COBWEBS, CHAOS AND BIFURCATIONS

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The cobweb model is one of the simplest economic models. The model describes the price behaviour in a single market. We investigate the dynamics of a nonlinear cobweb model with adaptive expectations, given by the following four equations, see Nerlove [6]:

$$q_t^d = D(p_t), \tag{1}$$

$$q_t^s = S(\hat{p}_t), \tag{2}$$

$$q_t^d = q_t^s, \tag{3}$$

$$\hat{p}_t = \hat{p}_{t-1} + w(p_{t-1} - \hat{p}_{t-1}) \qquad 0 \le w \le 1. \tag{4}$$

The first equation states that the demand for goods q_t^d at time t is a function of the current price p_t. Equation (2) states that the supply for goods q_t^s at time t is a function of the expected price $\hat{p}_t$ for period t. The market clearing eq. (3) states that supply equals demand at each time t. Equation (4) describes the expectation mechanism. Producers have adaptive expectations about price: If the actual price p_{t-1} is larger (smaller) than the previous expected price $\hat{p}_{t-1}$ then the new expected price $\hat{p}_t$ is revised upwards (downwards). Note that the new expected price is just a weighted average of the old expected price and the old price. The parameter w is called the expectations weight factor.

It will be useful to consider the case $w = 1$ first. We then have that $\hat{p}_t = p_{t-1}$, i.e. the expected price is just the previous price. This type of expectations is called naive expectations and was used in the original version of the cobweb model (which we will call the traditional cobweb model), see e.g. Ezekiel [3]. It is well known that, if the demand curve is decreasing and the supply curve is increasing, then only three types of dynamics are possible: convergence to a unique stable equilibrium price, convergence to stable period two oscillations or unbounded oscillations.

Recently, it has been shown by Artstein [1] and Jensen and Urban [5] that in the traditional cobweb model chaotic price behaviour can occur if at least one of the supply and demand curves is non-monotonic. In order to understand the occurrence of chaos from a geometric point of view, recall that the price–quantity time paths in the traditional cobweb model can graphically be represented as a back-and-forth movement, along horizontal and vertical line segments, between the supply

and demand curves. If at least one of the supply and demand curves is non-monotonic, then it is possible that such a graphical construction of a price–quantity time path is aperiodic, i.e. it never repeats itself and it does not "converge" to a periodic construction. The result is then an aperiodic or chaotic cobweb, see fig. 1(a).
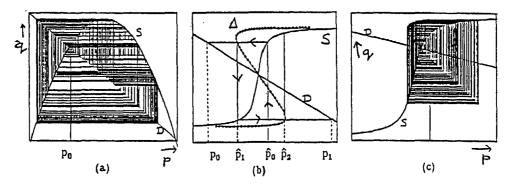

Fig. 1. Chaotic cobwebs.

We now turn to the cobweb model with adaptive expectations. We assume that both the supply and demand curves are monotonic. First we consider the following problem: Is chaotic price behaviour possible?

First we will address this problem from a geometric point of view. We introduce a curve Δ, see fig. 1(b), describing the new expected price p as a function of a given quantity of goods q. More precisely, a point (p, q) belongs to Δ, if $p = wD^{-1}(q) + (1 - w)S^{-1}(q)$. In fact the curve Δ is just a weighted average in the horizontal direction of the demand and the supply curves. If at least one of the supply and demand curves is nonlinear (but both are monotonic), then the curve Δ will also be nonlinear and it may be non-monotonic. Graphically, the (expected) price–quantity time paths in the cobweb model with adaptive expectations can be represented as a back-and-forth movement along horizontal and vertical line segments between the supply curve S and the curve Δ, see figs. 1(b) and 1(c). The fact that the curve Δ may be non-monotonic then explains the possibility of chaotic (expected) price behaviour, even if both the supply and demand curves are monotonic.

In order to analyze the dynamics of the model in more detail, from eqs. (1) to (4) we derive a difference equation $\hat{p}_{n+1} = f(\hat{p}_n)$, generating the expected price behaviour. We assume that the demand curve D is linearly decreasing, i.e. $D(p) = a - bp$, $b > 0$. The map f is then given by

$$f(x) = -wS(x)/b + (1-w)x + aw/b. \tag{5}$$

Note that the price dynamics are qualitatively the same as the expected price dynamics. We analyze the (expected) price dynamics generated by the map f, for a fairly

general class of nonlinear, increasing, S-shaped supply curves S. More precisely, we make the following assumptions, cf. Chiarella [2]:

    (i) the supply curve S is increasing and bounded;

    (ii) the supply curve S has a unique inflection point p^*, for which the slope of S is maximal.

It is convenient to choose the inflection point as the origin. As an example we consider the supply curve $S_\lambda(\hat{p}) = \arctan(\lambda\,\hat{p})$; the parameter λ tunes the "steepness of the S-shape". For this choice of the supply curve, the map f in (5) has two critical points and a unique fixed point x_{eq}.

From the economic point of view the following question is important: How does the price behaviour of the model depend on the height (parameter a) and the slope (parameter b) of the demand curve, the S-shape (parameter λ) of the supply curve and the expectations weight factor w? We will concentrate on the dependence w.r.t. the parameters a and λ.

In fig. 2 we present several bifurcation diagrams of the map f, w.r.t. the parameter a, for different choices of λ. For $\lambda = 3$ and $\lambda = 3.6$ there are finitely many period doubling and period halving bifurcations, but there is no chaos. For $\lambda = 3.9$ and $\lambda = 4$ chaos does occur, but there is always some regularity in the time paths; for example for $\lambda = 4$ in the chaotic region the expected price constantly jumps from a small to a large and then to a small value again, etc. (2-cyclic chaos). For $\lambda = 4.8$ chaos also occurs, but now the 2-cyclic regularity has disappeared. Apparently, increasing λ makes the model more unstable and the price behaviour more complicated.
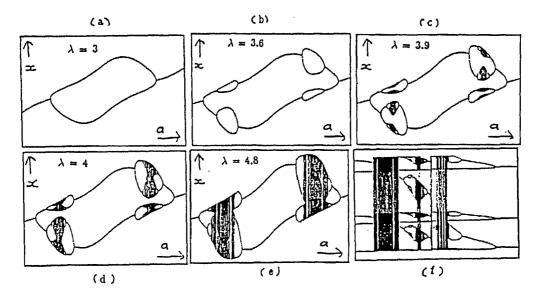

Fig. 2. Bifurcation diagrams w.r.t. the parameter a.

We also present theoretical results for a general class of S-shaped supply curves $S_\lambda(x) = g_\lambda(x)$, where g is a smooth map, which is increasing, bounded and has a unique inflection point in the origin, where the slope of g is maximal. For b and w fixed, and λ sufficiently large, the following holds (compare fig. 2(e)):

(a) there exist $a_{min} < 0 < a_{max}$, such that f has a stable equilibrium for $a \leq a_{min}$ and for $a \geq a_{max}$;

(b) there exists an interval of a-values, containing $a = 0$, for which the map f has a stable period-2 orbit;

(c) there exist two intervals A_1 and A_2 of a-values, with $A_1 \subset (a_{min}, 0)$ and $A_2 \subset (0, a_{max})$ such that the map f is chaotic for all a in A_i, $i = 1, 2$.

A proof of these results is given in Hommes [4].

We would like to point out that the period doubling and halving bifurcations may interchange many times. We have found a numerical example in which, from the stable equilibrium for $a < a_{min}$ to the other stable equilibrium for $a > a_{max}$, period doubling and period halving bifurcations interchange at least 19 times (!), cf. fig. 2(f). This is in sharp contrast to the simple economic assumptions from which we started, in particular the linearity of the demand curve and the assumptions about the slope of the supply curve.

Finally, we would like to emphasize that the cobweb model with adaptive expectations illustrates, in a nice way, how monotonic nonlinear curves may lead to the occurrence of chaotic behaviour in a simple economic model.

References

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