (3,1)$, and $v_3 = (2, 1+p)$. For all $p \in (0,1)$ they are the vertices of the only obtuse triangle in the partition, and the discrete Laplacian does not have a nonnegative inverse. For example, for $p = \frac{1}{2}$ this inverse equals

$$\begin{pmatrix} \frac{63}{248} & -\frac{1}{248} & \frac{1}{16} \\ -\frac{1}{248} & \frac{63}{248} & \frac{1}{16} \\ \frac{1}{16} & \frac{1}{16} & \frac{37}{160} \end{pmatrix}.$$

It can easily be shown that each nonpositive continuous function $f \neq 0$ whose support does not intersect the supports of the finite element functions that vanish at v_1 gives rise to an approximation u_h of u that is positive at v_2, hence violating the discrete maximum principle.

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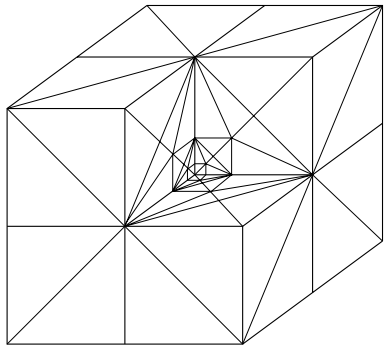

Fig. 15 *Locally refined partition of the Fichera domain into path-tetrahedra.*

It should be mentioned that an alternative approach to enforcing the validity of discrete maximum principles is to modify the finite element method in such a way that obtuse angles do not stand in their way anymore. This approach was followed by Burman and Ern in [15, 16]. The modified method, however, also needs a modified analysis and implementation.

Remark 25. In general, d-linear finite element approximations with respect to partitions into block-type elements may not satisfy the discrete maximum principle for $d \geq 2$. In fact, for $d \geq 4$ the finite element stiffness matrix cannot be diagonally dominant [48]. Problems with $d \geq 4$ are encountered in financial mathematics and theoretical physics (see references in [10]).

In [51] the Coxeter trisection from Figure 11 is used recursively to construct local nonobtuse refinements of partitions in a neighborhood of vertices of a polyhedron by means of path-tetrahedra. For instance, in Figure 15 we see such a refinement near a reentrant corner, where singularities of the solution are expected. Kuhn's triangulation is employed for preconditioning of large problems solved by multigrid methods (see [3, 6]). In [10] it is used to prove gradient superconvergence of linear finite elements in $\mathbb{R}^d$. Bossavit in [9] uses partitions formed by the Sommerville nonobtuse tetrahedra to approximate solutions of electric network problems. Again, just one poorly shaped triangle or tetrahedron may increase the condition number of the matrix associated with the discretization (see [78, sect. 5]). Acute-type partitions are required in the tent-pitcher algorithm (see [90] and [24, p. 238]).

Nonobtuse partitions are also employed in the finite volume method [1]. For instance, in [25] this method requires a strict Delaunay triangulation, which means that the circumscribed ball about each simplex from the partition does not fully contain another simplex from the partition. In the box method, integration over elements from the Delaunay triangulation is replaced by integration over Voronoi cells (boxes) or similar regions centered at nodal points. If the triangulation is nonobtuse, then each cell (box) is entirely included in the patch of triangles surrounding a given inner nodal point, which is necessary for patches close to the boundary of the domain. As mentioned in [14, p. 481], even one obtuse triangle can lead to a large spike in the numerical solution.

In [50], we present sufficient conditions for the existence of partitions into path-tetrahedra with an arbitrarily small mesh size, as formulated in the following theorem.

THEOREM 26. *Let each tetrahedron in a nonobtuse partition contain its circumcenter. Then there exists a family of refined partitions into path-tetrahedra.*

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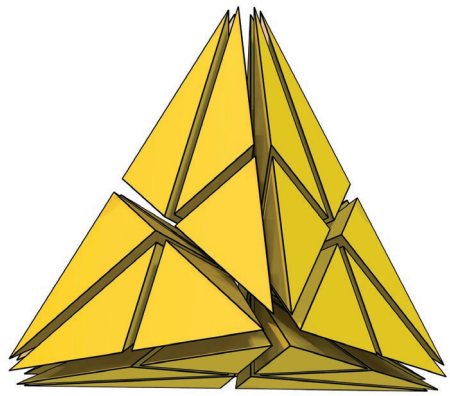

Fig. 16 *Partition of an acute tetrahedron containing its circumcenter into path-tetrahedra.*

Its proof is constructive. Each face (cf. Theorem 13) is first partitioned into four or six right triangles whose common vertex is the center of its circumscribed circle. Then each tetrahedron from the initial partition is partitioned into path-tetrahedra, by taking the convex hulls of the right triangles on its surface with its circumcenter (see Figure 16). Such a refinement technique is called yellow (see [50]). In this case, common faces of adjacent tetrahedra from the initial partition are partitioned in the same manner. The proof then proceeds by induction, since, by Theorems 15, 16, and 18, the assumptions of Theorem 26 are satisfied for any refined partition.

In [59] the nonobtuseness assumption in Theorem 26 is replaced by a weaker condition that requires that only faces are nonobtuse (see Theorem 13).

5. Further Applications. Acute and nonobtuse simplices also show up in other areas of mathematics. In algebra one considers groups of symmetries of the Platonic bodies and their generators. For instance, there are four generators for the cube. The corresponding four planes of symmetries bound a path-tetrahedron which is called the *fundamental domain.* All planes of symmetry divide the Platonic bodies into path-tetrahedra (see Figure 17).

Ortho-simplices served as a tool for the evaluation of an integral. Using the trisection of a path-tetrahedron (see Figure 11) into three smaller path-tetrahedra, Coxeter proved [19] that

$$\int_1^6 \frac{\sec^{-1} x}{(x+2)\sqrt{x+1}} \left(\frac{1}{\sqrt{x+3}} + 2 \right) \mathrm{d}x = \frac{2}{15}\pi^2.$$

A large number of (computational) geometric applications of nonobtuse simplices (in particular, path-tetrahedra) is given in the *Geometry Junkyard* [100]. Using graph theory it is shown in [31, Chap. 14] how to use path-simplices to establish the structure of electric networks. The main theorem (based on [30]) finds the possible networks composed only from resistors inside a "black box" having $n \in \{2, 3, \dots\}$ outlets. Also in geodesy it is convenient to use acute triangulations. The closer the triangle is to an equilateral triangle, the more accurately we can establish the coordinates of particular triangulation points by means of measurement of lengths of edges (and angles). Finally, nonobtuse simplices are also used in computer graphics [24], mathematical

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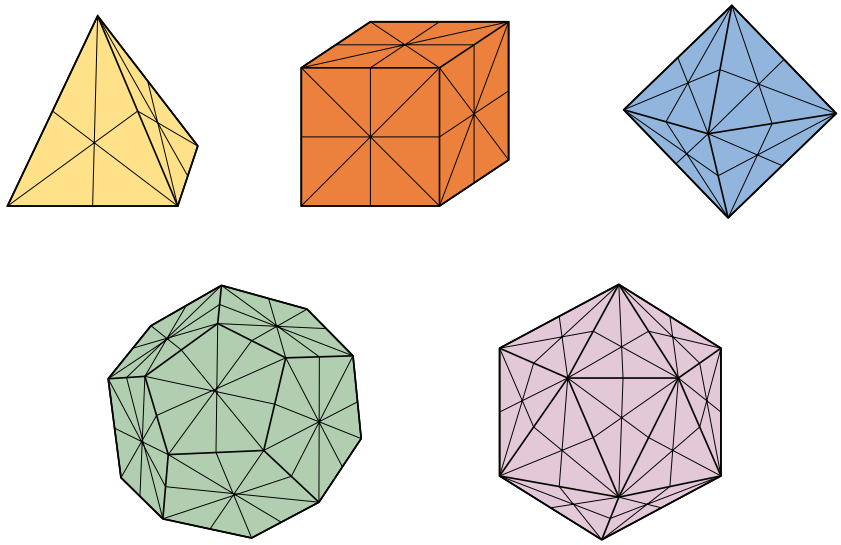

Fig. 17 *Partition of the Platonic bodies into path-tetrahedra.*

genetics [66], crystallography [32, 77], in the finite difference method [49], and in the Monte Carlo method for solving partial differential equations [97, p. 210]. There are surely numerous applications and occurrences of nonobtuse and acute simplices that are not listed in this paper.

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