94.2.4. *An Inequality Between Perpendicular Least-Squares and Ordinary Least-Squares*, proposed by H. Peter Boswijk and Heinz Neudecker. Let $Z = [Z_1 : Z_2]$ be an $n \times p$ matrix with $n > p$, and with $Z_i$ matrices of order $n \times p_i$, $i = 1,2$. Define $M_2 := I_n - Z_2 Z_2^+$, where $Z_2^+$ is the Moore–Penrose inverse of $Z_2$; if $\text{rank}(Z_2) = p_2$, then $M_2 = I_n - Z_2(Z_2'Z_2)^{-1}Z_2'$. Moreover, let $\lambda_1 \leq \cdots \leq \lambda_p$ denote the eigenvalues of $Z'Z$.

1. Prove that

$$\sum_{i=1}^{p_1} \lambda_i \leq \text{tr}(Z_1' M_2 Z_1), \tag{1}$$

where $\text{rank}(Z_2) = l \leq p_2$.

2. Let $S$ denote a $p \times p$ random matrix with a standard central Wishart distribution with $n$ degrees of freedom, i.e., $S \sim W(I_p, n)$, let $\mu_1 \leq \cdots \leq \mu_p$ denote its eigenvalues, and define

$$s_1 := \sum_{i=1}^{p_1} \mu_i. \tag{2}$$

Finally, let $s_2$ denote a random variable with a $\chi^2$ distribution with $p_1(n - p_2)$ degrees of freedom. Use (1) to prove that $s_1$ is stochastically dominated by $s_2$, i.e., for any $c > 0$,

$$P\{s_1 > c\} \leq P\{s_2 > c\}. \tag{3}$$

Remarks.

(i) The right-hand side of (1) is equal to the trace of the residual sum of squares from a multivariate ordinary least-squares (OLS) regression of $Z_1$ on $Z_2$. The left-hand side is the corresponding quantity for a multivariate version of perpendicular least-squares (PLS). Hence, (1) entails that PLS yields a smaller residual sum of squares than OLS.

(ii) An application of (3) can be found in reduced rank regression, cf. Anderson [1]. Consider the multivariate normal linear regression $Y = XB + U$, with $Y, U: n \times g$, $X: n \times k$, and $B: k \times g$, and with $n > g \geq k$. The likelihood ratio (LR) statistic for the hypothesis $\text{rank}(B) \leq r$, $r < k$, has an asymptotic $\chi^2$ distribution with $(g - r)(k - r)$ degrees of freedom under the null hypothesis, provided that $\text{rank}(B) = r$. On the other hand, if $\text{rank}(B) = q < r$, then the LR statistic has the same asymptotic distribution as the sum of the $(k - r)$ smallest eigenvalues of a $W(I_{k-q}, g - q)$ matrix. Result (3) shows that the latter distribution is more concentrated towards the origin, so that if $\chi_\alpha^2$ is the $100\alpha\%$ critical value of the $\chi^2((g - r)(k - r))$ distribution, then under the null hypothesis and as $n \to \infty$, $P\{LR > \chi_\alpha^2\} \to \alpha_q \leq \alpha$. Thus the size of the test is controllable (cf. Cragg and Donald [2, Theorem 2] for a related result).

*REFERENCES*

1. Anderson, T.W. Estimating linear restrictions on regression coefficients for multivariate normal distributions. *Annals of Mathematical Statistics* 22 (1951): 327–351.
2. Cragg, J.G. & S.G. Donald. Testing identifiability and specification in instrumental variable models. *Econometric Theory* 9 (1993): 223–240.

94.2.5. *Eigenvalues of the Product of Non-negative Definite Matrices*, proposed by Götz Trenkler. Let $A$ and $B$ be two non-negative definite matrices. Show that the eigenvalues of $AB$ are real and non-negative. What happens if $A$ and $B$ are positive definite? Warning: $AB$ is not necessarily non-negative definite!

94.2.6. *Convergence of a Nonlinear Time Series Model*, proposed by Peter C.B. Phillips. In the model

$$X_t = (\tfrac{1}{2} + \varepsilon_t)X_{t-1}, \qquad t = 1,2,\ldots$$

the shocks $\varepsilon_t$ are independent and identically distributed with mean zero and variance $\frac{1}{4}$, and $X_0$ is a random variable with zero mean and finite variance $\sigma^2 > 0$.

Show that $Z_t = 2^{t/2}X_t$ converges almost surely as $t \to \infty$. (Hint: Use the martingale convergence theorem for a suitable function of $Z_t$.) Hence, show that $X_t \to_{\text{a.s.}} 0$ as $t \to \infty$.

## SOLUTIONS

93.2.1. *The Maximum Rank Correlation Estimator and the Rank Estimator in Binary Choice Models*—Solution, proposed by Frank A.G. Windmeijer. With the underlying latent variable $Y^*$, the probability model can be specified as

$$P(Y_i = 1 \mid X_i) = P(Y_i^* \geq 0 \mid X_i) = P(\varepsilon_i \leq X_i'\beta_0 \mid X_i) = F_\varepsilon(X_i'\beta_0)$$
$$i = 1,\ldots,n;$$

where $(X_1,\varepsilon_1),\ldots,(X_n,\varepsilon_n)$ is a sample from a distribution $P$ on $\mathbf{R}^{k+1}$.

$R_n(\beta)$ can now be written in the form

$$R_n(\beta) = \int r(z,x,\varepsilon;\beta)\, dP_n(z,\eta)\, dP_n(x,\varepsilon),$$

where

$$r(z,x,\varepsilon;\beta) = 1_{\{\varepsilon \leq x'\beta_0\}} 1_{\{x'\beta \geq z'\beta\}} + 1_{\{\varepsilon > x'\beta_0\}} 1_{\{x'\beta < z'\beta\}},$$

and $P_n$ is the empirical measure of the pairs $(X_i,\varepsilon_i)$.

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