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7 Microscopic Degeneracies and a Counting Formula

Our main reason for being interested in the $\mathcal{N} = 4$, $d = 4$ string theory compactification introduced in chapter 3 lies in the possibility of studying the supersymmetric states in great details. In particular, the presence of many supersymmetries and a long chain of dualities relating different corners of moduli space makes possible a microscopic understanding of the supersymmetric spectrum of the theory, and this is something that cannot be said for a generic $\mathcal{N} = 2$, $d = 4$ string theory.

In this chapter we will review the microscopic counting of BPS states in the present theory. In section 7.1 we recall the microscopic origin of the $1/2$- and $1/4$-BPS states, and in particular we will see how a microscopic counting formula for dyonic states can be derived using the known D1-D5-P degeneracies. In section 7.2 we review various mathematical properties of this counting formula, which are connected to each other by their relations to a certain Borcherds- (or generalised-) Kac-Moody algebra. These properties will be important for our physical discussion in the next chapter.

7.1 Microscopic Degeneracies

In this section we will discuss the microscopic counting of the $1/2$- and $1/4$-BPS states of the theory, exploiting the chain of dualities introduced in section 3.3.2.

7.1.1 $1/2$- and $1/4$-BPS Solutions

The central charge in the $\mathcal{N} = 4$ supersymmetry algebra can be written as

$$\hat{Z} = \frac{1}{\sqrt{\lambda_2}}(P_L - \lambda Q_L)^m \Gamma_m \quad , \quad m = 1, .., 6 \, , \tag{7.1.1}$$

where $\lambda = \lambda_1 + i\lambda_2$ is the complex scalar which is a part of the $\mathcal{N} = 4, d = 4$ supergravity multiplet introduced in (3.3.1). In the heterotic frame it is the usual axion-dilaton field while in the IIA/$K3 \times T^2$ frame the Kähler moduli of the torus, in the IIB/$K3 \times T^2$ frame the complex moduli of the torus. And P_L, Q_L denote the moduli-dependent left-moving charges given in (3.3.3). Here and from now on all the moduli fields should be understood as being evaluated at spatial infinity.

As mentioned in (2.2.24), there are two BPS bounds in $\mathcal{N} = 4, d = 4$ supersymmetry algebra. Indeed, from

$$\hat{Z}^\dagger \hat{Z} = \frac{1}{\tau_2}|P_L - \tau Q_L|^2 \mathbb{1} - 2iP_L^m Q_L^n \Gamma_{mn}$$

and the fact that the operator $iP_L^m Q_L^n \Gamma_{mn}$ satisfies

$$(iP_L^m Q_L^n \Gamma_{mn})^2 = |P_L \wedge Q_L|^2 \equiv Q_L^2 P_L^2 - (Q_L \cdot P_L)^2 \ , \tag{7.1.2}$$

one concludes that $\hat{Z}^\dagger \hat{Z}$ has the following two eigenvalues

$$\begin{aligned} |Z_{P,Q}|^2 &= \frac{1}{\tau_2}|P_L - \tau Q_L|^2 + 2|P_L \wedge Q_L| \\ \text{and} \qquad |Z'_{P,Q}|^2 &= \frac{1}{\tau_2}|P_L - \tau Q_L|^2 - 2|P_L \wedge Q_L| \ . \end{aligned} \tag{7.1.3}$$

Therefore the 1/4-BPS states of the theory satisfy

$$M_{P,Q} = |Z_{P,Q}| > |Z'_{P,Q}| \ ,$$

while states that preserve half of the supersymmetries must have

$$|P_L \wedge Q_L| = 0 \quad \Leftrightarrow \quad P \parallel Q \ . \tag{7.1.4}$$

7.1.2 Microscopic Degeneracies of 1/2-BPS States

Let's begin with the microscopic counting of states which preserve half of the supersymmetries. From the supersymmetry algebra we have seen that the electric and magnetic charges have to be parallel to each other (7.1.4). Together with the co-prime condition (6.6.7) this means that we can always find the S-duality transformation such that the charges are purely magnetic, namely now we can put $Q = 0$ without loss of generality. The microscopic degeneracy therefore becomes a function of only one T-duality invariant $P^2/2$.

In other words, to count these 1/2-BPS states it is enough to count the perturbative heterotic string states, for example the momentum and winding

modes along the internal six-torus listed in Table 3.1. Recall that the right-moving sector of the heterotic string theory, which is the same as the open bosonic string theory, has non-vanishing zero point energy level -1. The mass shell condition and the level matching condition of the heterotic string therefore reads

$$m^2 = N_L + \frac{1}{2}P_L^2 = N_R - 1 + \frac{1}{2}P_R^2 \,. \tag{7.1.5}$$

Furthermore, supersymmetry requires the supersymmetric (left-moving) side of the string to be at its ground state, namely $N_L = 0$. Combining these we conclude the right-moving oscillator number is given in terms of the charges as

$$N_R = 1 + \frac{1}{2}\left(P_L^2 - P_R^2\right) = 1 + \frac{1}{2}P^2 \,. \tag{7.1.6}$$

Recall that $\frac{1}{2}P^2 \in \mathbb{Z}$ because the charge lattice $\Gamma^{6,22}$ is even and self-dual (unimodular).

There are 24 bosonic oscillators in the right-moving sector, which can be understood as the 24 bosonic oscillators in the light-cone quantisation of the bosonic string theory, which implies that the generating function of the degeneracies of the above 1/2-BPS states is

$$\sum_{\frac{P^2}{2}\in\mathbb{Z}} d(P)\, q^{1+\frac{P^2}{2}} = \prod_{n=1}^{\infty}\left(\frac{1}{1-q^n}\right)^{24} ,$$

or equivalently

$$d(P) = \oint d\sigma\, \frac{e^{-\pi i P^2 \sigma}}{\eta^{24}(\sigma)} \,, \tag{7.1.7}$$

where $\eta(\sigma)$ is the Dedekind eta-function

$$\eta(\sigma) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n) \quad , \quad q = e^{2\pi i\sigma} \,.$$

The 1/2-BPS states in this context are sometimes called the Dabholkar-Harvey states [134]. Notice that from the modular transformation of the eta-function one can see that the asymptotic growth of degeneracy is

$$\log d(P) \sim \sqrt{P^2/2} \tag{7.1.8}$$

and scales linearly with the charges, which is slower than the quadratic growth one expects for the Bekenstein-Hawking entropy of a four-dimensional black hole. In this sense we say that 1/2-BPS states are "small" and form "small" black holes.

7.1.3 Microscopic Degeneracies of 1/4-BPS States

More interesting and more complicated are the dyonic states, meaning states with both magnetic and electric charges non-vanishing in all duality frames and therefore must preserve only four of the sixteen supercharges, as can be seen from the supersymmetry algebra (7.1.4). In this part of the section, following [135, 136], we will derive a microscopic formula counting these 1/4-BPS states, which will be play an important role in the following chapters of the thesis.

Under the assumption (6.6.7) that the degeneracies depend only on the three quadratic invariants $P^2/2, Q^2/2, P \cdot Q$, it is enough to understand the degeneracies of the states with charges highlighted in Table 3.1. Namely, let's now consider states with the following charges in the type IIB frame: Q_1 D1 and Q_5 D5 strings with k units of momenta along the first circle, together with a Taub-NUT along the third circle and $\tilde{k}$ units of momenta along that direction, assuming that the size of the third circle is large compared to the rest of the internal directions[1].

One can immediately work out the three invariants for these charges

$$\begin{aligned} P^2 &= 2Q_1Q_5 \\ Q^2 &= 2k \\ P \cdot Q &= \tilde{k}\,, \end{aligned}$$

and see that for given $P^2/2, Q^2/2, P \cdot Q$ we can always find the corresponding D1, D5 charges, and momenta $Q_1, Q_5, k, \tilde{k}$.

The advantage of studying this relatively simple system is that its microscopic description is relatively well understood, namely the D1-D5-P system in five dimensions. Especially, since we are interested in the index counting the graded (in terms of bosons and fermions) degeneracies of the BPS states which has rigidity properties upon deformations of the theory, we can map the system to a regime with very different coupling constants while still being able to trust the counting from the microscopic theory.

Going back to the Table 3.1, let's first decompactify the circle $S^1_{(3)}$, meaning that we take the limit that the circle is very large in the four-dimensional Planck unit and the theory becomes the five-dimensional theory obtained by compactifying type IIB string theory on $S^1_{(1)} \times K3$. In the five-dimensional description, the KK monopole becomes a Taub-NUT space (1.3.7) with the

[1]Notice that we use Q_1 and Q_5 to denote the components of the charge vector corresponding to D1 and D5 branes respectively, but not the actual number of branes wrapped. The relevant subtlety here is that there is also the geometrically induced D1 charge when a D5 brane is wrapped around the K3.

$S^1_{(3)}$-direction being the direction of the circle fibration. Secondly, we assume that the internal circle $S^1_{(1)}$ is much larger than the size of the $K3$ manifold, then it was proposed that the six-dimensional world-volume theory of the D5-D1 branes is reduced to a two-dimensional supersymmetric sigma model, whose target space is the symmetric product of $\frac{P^2}{2}+1$ copies of $K3$ [137, 113, 138]. Furthermore, there are decoupled modes which are present even when $P^2=0$, corresponding to the closed string modes localised at the tip of the Taub-NUT that may also carry momenta along the internal circle, and the center of mass modes of the D1-D5 system, which may carry momenta along the internal and the Taub-NUT circle. In other words, we can break the theory into three separated parts David:2006yn,Dabholkar:2008zy

$$\Sigma(\mathrm{TN}_1)\times\Sigma(\mathrm{C.O.M.})\times\Sigma(S^{Q_1Q_5+1}K3)\ . \tag{7.1.9}$$

From Table 3.1 we can see that the first part can be dualized to a perturbative heterotic string system and is therefore again counted by the partition function

$$\frac{1}{\eta^{24}(\sigma)}=\frac{1}{q}\prod_{m\geq 1}\frac{1}{(1-q^m)^{24}}\ . \tag{7.1.10}$$

The contribution of the second part is computed to be [135, 133]

$$\frac{1}{(y^{1/2}-y^{-1/2})^2}\prod_{m\geq 1}\frac{(1-q^m)^4}{(1-q^my)^2(1-q^my^{-1})^2}\ . \tag{7.1.11}$$

Now let's look at the third factor of the CFT. Recall that the K3 elliptic genus has the following Fourier expansion

$$\begin{aligned}\chi(\sigma,z)&=\mathrm{Tr}_{\mathrm{RR}}\,(-1)^F\,e^{2\pi izJ_0}\,e^{2\pi i\tau(L_0-\frac{c}{24})}\,e^{-2\pi i\bar\tau(\tilde L_0-\frac{c}{24})}\\&=\sum_{n\in\mathbb{Z}_+,\ell\in\mathbb{Z}}c(4n-\ell^2)q^ny^\ell\ ,\end{aligned} \tag{7.1.12}$$

with $c(-1)=2, c(0)=20$.

The elliptic genus of the symmetric products of $K3$ has the following generating function given in terms of the Fourier coefficients of the K3 elliptic genus [43]

$$\sum_{N\geq 0}p^N\chi(S^NK3;q,y)=\prod_{n>0,m\geq 0,\ell}\left(\frac{1}{1-p^nq^my^\ell}\right)^{c(4nm-\ell^2)}\ , \tag{7.1.13}$$

and this is the contribution from the symmetric product part of the CFT. Identifying the CFT and the spacetime data as

$$k = \frac{Q^2}{2} = L_0 - \bar{L}_0 = L_0 - \frac{c}{24} = \text{momenta along internal circle}$$
$$\tilde{k} = P \cdot Q = J_0 = \text{momenta along TN circle}$$

and combining the three factors, we conclude that the generating function for the degeneracies of the 1/4-BPS states is

$$\sum_{P,Q} (-1)^{P\cdot Q}\, D(P,Q)\, e^{\pi i (P^2\rho + Q^2\sigma + 2P\cdot Q\nu)} = \frac{1}{pqy} \prod_{(n,m,\ell)>0} \left(\frac{1}{1-p^n q^m y^\ell}\right)^{c(4nm-\ell^2)}$$
$$p = e^{2\pi i\rho} \quad , \quad q = e^{2\pi i\sigma} \quad , \quad y = e^{2\pi i\nu} \tag{7.1.14}$$

and $(n,m,\ell) > 0$ means $n, m \geq 0,\ \ell \in \mathbb{Z}$ but $\ell < 0$ when $n = m = 0$.

In particular, the above formula has been shown [139] to reproduce the asymptotic growth which agrees with the macroscopic black hole entropy [140, 141]

$$S(P,Q) = \pi\sqrt{P^2Q^2 - (P\cdot Q)^2}\ . \tag{7.1.15}$$

This is the dyon counting formula, sometimes referred to as the Dijkgraaf-Verlinde-Verlinde formula, conjectured more than ten years ago [139].

7.2 The Counting Formula and a Borcherds-Kac-Moody Algebra

The above dyon counting formula (7.1.14) turns out to have many seemingly unrelated mathematical properties, such as being an automorphic form, having an infinite product expansion, and being the "lift" of a modular form related to the elliptic genus of K3. For later use we will now review the relevant mathematical properties of the following object appearing at the right-hand side of (7.1.14)

$$\Phi(\Omega) = pqy \prod_{(n,m,\ell)>0} \left(1 - p^n q^m y^\ell\right)^{c(4nm-\ell^2)} \tag{7.2.1}$$
$$\Omega = \begin{pmatrix} \sigma & -\nu \\ -\nu & \rho \end{pmatrix} \quad , \quad p = e^{2\pi i\rho}\ ,\ q = e^{2\pi i\sigma}\ ,\ y = e^{2\pi i\nu}\ ,$$

using the presence of a Borcherds-Kac-Moody algebra as the theme connecting these various properties.

7.2.1 Dyons and the Weyl Group

In this subsection we will introduce a vector space of Lorentzian signature which appears naturally in the dyon-counting problem. In particular we consider the vector of quadratic invariants in this vector space, and define a basis for these "charge vectors". This basis defines a Lorentzian lattice of signature $(2,1)$ and generates a group of reflection with respect to them. We then briefly argue the physical relevance of this group while leaving the details for later sections.

In the above formula (7.2.1) we have written the inverse partition function Φ as a function of a 2×2 symmetric complex matrix Ω. Indeed, anticipating the important role played by the S-duality group $PSL(2,\mathbb{Z})$ (3.3.7), it will turn out to be convenient to introduce a space $M_2(\mathbb{R})$ of 2×2 symmetric matrices with real entries

$$M_2(\mathbb{R}) = \left\{ X \middle| \; X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}, \; X = X^T \, , x_{ab} \in \mathbb{R} \right\} \; . \tag{7.2.2}$$

Besides the matrix Ω, the left-hand side of the counting formula (7.1.14) involves another matrix $\Lambda_{P,Q}$ constituted of the three T-duality invariants $(P^2, Q^2, P\cdot Q)$

$$\Lambda_{P,Q} = \begin{pmatrix} P\cdot P & P\cdot Q \\ P\cdot Q & Q\cdot Q \end{pmatrix} \; . \tag{7.2.3}$$

Since this is the vector of invariants of charges that determine the counting of states, in the following we will often refer to this vector $\Lambda_{P,Q}$ in the vector space $M_2(\mathbb{R})$ also as the "charge vector".

From the expression for the Bekenstein-Hawking entropy (7.1.15) for a 1/4-BPS dyonic black hole, which is manifestly invariant under the S-duality group (3.3.7), we see that the vector $(P^2, , Q^2, \; P\cdot Q)$ naturally lives in a space of Lorentzian signature $(+,+,-)$ on which the S-duality group acts as a Lorentz group $PSL(2,\mathbb{Z}) \sim SO^+(2,1;\mathbb{Z})$, where the "+" denotes the time-orientation preserving component of the group.

This motivates us to equip the vector space of 2×2 symmetric real matrices (7.2.2) with the following metric

$$(X,Y) = -\,\epsilon^{ac}\,\epsilon^{bd}\,x_{ab}\,y_{cd} \; = -\mathrm{det}Y\,\mathrm{Tr}(XY^{-1}), \tag{7.2.4}$$

where ϵ is the usual epsilon symbol $\epsilon^{12} = -\epsilon^{21} = 1$.

Especially, the norm of a vector is given by[2]

$$\|X\|^2 = (X,X) = -2\,\mathrm{det}X \; . \tag{7.2.5}$$

[2]The "-2" factor here and in many places later is due to the fact that we choose the normalisation of the metric to be consistent with the familiar convention for Kac-Moody algebras that the length squared of a real simple root is 2.

One can immediately see that this is indeed a vector space of signature (2,1), in which the diagonal entries of the 2×2 matrix play the role of the light-cone directions. As mentioned earlier, an element of the S-duality group $PSL(2,\mathbb{Z})$ acts as a Lorentz transformation on this space: for any real matrix γ with determinant one, one can check that the following action

$$X \to \gamma(X) := \gamma X \gamma^T \tag{7.2.6}$$

is indeed a Lorentz transformation satisfying $\|X\|^2 = \|\gamma(X)\|^2$.

Using this metric, the entropy of a dyonic black hole (7.1.15) becomes nothing but given by the length of the charge vector $\Lambda_{P,Q}$ as

$$S(P,Q) = \pi\sqrt{-\frac{1}{2}\|\Lambda_{P,Q}\|^2} \ . \tag{7.2.7}$$

Similarly, the counting formula (7.1.14) can now (at least formally) be rewritten as the following contour integral

$$D(P,Q) = (-1)^{P\cdot Q} \oint_{\mathcal{C}} d\Omega \, \frac{e^{\pi i(\Lambda_{P,Q},\Omega)}}{\Phi(\Omega)} \ . \tag{7.2.8}$$

Next we would like to consider a basis for the charge vectors $\Lambda_{P,Q}$. From the fact that P^2, Q^2 are both even, it is easy to check that for any dyonic charge which permits a black hole solution, namely for all (P,Q) with $S(P,Q) > 0$, the charge vector $\Lambda_{P,Q}$ is an integral positive semi-definite linear combination of the following basis vectors

$$\alpha_1 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \quad , \quad \alpha_2 = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} \quad , \quad \alpha_3 = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \ . \tag{7.2.9}$$

In other words, for all black hole dyonic charges we have

$$\Lambda_{P,Q} \in \Gamma_+ := \{\mathbb{Z}_+\alpha_1 + \mathbb{Z}_+\alpha_2 + \mathbb{Z}_+\alpha_3\} \ . \tag{7.2.10}$$

As a side remark we note that there is another place where the positive part Γ_+ of the lattice generated by the three vectors $\alpha_{1,2,3}$ appears naturally. Consider the integral vector

$$\alpha = \begin{pmatrix} 2n & \ell \\ \ell & 2m \end{pmatrix} \quad , \quad n, m, \ell \in \mathbb{Z} \ , \tag{7.2.11}$$

from the fact that the Fourier coefficients of the K3 elliptic genus satisfies $c(k) = 0$ for $k < -1$ (2.1.51), one can easily show that the microscopic partition function can be rewritten in the following suggestive form

$$\Phi(\Omega) = e^{-2\pi i(\varrho,\Omega)} \prod_{\alpha\in\Gamma_+} \left(1 - e^{-\pi i(\alpha,\Omega)}\right)^{c(-\|\alpha\|^2/2)} \ , \tag{7.2.12}$$

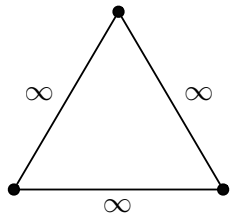

Figure 7.1: The Coxeter graph of the hyperbolic reflection group generated by (7.2.15). See (8.9) for the definition of the Coxeter graph.

where

$$\varrho = \frac{1}{2}\sum_{i=1}^{3}\alpha_i \tag{7.2.13}$$

is the Weyl vector corresponding to the above basis vectors, a name that will be justified later in section 7.2.3.

The matrix of the inner products of the above basis is

$$(\alpha_i, \alpha_j) = \begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{pmatrix} . \tag{7.2.14}$$

We can now define in the Lorentzian vector space $M_2(\mathbb{R})$ the group W generated by the reflection with respect to the spacelike vectors $\alpha_{1,2,3}$:

$$s_i : X \to X - 2\frac{(X, \alpha_i)}{(\alpha_i, \alpha_i)}\alpha_i \quad , \quad i = 1, 2, 3 . \tag{7.2.15}$$

This group turns out to be a hyperbolic Coxeter group with the Coxeter graph shown in Fig 7.1. The definition and basic properties of Coxeter groups can be found in the Appendix 8.9. We will from now on refer to this group as the Weyl group, anticipating the role it plays in the Borcherds-Kac-Moody algebra discussed in the following sections. In particular, we will denote as Δ_+^{re} the set of all positive roots of the Weyl group (8.9.2)

$$\Delta_+^{re} = \{\alpha = w(\alpha_i), w \in W, i = 1, 2, 3\} \cap \Gamma_+ = \{\mathbb{Z}_+\alpha_1 + \mathbb{Z}_+\alpha_2 + \mathbb{Z}_+\alpha_3\} , \tag{7.2.16}$$

as it will turn out to be the set of all positive *real* (*i.e.* spacelike) roots of the Borcherds-Kac-Moody algebra discussed in section 7.2.3.

The physical relevance of this group can be intuitively understood in the following way. We have seen that the S-duality group $PSL(2, \mathbb{Z})$, which acts like (3.3.7), is a symmetry group of the theory. We can further extend this

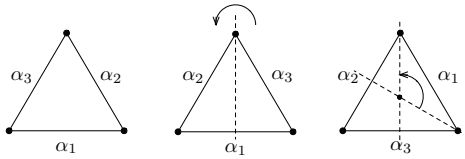

Figure 7.2: The dihedral group D_3, which is the symmetry group of an equilateral triangle, or the outer automorphism group of the real roots of the Borcherds-Kac-Moody algebra (the group of symmetry mod the Weyl group), is generated by an order two element corresponding to a reflection and an order three element corresponding to the 120° rotation.

symmetry group with the spacetime parity reversal transformation

$$\lambda \to -\bar{\lambda} \quad , \quad \begin{pmatrix} P \\ Q \end{pmatrix} \to \begin{pmatrix} P \\ -Q \end{pmatrix} \tag{7.2.17}$$

and thereby extend the group $PSL(2,\mathbb{Z})$ to $PGL(2,\mathbb{Z})$. From the point of view of the Lorentzian space $M_2(\mathbb{R}) \sim \mathbb{R}^{2,1}$, the above element, when acting as (7.2.6), augments the restricted (time-orientation preserving) Lorentz group with the spatial reflection. Notice that the requirement that the inverse of an element $\gamma \in PGL(2,\mathbb{Z})$ is also an element implies $\det\gamma = \pm 1$. Explicitly, this group acts on the charges and the (heterotic) axion-dilaton as

$$\begin{pmatrix} P \\ Q \end{pmatrix} \to \begin{pmatrix} P_\gamma \\ Q_\gamma \end{pmatrix} := \gamma \begin{pmatrix} P \\ Q \end{pmatrix} \quad , \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PGL(2,\mathbb{Z})$$
$$\lambda \to \lambda_\gamma := \frac{a\lambda + b}{c\lambda + d} \ \left(\frac{a\bar{\lambda} + b}{c\bar{\lambda} + d} \right) \ \text{when} \ \ ad - bc = 1 \ (-1) \ . \tag{7.2.18}$$

As we will prove now, this is nothing but the semi-direct product of the Weyl group W and the automorphism group of its fundamental domain (the fundamental Weyl chamber), which is in this case the dihedral group D_3 that maps the regular triangle whose boundaries are orthogonal to $\{\alpha_1, \alpha_2, \alpha_3\}$ to itself. Explicitly, the D_3 is the group with six elements generated by the following two generators: the order two element corresponding to the reflection

$$\alpha_1 \to \alpha_1 \quad , \quad \alpha_2 \leftrightarrow \alpha_3 \tag{7.2.19}$$

and order three element corresponding to the 120° rotation

$$\alpha_1 \to \alpha_2 \quad , \quad \alpha_2 \to \alpha_3 \quad , \quad \alpha_3 \to \alpha_1 \tag{7.2.20}$$

of the triangle. See Figure 7.2.

Recall that the usual S-duality group $PSL(2,\mathbb{Z})$ is generated by the two elements S and T, with the relation $S^2 = (ST)^3 = 1$. In terms of 2×2 matrices, they are given by

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad , \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} .$$

The extended S-duality group $PGL(2,\mathbb{Z})$ is then generated by the above two generators, together with the other one corresponding to the parity reversal transformation (7.2.17)

$$R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

On the other hand, in terms of these matrices and the $PGL(2,\mathbb{Z})$ action (7.2.6) on the vectors in the vector space $M_2(\mathbb{R}) \sim \mathbb{R}^{2,1}$, one of the three generators of the Weyl group W, corresponding to the reflection with respect to the simple root α_1, is given by

$$s_1 : X \to X - 2\frac{(X,\alpha_1)}{(\alpha_1,\alpha_1)}\,\alpha_1 = R(X) . \tag{7.2.21}$$

For the dihedral group D_3, the reflection ($\alpha_2 \leftrightarrow \alpha_3$) generator is given by

$$X \to RS(X) \tag{7.2.22}$$

and the order three 120° rotation generator is given by

$$X \to RSTR(X) . \tag{7.2.23}$$

From the expression for these three generators one can deduce the rest of the elements of W. For example, the reflections s_2, s_3 with respect to the other two simple roots α_2, α_3 are given by R conjugated by the appropriate power (1 and 2 respectively) of the rotation generator $RSTR$.

In particular, we have shown that the extended S-duality group can be written as

$$PGL(2,\mathbb{Z}) \cong O^+(2,1;\mathbb{Z}) \cong W \rtimes D_3 . \tag{7.2.24}$$

This means that the Weyl group is a normal subgroup of the group $PGL(2,\mathbb{Z})$, namely that the conjugation of a Weyl group element with any element of $PGL(2,\mathbb{Z})$ is again a Weyl group element.

This relation between the symmetry of the root system of the present Weyl group and the physical phenomenon of crossing the walls of marginal stability will be further explored later in section 8.6.

7.2.2 K3 Elliptic Genus and the Siegel Modular Form

In this subsection we will focus on the automorphic property of the microscopic partition function $\Phi(\Omega)$. As we have discussed in the previous subsection, the theory has an extended S-duality group $PGL(2,\mathbb{Z})$, which acts naturally on the argument Ω of the partition function as (7.2.6). We therefore expect $\Phi(\Omega)$ to transform nicely under this group action. But it will turn out that this function has a much larger automorphic group $Sp(2,\mathbb{Z}) \supset SL(2,\mathbb{Z})$ under which it displays a nice transformation property. We will now motivate and explain the presence of this automorphic group from a mathematical point of view. The material covered here can be found in, for example, [44, 142]

As it stands in equation (7.2.1), $\Phi(\Omega)$ is a function of the 2×2 symmetric complex matrix Ω. But as $c(n)$ grows with n, it is clear that in order for the function to be convergent Ω should be restricted to lie in the Siegel upper-half plane, obtained by complexifying the vector space $M_2(\mathbb{R})$ introduced before and taking only the future light-cone for the imaginary part

$$\begin{aligned} &\Omega \in M_2 + iV^+ \\ &V^+ = \{X \in M_2, \|X\|^2 < 0\,, \mathrm{Tr}X > 0\}\ . \end{aligned} \tag{7.2.25}$$

See Figure 8.1. In other words, $\Phi(\Omega)$ should be considered as a function on the space $M_2(\mathbb{R}) + iV^+$. But there is another equivalent presentation of this space, namely the Grassmannian of a higher dimensional space

$$M_2 + iV^+ = \frac{O(3,2)}{O(3)\times O(2)} = \{u \in \mathbb{C}^5, \langle u,u\rangle = 0, \langle u,\bar{u}\rangle < 0\}/(u \sim \mathbb{C}^* u)\ . \tag{7.2.26}$$

To see the second equivalence, simply observe that the real and imaginary part of u are indeed two mutually perpendicular timelike vectors which span a maximally timelike surface in the total space $\mathbb{R}^{3,2}$, a phenomenon we have used in our discussion of the K3 moduli space in section 3.2. To see the first equivalence, separate the five-dimensional one $\mathbb{R}^{3,2}$ into $\mathbb{R}^{2,1}\oplus\mathbb{R}^{1,1}$ with the inner product

$$\langle (X; z_+, z_-), (X; z_+, z_-)\rangle = \|X\|^2 + 2z_+z_-\ ,$$

then using the $\mathbb{C}^*$ identification we obtain the following one-to-one mapping between a vector $\Omega \in M_2 + iV^+$ and $u \in \frac{O(3,2)}{O(3)\times O(2)}$:

$$\Omega \to u(\Omega) = (\Omega; 1, -\frac{1}{2}\|\Omega\|^2)\ .$$

From this point of view, $\Phi(\Omega)$ is a function on the coset space $O(3,2)/O(3)\times O(2)$, and it is therefore not surprising that $\Phi(\Omega)$ should have automorphic proper-

ties with respect to the automorphism group $SO^+(3,2;\mathbb{Z}) \cong Sp(2,\mathbb{Z})/\{\pm\mathbb{1}_4\}$. For the explicit map between these two groups, see for example [143].

To be more precise, what we have here is actually a special case of the following theorem of R. Borcherds (Theorem 10.1 of [44]).

Theorem 7.2.1 *Let $g(\tau) = \sum f(4n)q^n$ be a meromorphic modular form with all poles at the cusps of weight $-s/2$ for $SL(2,\mathbb{Z})$ with integer coefficients, with $24|f(0)$ if $s=0$. There is a unique vector ϱ in a Lorentzian lattice $\Gamma = \Gamma^{s+1,1}$ such that*

$$F(\Omega) = e^{-\pi i(\Omega,\varrho)} \prod_{\alpha\in\Gamma_+} \left(1 - e^{-\pi i(\Omega,\alpha)}\right)^{f(-\frac{1}{2}\|\alpha\|^2)} \tag{7.2.27}$$

is a meromorphic automorphic form of weight $f(0)/2$ for $O(s+2,2;\mathbb{Z})$.

Furthermore, define a rational quadratic divisor to be the following zero locus

$$\langle(\alpha;2r,2s),u(\Omega)\rangle = \langle(\alpha;2r,2s),(\Omega;1,-\frac{1}{2}\|\Omega\|^2)\rangle = 0 \quad , \quad r,s\in\mathbb{Z} \tag{7.2.28}$$

for

$$\langle(\alpha;2r,2s),(\alpha;2r,2s)\rangle > 0 \ ,$$

then all the zeros and poles of F lie on the rational quadratic divisors with the multiplicities of the zeros being

$$\sum_{n>0} f(-\frac{n^2}{2}\langle(\alpha;2r,2s),(\alpha;2r,2s)\rangle) \ .$$

In some cases the above product formula is known to be the denominator formula of a certain Borcherds-Kac-Moody algebra. In this case the vector ϱ is the Weyl vector of the algebra.

To see how this theorem applies to our $\Phi(\Omega)$, let's first recall a few facts about the elliptic genus of K3. As was discussed in section 2.1.5, the elliptic genus has a theta-function decomposition given in (2.1.38). From the transformation properties of the theta-function we conclude that $h_\mu(\tau)$'s transform as modular forms of weight $-1/2$.

For the K3 case, we know the full answer in terms of Eisenstein series (2.1.51). Now consider the case when the modular form in the above theorem

is given by $h_\mu/2$, where

$$\begin{aligned} 2\phi_{0,1}(\tau,z) &= \chi_{K3}(\tau,z) = \sum_{\mu=0,1} h_\mu(\sigma)\theta_\mu(\sigma,z) \\ &= 2y^{-1} + 20 + 2y + \mathcal{O}(q) \ , \\ h_\mu(\sigma) &= c(4n-\mu^2)q^{n-\frac{\mu^2}{4}} \\ \theta_\mu &= \sum_{\ell\in\mathbb{Z}} q^{(\ell+\frac{\mu}{2})^2}\, y^{\mu+2\ell} \ . \end{aligned}$$

By taking $f(n) = \frac{1}{2}c(n)$ and comparing the result to (7.2.12) we see that $\Phi(\Omega) = (F_5(\Omega))^2$ is a weight 10 automorphic form for the group $SO^+(3,2;\mathbb{Z}) \cong Sp(2,\mathbb{Z})/\{\pm\mathbb{1}_4\}$, which is also the modular group of a genus two Riemann surface. In other words, $\Phi(\Omega)$ transforms as

$$\Phi(\Omega) \to \big(\det(C\Omega + D)\big)^{10}\, \Phi(\Omega) \tag{7.2.29}$$

when

$$\Omega \to (A\Omega + B)(C\Omega + D)^{-1} \ ,$$

for 2×2 matrices of integral entries satisfying the symplectic condition

$$AB^T = B^T A \quad , \quad CD^T = D^T C \quad , \quad AD^T - BC^T = \mathbb{1}_{2\times 2} \ .$$

In particular, this gives a product formula for the Igusa cusp form of weight 10, which is one of the five generators of the ring of genus two modular forms of all weights, as [143]

$$\Phi(\Omega) = \prod_{(a,b)\ \text{even}} \theta^2_{a,b}(\Omega) = e^{-2\pi i(\varrho,\Omega)} \prod_{\alpha\in\Gamma_+} \Big(1 - e^{-\pi i(\alpha,\Omega)}\Big)^{c(-\|\alpha\|^2/2)} \tag{7.2.30}$$

where the product of the theta function is taken over all $a, b \in (\mathbb{Z}/2\mathbb{Z})^2$ with $a^T b = 0 \bmod 2$.

Furthermore, using the second part of the above theorem, we see that all the zeros of $\Phi(\Omega)$ are of multiplicity two and lie on the rational quadratic divisor

$$\frac{1}{2}\langle(\alpha;2r,2s),(\Omega;1,-\frac{1}{2}\|\Omega\|^2)\rangle = r(\rho\sigma - \nu^2) + n\rho + m\sigma + \ell\nu + s = 0 \ ,$$

with

$$\frac{1}{2}\langle(\alpha;2r,2s),(\alpha;2r,2s)\rangle = \ell^2 - 4nm + 4rs = 1 \quad , \quad \ell,n,m,r,s \in \mathbb{Z} \ . \tag{7.2.31}$$

In the counting formula (7.2.8), these zeros lead to double poles in the integrand.

It is clear that all the above zeor are related to each other by $Sp(2,\mathbb{Z})$ transformations. Indeed, when one identifies Ω with the period matrix of a genus two surface, the poles in $1/\Phi$ occur precisely at those values of Ω at which the genus two surface degenerates into two disconnected genus one surfaces through the pinching of a trivial homology cycle. These degenerations are labelled by elements of $Sp(2,\mathbb{Z})/\{\pm\mathbb{1}_4\}$ and are characterized by the fact that the transformed period matrix is diagonal. From this consideration and from the knowledge that (Ω,α_1) describes such a degeneration, we see that the location of the above rational quadratic divisor can also be written as

$$\big((A\Omega+B)(C\Omega+D)^{-1},\alpha_1\big)=0\ . \tag{7.2.32}$$

7.2.3 The Borcherds-Kac-Moody Superalgebra and the Denominator Formula

After the discussion in the last two sections about the related mathematical properties of the counting formula, now we are ready to see how it is associated with a Borcherds-Kac-Moody, or generalised Kac-Moody, superalgebra. A Borcherds-Kac-Moody superalgebra is a generalisation of the usual Lie algebra by the following facts: (i) the Cartan matrix is no longer positive-definite ("Kac-Moody"), (ii) there are also the so-called "imaginary" simple roots with lightlike or timelike length ("Borcherds"), (iii) it is $\mathbb{Z}_2$-graded into the "bosonic" and the "fermionic" part ("super"). We will summarise the important properties of these algebras that we will use later. See [144] or the appendix of [143] for a more systematic treatment of the subject.

Consider a set $I=\{1,2,\cdots,n\}$ and a subset $S\subset I$. A generalised Cartan matrix is a real $n\times n$ matrix $A=(h_i,h_j)$ that satisfies the following properties

1. either $A_{ii}=2$ or $A_{ii}\leq 0$.

2. $A_{ij}<0$ if $i\neq j$, $A_{ij}\in\mathbb{Z}$ if $A_{ii}=2$.

   Furthermore we will restrict our attention to the special case of BKM algebra without odd real simple roots, which means

3. $A_{ii}\leq 0$ if $i\in S$.

Then the BKM superalgebra $\mathfrak{g}(A,S)$ is the Lie superalgebra with even generators $\mathfrak{h}$, e_i, f_i with $i\in I-S$ and odd generators e_i, f_i with $i\in S$, satisfying

the following defining relations

$$
\begin{aligned}
&[e_i, f_j] = \delta_{ij} h_i \\
&[h, h'] = 0 \\
&[h, e_i] = (h, h_i)\, e_i \quad , \quad [h, f_i] = -(h, h_i)\, f_i \\
&(\mathrm{ad} e_i)^{1-A_{ij}} e_j = (\mathrm{ad} f_i)^{1-A_{ij}} f_j = 0 \quad \text{if } A_{ii} = 2, i \neq j \\
&[e_i, e_j] = [f_i, f_j] = 0 \quad \text{if } A_{ij} = 0\,.
\end{aligned}
$$

Another important concept we need is that of the root space. For later use we have to introduce more terminologies. The *root lattice* Γ is the lattice (the free Abelian group) generated by α_i, $i \in I$, with a real bilinear form $(\alpha_i, \alpha_j) = A_{ij}$. The Lie superalgebra is graded by Γ by letting $\mathfrak{h}$, e_i, f_i have degree 0, α_i and $-\alpha_i$ respectively. Then a vector $\alpha \in \Gamma$ is called a *root* if there exists an element of $\mathfrak{g}$ with degree α. A root α is called *simple* if $\alpha = \alpha_i$, $i \in I$, *real* if it's spacelike $\|\alpha\|^2 > 0$ and *imaginary* if otherwise. It is called *even (odd)* if the elements in $\mathfrak{g}$ with degree α are generated by the even (odd) generators, and *positive (negative)* if it is a positive- (negative-) semi-definite linear combinations of the simple roots. It can be shown that a root is either positive or negative, and either even or odd. Furthermore, the *Weyl group* W of $\mathfrak{g}$ is the group generated by the reflection in $\Gamma \otimes \mathbb{R}$ with respect to all real simple roots. A Weyl vector ϱ is the vector with the property

$$
(\varrho, \alpha_i) = -\frac{1}{2}(\alpha_i, \alpha_i)
$$

for all simple real roots α_i. It is easy to see that, for the case discussed in section 7.2.1, the vector (7.2.13) is indeed the Weyl vector satisfying the above condition.

Just as for ordinary Kac-Moody algebras, there is the following so-called denominator formula

$$
e(-\varrho) \prod_{\alpha \in \Delta_+} \big(1 - e(-\alpha)\big)^{\mathrm{mult}\alpha} = \sum_{w \in W} \epsilon(w) w(e(-\varrho)\, S)\,, \tag{7.2.33}
$$

where Δ_+ is the set of all positive roots, $\epsilon(w) = (-1)^{\ell(w)}$ where $\ell(w)$ is the length of the word w in terms of the number of generators, defined in (8.9.3)[3]. There are differences between this product formula and the usual Weyl denominator formula due to, first of all, the fact that it's "super". Concretely,

[3]Warning: the conventions of the signs of the above formula, in particular the signs of the Weyl vector, do vary in the existing literature.

we have used the following definition for the "multiplicity" of roots $\mathrm{mult}\alpha$:

$$\mathrm{mult}\alpha = \mathrm{dim}\mathfrak{g}_\alpha(-\mathrm{dim}\mathfrak{g}_\alpha) \quad \text{when} \;\; \alpha \;\; \text{is even (odd)} \; .$$

Furthermore, there is a correction term S on the right-hand side of the formula due to the presence of the imaginary roots. The exact expression for S is rather complicated for generic BKM superalgebras and can be found in [144, 143].

In the above formula, the $e(\mu)$'s are formal exponentials satisfying the multiplication rule $e(\mu)e(\mu') = e(\mu+\mu')$. Taking $e(-\mu) \to e^{-\pi i(\mu,\Omega)}$, the left-hand side becomes the product formula of (7.2.27) with $\mathrm{mult}\alpha = f(-\frac{1}{2}\|\alpha\|^2)$.

Now we will concentrate on the case discussed in the last subsection. We have seen that (7.2.30)

$$\Phi(\Omega)^{1/2} = e^{-\pi i(\varrho,\Omega)} \prod_{\alpha\in\Delta^+} \left(1 - e^{-\pi i(\alpha,\Omega)}\right)^{\frac{1}{2}c(-\|\alpha\|^2/2)} \; . \tag{7.2.34}$$

From the transformation property of the above automorphic form under the Weyl group W introduced in (7.2.15), we can also rewrite it in a form as the right-hand side of (7.2.33). From this equivalence one can therefore read out the set of even and odd imaginary simple roots and therefore construct a "automorphic-form corrected" Borcherds-Kac-Moody superalgebra, whose denominator is the Siegel modular form $\Phi(\Omega)^{1/2}$ of weight five and have

$$\mathrm{mult}\alpha = \frac{1}{2}\, c(-\|\alpha\|^2/2) \; , \tag{7.2.35}$$

where $c(n)$ is the Fourier coefficient of the K3 elliptic genus.

By construction, the real simple roots can be chosen to be the basis of the charge vector appearing in the dyon counting formula (7.2.9), while the Weyl group is the one generated by the three reflections with respect to them (7.2.15). In particular, the part of the generalised Cartan matrix corresponding to the simple roots is given by (7.2.14). The above expression for the root multiplicity together with the property $c(n) = 0$ for $n < -1$ is indeed consistent with the fact that all real roots have length $\|\alpha\|^2 = 2$. The set of all real positive roots is denoted as Δ_+^{re} as was announced in (7.2.16).

The fact that the dyons are counted with a generating function which is simply the square of the denominator formula of a generalized Kac-Moody algebra, and that the charge vectors naturally appear as elements of its root lattice, strongly suggests a physical relevance of this superalgebra in the BPS sector of the theory. Later we will see how features of this algebra appear in a physical context and elucidate (part of) the role of this algebra.