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Supermassive black holes as giant Bose-Einstein condensates

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Abstract – The Schwarzschild metric has a divergent energy density at the horizon, which motivates a new approach to black holes. If matter is spread uniformly throughout the interior of a supermassive black hole, with mass $M \sim M_\star = 2.34\,10^8 M_\odot$, it may arise from a Bose-Einstein condensate of densely packed H atoms. Within the relativistic theory of gravitation with a positive cosmological constant, a bosonic quantum field is coupled to the curvature scalar. In the Bose-Einstein condensed ground state an exact, self-consistent solution for the metric is presented. It is regular with a specific shape at the origin. The redshift at the horizon is finite but large, $z \sim 10^{14} M_\star/M$. The binding energy remains as an additional parameter to characterize the BH; alternatively, the mass observed at infinity can be any fraction of the rest mass of its constituents.

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On the basis of the Schwarzschild, Kerr and Kerr-Newman metrics, it is generally believed that black holes (BHs) are singular objects with all matter localized in the center or, if rotating, on an infinitely thin ring. Recent approaches challenge this unintuitive assumption and consider matter just spread throughout the interior [1–3]. Here we shall follow this line of research. To start, let us just look at some orders of magnitude. For solar-mass neutron stars it is known that the density is about the nuclear density. Solar-mass black holes are about 10 times smaller, thus 1000 times more compact. Clearly, this begs for a quantum chromodynamics description in curved space.

We shall focus on the other extreme, superheavy BHs. They occur in the center of each galaxy and weigh about $M_{\rm BH} = 0.0012\, M_{\rm bulge}$ [4]. Let us assume that they consist of hydrogen atoms and that mass and particle number are related as $M \equiv \nu N m_{\rm H}$ with some $\nu \leqslant 1$. If we neglect rotation, the radius is $R = GM/c^2$ (see below). We may compare the BH density $3N/4\pi R^3$ with the one of densely packed, non-overlapping H atoms, that is, with the Bohr density $n_{\rm B} \equiv 3/4\pi a_0^3$, with $a_0 = 0.529$ Å the Bohr radius. This yields a mass $M_\star = c^3 (a_0^3/G^3 m_{\rm H})^{1/2}$ (we take $\nu = 1$ here), which lies in the range of observed supermassive black holes, $M_\star = 4.66\,10^{38}\,{\rm kg} = 2.34\,10^8\, M_\odot$.

Next comes the question of how matter can withstand the enormous pressure normally associated with such high densities. It was proposed originally by Sacharov that the vacuum equation of state $p = -\rho$ could describe matter at superhigh densities [5]. Laughlin and coworkers assume that matter near the horizon could be in its Bose-Einstein condensed (BEC) phase, modeled by the vacuum equation of state [1]. Dymnikova considers BHs obeying it in the interior, which, however, have one or two horizons [2]. Mazur and Mottola take the BEC idea over to the interior, and investigate a "gravo-star", of which the interior obeys the vacuum equation of state, and which is surrounded by a thin shell of normal matter having the stiff equation of state $p = +\rho$. This solution is regular everywhere [3].

We shall demonstrate that a supermassive BH can exist as a self-gravitating hydrogen cloud, in a Bose-Einstein condensed phase. We study the problem in a series of improved starting points: assume a stiff equation of state, self-consistently solve a quantum field coupled to the curvature scalar, first for a uniform ground-state wave function and next for a space-dependent one. Hereto we have to employ the relativistic theory of gravitation (RTG), which reproduces all weak gravitational effects in the solar system [6,7] as well as the ΛCDM cosmology [8].

We consider a static metric with spherical symmetry,

$$ds^2 = U(r)c^2 dt^2 - V(r)dr^2 - W^2(r)(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{1}$$

The gravitational energy density arises from the Landau-Lifshitz pseudo-tensor [9], generalized to become a tensor

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in Minkowski space [8,10]. For (1) it takes the form

$$t^{00} = \frac{c^4 W^2}{8\pi G r^6}\left(-\frac{r^2 V' W W'}{V} + r^3 V' - 5r^2 W'^2 + \frac{2r^3 V W'}{W} + 8rWW' - 2r^2 V - 3W^2\right). \quad (2)$$

Let us start with the general theory of relativity (GTR). The Schwarzschild metric reads in the harmonic gauge

$$U_S = \frac{1}{V_S} = 1 - \frac{2M}{W_S} = \frac{r-M}{r+M}, \qquad W_S = r + M. \quad (3)$$

(We put $G = c = \hbar = 1$.) It is singular at the horizon $r_h = M$ and involves the gravitational energy density

$$t^{00} = \frac{1}{4\pi r^2}\frac{\mathrm{d}}{\mathrm{d}r}\frac{M(r+M)^3(2r+M)}{2r^3(r-M)}. \quad (4)$$

Its quadratic divergence at r_h presents a hitherto overlooked peculiarity, that induces a negative infinite contribution to the total energy. For this reason, we shall switch to the RTG with matter not located at the singularity $r = 0$, but just spread out within the horizon.

Quantum field theory of Bose-Einstein condensed black holes. – Let our H atoms be described by a bosonic field

$$\hat{\psi}(\mathbf{r}, t) = \sum_i \hat{a}_i \psi_i(\mathbf{r}) e^{-iE_i t},$$

where $i = \{n, \ell, m\}$, $[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}$ and eigenfunctions factor as $\psi_i(\mathbf{r}) = \phi_n(r) Y_{\ell m}(\theta, \phi)$. The rotating wave approximation then leads to the Lagrangian [11]

$$L_{\mathrm{mat}} = g^{\mu\nu}\partial_\mu \hat{\psi}^\dagger \partial_\nu \hat{\psi} - (m^2 + \xi R)\hat{\psi}^\dagger \hat{\psi} - \frac{\lambda}{4}\hat{\psi}^{\dagger 2}\hat{\psi}^2. \quad (5)$$

For a field in curved space the renormalization group generates a coupling to the Ricci curvature scalar R [12]. Its strength ξ is for now a phenomenological parameter. The dimensionless coupling $\lambda = 8m^2 cg/\hbar^3$ with $g = 4\pi\hbar^2 a_s/m$ models the two-particle interaction by the scattering length. For hydrogen in flat space one has [13]

$$a_s = 0.32\, a_0 \quad \text{singlet state}, \qquad a_s = 1.34 a_0 \quad \text{triplet}.$$

We shall continue with the singlet value $\lambda = 0.81\, 10^7$.

With $\Psi_0 = (2E_0 N_0)^{1/2}\psi_0 e^{-iE_0 t}$ for N_0 ground-state atoms, the relativistic Gross-Pitaevskii equation reads

$$\left(\frac{1}{\sqrt{-g}}\partial_\mu \sqrt{-g} g^{\mu\nu}\partial_\nu + m^2 + \xi R + \frac{\lambda|\Psi_0^2|}{4E_0}\right)\Psi_0 = 0. \quad (6)$$

A homogeneous ground state, $\Psi_0(r,t) = \Psi_0 e^{-iE_0 t}$ occurs when

$$-\frac{E_0^2}{U} + m^2 + \xi R + \frac{\lambda}{4E_0}|\Psi_0|^2 = 0. \quad (7)$$

We focus on the RTG, which describes gravitation as a field in Minkowski space [6,7] and possesses the same gravitational energy momentum tensor and thus also the gravitational energy density eq. (2) [8]. It extends the Hilbert-Einstein action with the cosmological term and a bimetric coupling between the Minkowski (γ) and Riemann (g) metrics,

$$L = -\frac{R}{16\pi} - \rho_\Lambda + \frac{1}{2}\rho_{\mathrm{bi}}\gamma_{\mu\nu} g^{\mu\nu} + L_{\mathrm{mat}}. \quad (8)$$

(For $\rho_{\mathrm{bi}} = 0$ it is just a field theory for the GTR.) One has

$$\rho_{\mathrm{tot}} = \rho + \rho_\Lambda + \frac{\rho_{\mathrm{bi}}}{2U} - \frac{\rho_{\mathrm{bi}}}{2V} - \frac{\rho_{\mathrm{bi}} r^2}{W^2},$$
$$p_i^{\mathrm{tot}} = p_i - \rho_\Lambda + \frac{\rho_{\mathrm{bi}}}{2U} - \frac{\rho_{\mathrm{bi}}}{2V} + \frac{\rho_{\mathrm{bi}} r^2}{W^2} \quad (9)$$

with $i = r, \theta, \phi$. The value $\rho_{\mathrm{bi}} = \rho_\Lambda$ is imposed to have a Minkowski metric in the absence of matter. One may fix them to the observed positive cosmological constant [8]. However, historically the opposite choice $\rho_\Lambda < 0$ was considered and the cosmological data were described by an additional inflaton field [7]. So the sign of ρ_{bi} is not known yet; we show that solving a realistic black hole settles this issue. The new point of the RTG is that $g_{00} = U$ can be very small. Despite the smallness of ρ_{bi}, the ρ_{bi}/U-terms become relevant near the horizon [6,8], bringing

$$R = -8\pi T_{\mathrm{tot}} = 8\pi\left(-\rho + p_r + p_\theta + p_\phi + \frac{\rho_{\mathrm{bi}}}{U}\right). \quad (10)$$

To start, let us consider the stiff equation of state

$$\rho = \frac{1}{2}\rho_c\left(\frac{U_c}{U} + 1\right), \qquad p_i = p \equiv \frac{1}{2}\rho_c\left(\frac{U_c}{U} - 1\right), \quad (11)$$

where $\rho_c = 3/32\pi M^2$. For $U_c = 0$ it is the vacuum equation of state $p = -\rho =$ const. One gets

$$R = 8\pi\frac{\rho_c U_c + \rho_{\mathrm{bi}}}{U} - 16\pi\rho_c. \quad (12)$$

Equation (7) has a solution due to the ξR-term. The constant and $1/U$-terms imply

$$\xi = \xi_0\left(1 + \frac{\lambda|\Psi_0|^2}{4E_0 m^2}\right), \qquad E_0^2 = 8\pi\xi(\rho_c U_c + \rho_{\mathrm{bi}}), \quad (13)$$

respectively. The dimensionless parameter

$$\xi_0 = \frac{2}{3}m^2 M^2 = 1.80\, 10^{54}\left(\frac{M}{M_\star}\right)^2, \quad (14)$$

appears to be large, but since $R \sim 1/M^2$ the combination ξR is just of order m^2, making its effect of order unity. The metric can be solved as below; we omit details.

Self-consistent field theory. – Rather than imposing an equation of state, the material energy momentum tensor $T_m^{\mu\nu}$ should be derived from first principles, *i.e.* from the quantum field theory for the H atoms. Its energy density reads, if we exclude the effect of the ξR-term,

$$\rho_m = \frac{\langle\partial_t\psi^\dagger\partial_t\psi\rangle}{U} + \frac{\langle\partial_r\psi^\dagger\partial_r\psi\rangle}{V} + \frac{\langle\partial_\theta\psi^\dagger\partial_\theta\psi\rangle}{W^2} + \frac{\langle\partial_\phi\psi^\dagger\partial_\phi\psi\rangle}{W^2\sin^2\theta} + m^2\langle\psi^\dagger\psi\rangle + \frac{\lambda}{4}\langle\psi^{\dagger 2}\psi^2\rangle. \quad (15)$$

The pressures $(p_r^m, p_\theta^m, p_\phi^m)$ have this shape with signature $(++-----)$, $(+-+---)$, and $(+--+--)$, respectively. Spherical symmetry will imply that $p_\theta^m = p_\phi^m \equiv p_\perp^m$. For a uniform ground state p_m is isotropic,

$$(\rho_m, p_m) = \frac{1}{2}\left(\frac{E_0}{U} \pm \frac{m^2}{E_0}\right)|\Psi_0^2| \pm \frac{\lambda|\Psi_0^4|}{16E_0^2}. \tag{16}$$

They consist of a vacuum part $p = -\rho =$ const and a stiff part $p = +\rho \sim 1/U$, the types studied in [3] and above. In the non-relativistic ($E_0 = m$) and flat-space ($U = 1$) limit, they reduce for $\lambda = 0$ to $\rho_m = mc^2|\Psi_0^2|$ and $p_m = 0$.

Because of the ξR-term in (5), the Einstein equations embody a direct backreaction of matter on curvature, $G^{\mu\nu} = 8\pi(T_m^{\mu\nu} + T_\Lambda^{\mu\nu} + T_{\rm bi}^{\mu\nu}) - 16\pi\xi\langle\psi^\dagger\psi\rangle G^{\mu\nu}$. To connect to the standard notation, $G^{\mu\nu} = 8\pi T_{tot}^{\mu\nu}$, we define $T^{\mu\nu}$ by

$$T_{\rm tot}^{\mu\nu} = \frac{T_m^{\mu\nu} + T_\Lambda^{\mu\nu} + T_{\rm bi}^{\mu\nu}}{1+B} \equiv T^{\mu\nu} + T_\Lambda^{\mu\nu} + T_{\rm bi}^{\mu\nu}, \tag{17}$$

with direct backreaction strength of matter on the metric

$$B = 16\pi\xi\langle\psi^\dagger\psi\rangle = \frac{8\pi\xi|\Psi_0^2|}{E_0}. \tag{18}$$

For $\lambda = 0$ the curvature scalar follows from (16) as

$$R = \frac{8\pi}{1+B}\left(\frac{E_0|\Psi_0^2| + \rho_{\rm bi}}{U} - \frac{2m^2}{E_0}|\Psi_0^2|\right). \tag{19}$$

Solving eq. (7), we find two relations and a consequence,

$$B = 1, \qquad E_0^2 = 8\pi\xi\rho_{\rm bi}, \qquad |\Psi_0^2| = \frac{E_0}{8\pi\xi} = \frac{\rho_{\rm bi}}{E_0}. \tag{20}$$

(The GTR situation, reached by taking $\rho_{\rm bi} \to 0$ first, would not allow a meaningful solution.) The first identity expresses a 100% direct backreaction of matter on the metric. This motivates the introduction of the parameters [8]

$$\mu = \sqrt{16\pi\rho_{\rm bi}} = \sqrt{2\Lambda}, \qquad \bar\mu = \mu M = 7.90\,10^{-15}\frac{M}{M_\star}. \tag{21}$$

Instead of searching for a finite U, as for boson stars [14], we assume a very small U with $U(0) = 0$, coded by a parameter υ,

$$U = \frac{1}{2}\mu^2\upsilon^2 W^2. \tag{22}$$

In terms of the mass function $\mathcal{M}(r)$, defined by

$$V = \frac{W'^2}{1 - 2\mathcal{M}/W}, \tag{23}$$

the 00 and 11 Einstein equations take the form

$$\mathcal{M}' = 4\pi W' W^2 \rho_{\rm tot}, \quad \frac{W - 2\mathcal{M}}{2UW^2}\frac{U'}{W'} - \frac{\mathcal{M}}{W^3} = 4\pi\, p_{\rm tot}^r. \tag{24}$$

The Ansatz (22) solves them and yields, due to (20),

$$\upsilon = 1, \qquad \mathcal{M} = \frac{W}{4} + \frac{W^3}{16M^2}\frac{2m^2M^2}{3\xi}. \tag{25}$$

For the Schwarzschild black hole the horizon occurs when $\mathcal{M} = M$ for $W = 2M$. Concerning the outside metric, we will be close to that situation. This implies again that a mass M corresponds to $\xi = 2m^2M^2/3$.

Let us introduce the "Riemann" variables x and y by

$$x = \frac{W}{2M}, \qquad y = \sqrt{1 - x^2}, \tag{26}$$

so that $U = 2\bar\mu^2 x^2$. The $\rho_{\rm bi}$-terms in (8) violate general coordinate invariance and impose the harmonic gauge,

$$\frac{U'}{U} - \frac{V'}{V} + 4\frac{W'}{W} = \frac{4rV}{W^2}. \tag{27}$$

With (22), (23) and $\mathcal{M} = \frac{1}{2}M(x + x^3)$ from (25), it brings

$$\frac{2x'}{x} - \frac{2x''}{x'} - \frac{2xx'}{1-x^2} + \frac{4x'}{x} = \frac{8rx'^2}{x^2(1-x^2)}.$$

Going to the inverse function $r(x)$ makes it linear,

$$x^2(1-x^2)r'' + x(3 - 4x^2)r' = 4r. \tag{28}$$

The solution is then remarkably simple,

$$r = r_1\left(1 + \frac{y}{\sqrt5}\right)x^{\sqrt5 - 1}(1+y)^{-\sqrt5}, \tag{29}$$

(The second independent solution with $\sqrt5 \to -\sqrt5$ is singular.) This determines the metric function V,

$$V = \frac{2W'^2}{y^2} = \frac{5M^2}{2r_1^2}x^{4-2\sqrt5}(1+y)^{2\sqrt5}. \tag{30}$$

Putting these results together, it now follows that

$$\rho = \frac{3}{64\pi M^2}, \qquad p = -\frac{3}{64\pi M^2}, \tag{31}$$

as the $1/U$-terms cancel due to the relation $E_0^2 = 8\pi\xi\rho_{\rm bi}$. So, after all, we reproduce the vacuum equation of state.

To normalize $|\Psi_0|$, we need the $3d$ volume element in the future time direction, $\mathrm{d}\Sigma^\mu = \mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi n^\mu\sqrt{-g_3} \equiv \delta_0^\mu\mathrm{d}\mathcal{V}$, set by the timelike unit vector $n^\mu = \delta_0^\mu/\sqrt U$ and $g_3 = -VW^4 sin^2\theta$. This results in $\mathrm{d}\mathcal{V} = \mathrm{d}r\,\mathrm{d}\Omega\sqrt{V/U}\,W^2$.

The general inner product [12] $(\psi_1, \psi_2) = -i\int\mathrm{d}\Sigma^\mu \times (\psi_1\partial_\mu\psi_2^* - \psi_2^*\partial_\mu\psi_1)$ defines the orthonormality

$$(\psi_i, \psi_j) = (E_i + E_j)\int\mathrm{d}\mathcal{V}\,\psi_i\psi_j^* \equiv \delta_{ij}. \tag{32}$$

With $\mathrm{d}\mathcal{V} = \mathrm{d}y\,\mathrm{d}\Omega\,8M^3/(\bar\mu\upsilon)$ it yields

$$|\Psi_0^2| = 2E_0N_0|\psi_0^2| = \frac{N_0\upsilon\bar\mu}{32\pi M^3}, \tag{33}$$

having proper ground-state occupation, $\int\mathrm{d}\mathcal{V}\,|\Psi_0|^2 = N_0$.

In eq. (13) the correction term is of order

$$\bar\lambda \equiv \frac{\lambda}{32\pi m^2\xi} = \frac{3\lambda}{64\pi\, m^4M^2} \approx 7.58\,10^{-12}\frac{M_\star^2}{M^2}. \tag{34}$$

These corrections seem relevant for BHs with masses $M \sim 2.75\,10^{-6}M_\star \sim 645\,M_\odot$. Much less below $M_\star$ the hydrogen atoms will get ionized, calling for fermionic fields for protons and electrons, which by a BCS pairing can again undergo a BEC transition. This BCS-BEC scenario is beyond the aim of the present paper.

The exterior. – At the horizon $r_h \approx r_1$, $y_h \ll 1$ one has

$$U = 2\bar{\mu}^2, \quad V = \frac{5}{2}, \quad W = 2M, \quad W' = \frac{1}{2}\sqrt{5}\, y_h. \tag{35}$$

We have to connect this to the vacuum solution outside the BH. Well away from matter, the harmonic constraint brings the Schwarzschild shape (3), where $M \equiv \mathcal{M}(r_h)$ is the mass, essentially as observed at infinity. The values (35) are far from Schwarzschild's ones, even when r is near M (*e.g.*, $W'_S = 1$). The problem, nevertheless, appears to be consistent. Near r_h we need the deformation of the Schwarzschild metric which regularizes its singularity due to the bimetric coupling [6,7]. An elegant scaling form for small $\bar{\mu}$ was given by the present author, [8],

$$\begin{aligned} r &= M\frac{1+\eta(e^\xi+\xi+\log\eta+2)}{1-\eta(e^\xi+\xi+\log\eta+2)}, \qquad U = \eta e^\xi, \\ V &= \frac{e^\xi}{\eta(1+e^\xi)^2}, \qquad W = \frac{2M}{1-\eta e^\xi - \bar{\mu}^2(\xi+w_0)}. \end{aligned} \tag{36}$$

Here ξ is the running variable and η a small scale. For $\eta e^\xi = \mathcal{O}(1)$ it coincides with the Schwarzschild solution. Matching with the interior appears to be possible,

$$e^{\xi_h} = \sqrt{5}\bar{\mu}, \quad \eta = \frac{2}{\sqrt{5}}\bar{\mu}, \quad W' = e^{\xi_h} + \frac{\bar{\mu}^2}{\eta} = \frac{3}{2}\sqrt{5}\bar{\mu}, \tag{37}$$

implying $y_h = 3\bar{\mu}$. Taken together, the three regimes, interior, horizon and exterior, provide an exact solution of the problem. At the origin it exhibits the singularities

$$U = \bar{U}_1 r^{\gamma_\mu}, \quad V = \frac{1}{2}\gamma_\mu^2 \bar{W}_1^2 r^{\gamma_\mu - 2}, \quad W = \bar{W}_1 r^{\frac{1}{2}\gamma_\mu}, \tag{38}$$

where $\gamma_\mu = \frac{1}{2}(\sqrt{5}+1)$ is the golden mean. But if we take W as the coordinate, we have in the interior the shape

$$\mathrm{d}s^2 = \frac{1}{2}\mu^2 W^2 \mathrm{d}t^2 - \frac{2\mathrm{d}W^2}{1-W^2/4M^2} - W^2 \mathrm{d}\Omega^2, \tag{39}$$

which is regular at its origin, with the term $2\mathrm{d}W^2$ coding the above singularities.

We may rewrite the exterior solution by eliminating ξ,

$$\begin{aligned} r &= M\frac{1+U+(2\bar{\mu}/\sqrt{5})(\log U+2)}{1-U-(2\bar{\mu}/\sqrt{5})(\log U+2)}, \\ V &= \frac{U}{(U+2\bar{\mu}/\sqrt{5})^2}, \\ W &= \frac{2M}{1-U+2\bar{\mu}^2-\bar{\mu}^2\log(U/2\bar{\mu}^2)}. \end{aligned} \tag{40}$$

This describes the free space region $r \geqslant M$, where $2\bar{\mu}^2 \leqslant U \leqslant 1+\mathcal{O}(\bar{\mu})$. At the cosmic scale $r \sim 1/\mu$ Newton's law picks up the Yukawa-type factor $\cos\mu r$, due to the tachyonic nature of gravitation in the RTG with $\rho_{\rm bi} > 0$ [8].

The interior shape can also be expressed with U as running variable, where it lies in the range $(0, 2\bar{\mu}^2)$. Due to eq. (29) it also holds that

$$W = 2Mx = \frac{1}{\mu}\sqrt{2U}, \qquad y = \sqrt{1-\frac{U}{2\bar{\mu}^2}}. \tag{41}$$

The locus and the metric function V are given by (29), with $r_1 \approx M$, and (30), respectively.

Contributions to the energy. – With the weight $\mathrm{d}\mathcal{V}$ given below (32), the standard expression for the energy, $\int \mathrm{d}\mathcal{V}\rho$ scales as $1/\bar{\mu}$ and even diverges logarithmically at $r=0$. However, in the RTG the energy is determined by eq. (2). At the origin it diverges as $r^{\sqrt{5}-5}$, which is integrable. The gravitational energy inside the BH reads

$$U_{\rm grav,\,int} = 4\pi\int_0^M \mathrm{d}r\, r^2 t^{00} = -842.898\, M.$$

The total energy density reads $\Theta^{00} = t^{00} + VW^4\rho_{tot}/r^4$. We can calculate the material and the bimetric energy,

$$\begin{aligned} U_{\rm mat} &= 4\pi\int_0^M r^2\mathrm{d}r\frac{VW^4}{r^4}\,\frac{3}{64\pi M^2} = 169.431\, M, \\ U_{\rm bi} &= 4\pi\int_0^M r^2\mathrm{d}r\frac{VW^4}{r^4}\,\frac{1}{64\pi M^2 x^2} = 686.466\, M. \end{aligned}$$

Together they make up for

$$U_{\rm interior} = U_{\rm grav,\,int} + U_{\rm mat} + U_{\rm bi} = 13\, M. \tag{42}$$

The energy density in the skin layer first has a large positive and then a large negative part, due to the term $r^3 V'$. The integrated effect is obtained easily since, in the formulation of the Einstein equations in Minkowski space, the total energy density is a total derivative, [8]. The region $r > M$ thus yields $U_{\rm exterior} = -12M$. Together with the interior it confirms the total energy $U = Mc^2$, expected from the decay of the metric, $g_{00} = 1 - 2GM/c^2 r$.

Non-uniform ground state. – Till now we assumed that the ground-state wave function has a constant amplitude, and drops to zero at the horizon. Clearly, this cannot be exact. Considering $\Psi_0 \to \Psi_0(r)$, we first take into account that in deriving the Einstein equations, partial integrations are to be performed. This brings derivatives of $B \sim |\Psi_0|^2$, and induces an extra term $T_B^{\mu\nu}$,

$$(1+B)G_{\mu\nu} = 8\pi(T_m^{\mu\nu} + T_\Lambda^{\mu\nu} + T_{\rm bi}^{\mu\nu} + T_B^{\mu\nu}). \tag{43}$$

The elements of $(T_B)_\nu^\mu \equiv \mathrm{diag}(\rho_B, -p_r^B, -p_\perp^B, -p_\perp^B)$ are

$$\begin{aligned} \rho_B &= \frac{1}{8\pi}\left(\frac{B''}{V} + \frac{2rB'}{W^2} - \frac{B'U'}{2UV}\right), \\ p_r^B &= \frac{-1}{8\pi}\left(\frac{B'U'}{2UV} + \frac{2B'W'}{VW}\right), \\ p_\perp^B &= \frac{-1}{8\pi}\left(\frac{B''}{V} + \frac{2rB'}{W^2} - \frac{B'W'}{VW}\right). \end{aligned} \tag{44}$$

Equation (43) now leads to a total energy momentum tensor

$$T^{\mu\nu}_{\rm tot} = \frac{T^{\mu\nu}_m + T^{\mu\nu}_\Lambda + T^{\mu\nu}_{\rm bi} + T^{\mu\nu}_B}{1+B} \equiv T^{\mu\nu} + T^{\mu\nu}_\Lambda + T^{\mu\nu}_{\rm bi}. \tag{45}$$

$T^\mu_\nu \equiv \mathrm{diag}(\rho, -p_r, -p_\perp, -p_\perp)$ has at $\lambda = 0$ the elements

$$\rho = \frac{1}{8\pi(1+B)}\left(\frac{2B'W'}{VW} - \frac{B'V'}{2V^2} + \frac{B''}{V} + \frac{m^2B}{2\xi} + \frac{B'^2}{8\xi BV}\right),$$
$$p_r = \frac{1}{8\pi(1+B)}\left(-\frac{2B'W'}{VW} - \frac{B'U'}{2UV} - \frac{m^2B}{2\xi} + \frac{B'^2}{8\xi BV}\right),$$
$$p_\perp = \frac{1}{8\pi(1+B)}\left(\frac{B'W'}{VW} - \frac{2rB'}{W^2} - \frac{B''}{V} - \frac{m^2B}{2\xi} - \frac{B'^2}{8\xi BV}\right). \tag{46}$$

The Gross-Pitaevskii equation (6) reads in terms of B

$$-[(6\xi+1)B+1]\left(\frac{B''}{2V} + \frac{rB'}{W^2}\right) + \frac{B'^2}{4BV} + m^2B(1-B) + \bar\lambda m^2B^2 = \left(E_0^2 - \frac{1}{2}\xi\mu^2\right)\frac{B}{U}. \tag{47}$$

We can now first verify that the total energy momentum tensor is conserved due to the harmonic condition (27). The singular term B/U also drops out from (47) for

$$E_0^2 = 8\pi\xi\rho_{\rm bi} = \frac{1}{2}\xi\mu^2. \tag{48}$$

This was already used in (46) to cancel the $1/U$-terms. With $\xi \equiv 2m^2M_1^2/3$ and $\bar\mu \equiv \mu M_1$ it implies again $E_0 = \bar\mu m/\sqrt{3}$. For very small $B \ll 1/\xi \sim 10^{-54}$ there will be an exponential falloff, $B \sim \exp(-\sqrt{5}mr)$, so the horizon is a fraction of the Compton length thick. In this narrow range ρ, p_r and $p_\perp$ vanish smoothly. In the regime $B \gg 1/\xi$, eq. (47) simplifies and actually reduces to eq. (7),

$$-\frac{B''}{V} - \frac{2rB'}{W^2} = \frac{(1-\bar\lambda)B - 1}{2M_1^2}. \tag{49}$$

Here we can consider B as vanishing sharply, $B \sim r_h - r$, so that $T^{\mu\nu}_B \neq 0$, keeping the ultimate exponential tail and decay of ρ and the p_i in mind. Equation (46) then brings

$$\rho = \frac{-1}{8\pi(1+B)}\left(\frac{B'U'}{2UV} - \frac{4+2B+\bar\lambda(4B+3B^2)}{8M_1^2}\right),$$
$$p_r = \frac{-1}{8\pi(1+B)}\left(\frac{2B'W'}{VW} + \frac{B'U'}{2UV} + \frac{6B+3\bar\lambda B^2}{8M_1^2}\right), \tag{50}$$
$$p_\perp = \frac{1}{8\pi(1+B)}\left(\frac{B'W'}{VW} - \frac{4+2B+\bar\lambda(4B+3B^2)}{8M_1^2}\right).$$

Let us first return to the interior where $U = 2\bar\mu^2x^2$, $V = 8M_1^2x'^2/y^2$ and $W = 2M_1x$. Equation (49) can be written as

$$\frac{1}{4}x^2B_{yy} - yB_y + (1-\bar\lambda)B = 1. \tag{51}$$

To understand the structure of the problem, we again take $\lambda = 0$. Then for any A there is the solution

$$B(x) = 1 + Ay = 1 + A\sqrt{1-x^2}. \tag{52}$$

Expressing the shapes (50) in y, we have

$$\rho = -p_\perp = \frac{1}{64\pi M_1^2(1+B)}(2B + yB_y + 4),$$
$$p_r = \frac{-3}{64\pi M_1^2(1+B)}(2B - yB_y). \tag{53}$$

Surprisingly, their A-dependence factors out, keeping a vacuum equation of state $\rho = -p = 3/64\pi M_1^2$, so (52) is a non-uniform, exact solution of the same metric. The horizon $B = 0$ is now located at $r_h > r_1$ where $y_h = -1/A$. (Equation (29) continues to negative $y \approx \sqrt{5}(r_1 - r)/4r_1$ for $r > r_1$.) However, a problem shows up with the matching, since $W'(r_h) \sim -1/A$ cannot be of order $\bar\mu \sim 10^{-14}$ anymore. We thus have to deviate from the exact solution, which leads in general to a numerical problem. Analytically, this question can be considered for large A, by adding $1/A^2$ corrections to the previous solution. Expanding in $1/A$ at fixed $s \equiv A\sqrt{5}(r/r_1 - 1)/4$, we arrive at

$$B = 1 - s + \frac{\sqrt{5}}{2A}s^2 - \frac{7s^3}{6A^2} + \frac{b_1(s)}{A^2},$$
$$\frac{U}{2\bar\mu^2} = 1 - \frac{s^2}{A^2} + \frac{u_1(s)}{A^2},$$
$$\frac{2}{5}V = 1 - \frac{2\sqrt{5}}{A}s + \frac{13s^2}{A^2} + \frac{v_1(s)}{A^2}, \tag{54}$$
$$\frac{W}{2M_1} = 1 - \frac{s^2}{2A^2} + \frac{w_1(s)}{A^2}.$$

The additional terms, found to be

$$b_1 = b_{10} + b_{11}s, \qquad u_1 = u_{10} + 4w_{11}\ln(2-s),$$
$$v_1 = v_{10}, \qquad w_1 = w_{10} - w_{11}\ln(2-s), \tag{55}$$

produce an anisotropy, $\rho \neq -p_r \neq -p_\perp \neq \rho$. The horizon $B = 0$ is located at $s_h = 1 + \sqrt{5}/2A + \mathcal{O}(1/A^2)$, where $W' = \sqrt{5}(w_{11} - 1)/2A$. Clearly, $W' \sim \bar\mu$ from (37) can be attained by tuning $w_{11} = 1 + 3A\bar\mu + O(1/A)$. The maximum of W at $s = 0$ in the absence of the w_1-term has now been shifted to the horizon, which shows that the problem has a proper solution, with A remaining a free parameter. U has a maximum at $s = 1 - \sqrt{3}$. The mass seen at infinity, $M \equiv \mathcal{M}(r_h) = \frac{1}{2}W(r_h)$, coincides with M_1 to order $1/A^2$. At the horizon we can fix r_1 from (36),

$$r_1 = M_1\left[1 - \frac{4}{\sqrt{5}\,A} + \frac{4\bar\mu}{\sqrt{5}}\left(2 + \ln 2\bar\mu^2\right) + \mathcal{O}\left(\frac{1}{A^2}\right)\right]. \tag{56}$$

Properties of the solution. – The ground-state occupation number becomes upon neglecting the $1/A^2$ corrections

$$N_0 = \int \mathrm{d}\mathcal{V}|\Psi_0|^2 = 2\sqrt{3}\,\frac{M}{m}\int_{-1/A}^{1}\mathrm{d}y\,(1 + Ay). \tag{57}$$

We may write the two leading orders as

$$M = \nu N_0 m, \qquad \nu = \frac{1}{\sqrt{3}(2+A)} \approx \frac{1}{\sqrt{3}\,A}. \tag{58}$$

Clearly, the energy Mc^2 of the BH can be any fraction of the rest energy N_0mc^2 of the constituent hydrogen atoms. If ν starts at a value $\nu_c < 1$, our BH is likely approached in an explosive manner, possibly related to jets of quasars.

We found the sharpness of the horizon to be a fraction of the Compton length of H. On a much larger scale $\ell_{\rm grav}$ there is near-horizon growth of the metric functions $U/\eta \approx \eta V \approx e^{\xi}$, taking place in the millimeter range,

$$\ell_{\rm grav} = \frac{{\rm d}r}{{\rm d}\xi} = 2\eta M = \frac{4}{\sqrt{5}}\,\bar{\mu}\,M = 4.88\,10^{-3}\,\frac{M^2}{M_\star^2}\,{\rm m}. \qquad (59)$$

It is a realistic value, small compared to the size of the BH, and still large compared to the Bohr radius.

The interacting situation. – If the non-linearity λ is relevant, a numerical solution is called for. Assuming the same leading-order behaviors near $r = 0$, the above structure survives. Mass and particle number remain independent parameters. This analysis remains as a task for future.

Conclusion. – We have questioned the general wisdom that static BHs have all their mass in the center and cannot be described by present theories. Numerical estimates show that a picture of closely packed H atoms naturally applies to the supermassive BHs in the center of galaxies, $M \sim M_\star = 2.34\,10^8 M_\odot$. We present within the relativistic theory of gravitation (RTG; a colloquial term is: Not-so-General Relativity) an exact solution for a BH, whose interior is governed by quantum matter in its Bose-Einstein condensed phase. Its density decays algebraically in the bulk and exponentially near the horizon. This solution is matched with the Schwarzschild metric, which near the horizon is deformed in the RTG. Power law singularities occur at the origin, that get absorbed in the Riemann description of the metric. Elsewhere, the solution is regular. The redshift at the horizon is finite, though of the order $1/\bar{\mu} \sim 10^{14} M_\star/M$. To specify a BH requires not only the mass (and, in general, charge and spin), but also the rest energy of the constituent matter.

Our BH is a quantum fluid confined by its own gravitation. In the interior, time keeps its standard role. No Planckian physics is involved; Hawking radiation is absent and Bekenstein-Hawking entropy plays no role.

Our BH has one "hair". As one would expect for a classical theory of gravitation, when the quantum matter in the BH has reached a certain ground state, the classical metric allows the system still to go to a lower energy state. Indeed, the passage of celestial bodies will induce oscillations in the metric and emission of gravitational waves, which, upon re-equilibration, increase the binding energy, finally up to 100% of the rest energy of its constituents, Nmc^2. This property may explain the enormous jets and energy output of quasars and also be responsible for very-high-energy cosmic rays, $E > 4 \cdot 10^{19}$ eV [15].

An important question is whether formation of realistic supermassive BHs brings the matter indeed in or near the Bose-Einstein condensed ground state. Also the stability of the solution needs to be studied. It also remains to be seen whether the phenomenological value for the parameter ξ has a microscopic underpinning. Extension to finite temperatures, not presented here, will exhibit a $T^{3/2}$ fraction of thermal atoms.

Calculation of the normal-mode spectrum may lead to predictions that deviate from the ones of the GTR; this spectrum may be observed in the foreseeable future.

If we apply eq. (48) to Logunov's case $\rho_{\rm bi} < 0$, it follows that $\xi < 0$, so $B < 0$ due to (18). To avoid a singularity in *e.g.* (46), a lower bound $B(0) > -1$ will be required. As already indicated by the exact solution (52), B will then go to zero at some point well below M. This prevents fitting to the external metric and excludes our BH solution.

We failed to apply our approach to the GTR, technically because it lacks compensation for the singular $1/U$- terms. If no other solution exists for the considered physical situation, the GTR must be abandoned and replaced by another theory, the RTG being the first candidate. In view of its smaller symmetry group, this may have far reaching consequences for singularities in classical gravitation and for quantum approaches to gravitation, while Minkowski space-time needs no quantization.

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